# A Trip to the Minimal Surface Zoo

## Or: Selected Classification Problems in Minimal Surface Theory


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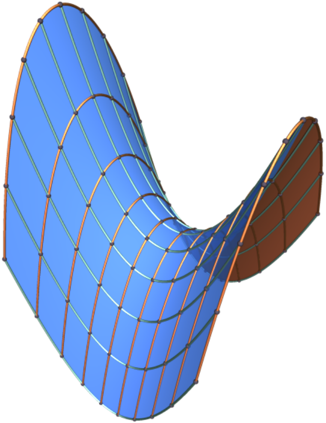
# Definition of a Minimal Surface

## Definition

A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with mean curvature $H \equiv 0$.



# Definition of a Minimal Surface

## Definition

A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with mean curvature $H \equiv 0$.

## Where does the name minimal come from?

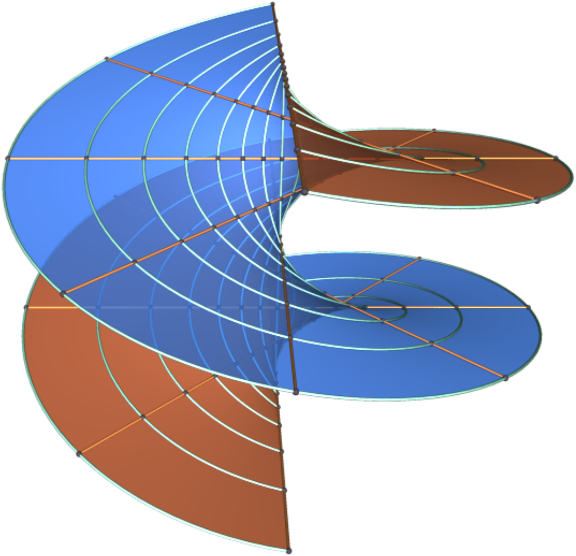
The condition $H \equiv 0$ is equivalent to the condition that a small, local deformation will *increase* the area.

The intersection of a minimal surface with sufficiently small balls is a surface patch which minimizes area with respect to the boundary.

# Examples - Plane

- Discoverer unknown
- Has every property you want

# Examples - Helicoid



- Meusnier (1776)
- Only ruled minimal surface
- Is simply connected

# Simply Connected Spaces

Suppose an ant on a surface ties a rope to a tree, then runs out (wherever she wants) holding the rope, finally coming back to the tree. She ties the other end to the tree and then pulls the rope all back to the tree. If she can pull all the rope back, we call the surface simply connected.

Mathematically, we say that a space is simply connected if it

- is path connected, and
- any closed loop can be continuously deformed (without breaking!) into any other loop.

The helicoid is simply connected.

# Typical Problem

### Problem

Classify all complete, embedded, simply connected minimal surfaces.

This problem was open until very recently, when four people proved:

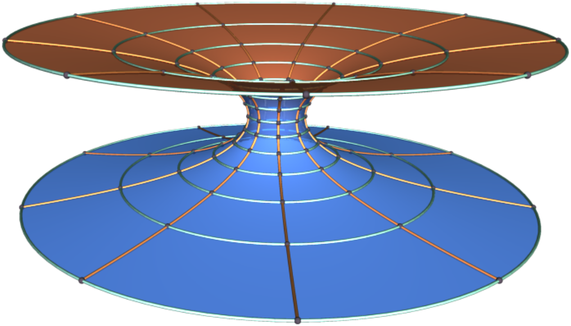
### Theorem

*The only complete, embedded, simply connected minimal surfaces are the plane and the helicoid.*

This is, in some sense, the most basic classification question!

# Not An Example

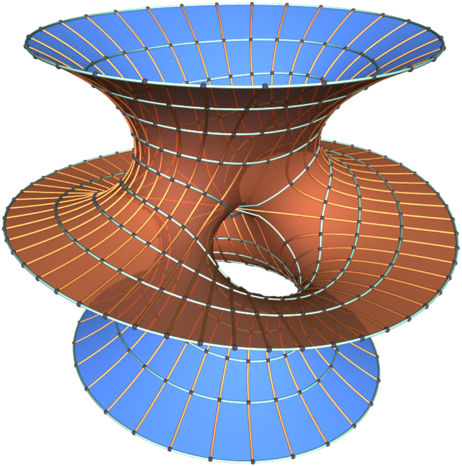
# Examples - Catenoid



- Euler (1741), Meusnier (1776)
- Only minimal surface of revolution (except for the plane)

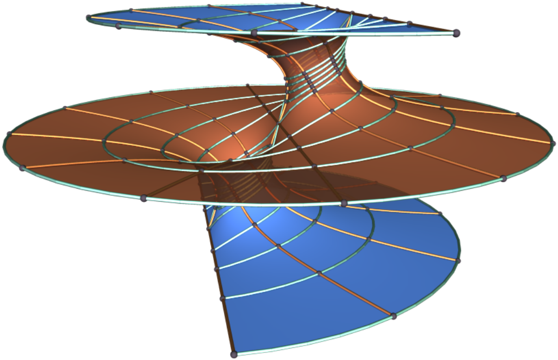
# Examples - Costa's Surface

From 1700 - 1992, the only known minimal surfaces either were the catenoid, helicoid, or plane; or they had infinite topology (in some sense: infinitely many holes).



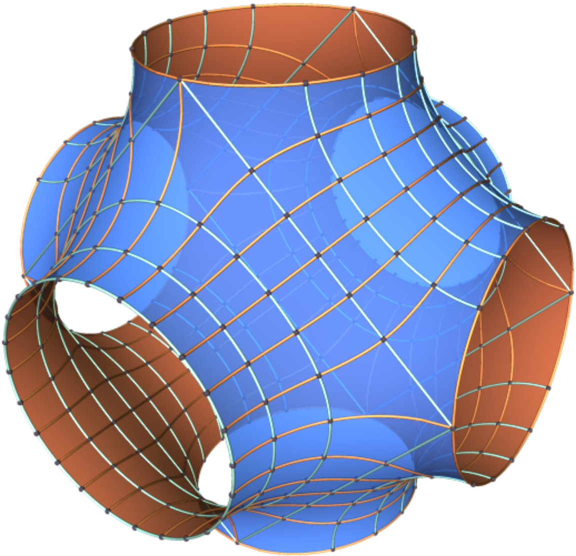
- Costa (1992) as a graduate student
- Complete, embedded, topologically a torus
- Rejuvenated the study of minimal surfaces

# Examples - Riemann



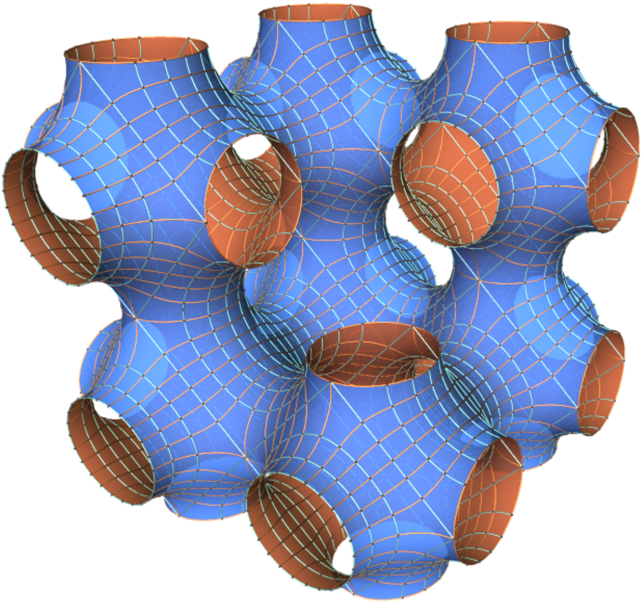
- Riemann (pre-1866)
- Singly periodic surface
- Foliated by (generalized) circles

# Examples - P Surface

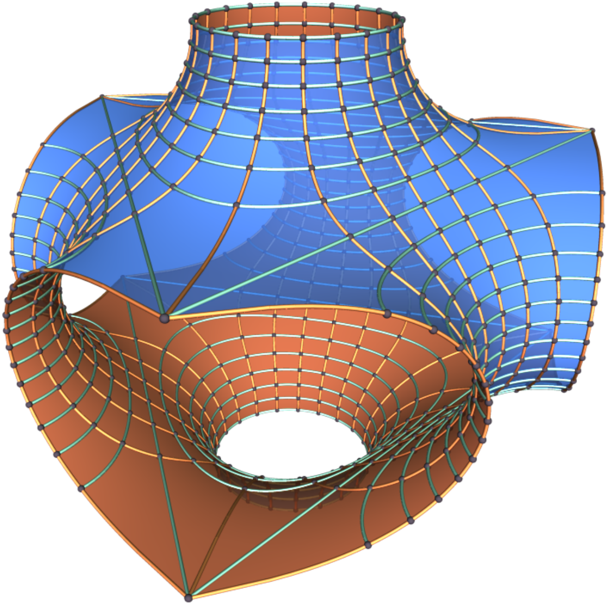


- Schwarz (1865)
- Triply periodic surface; cubical lattice
- Tiled by right angled hexagons

# Examples - P Surface

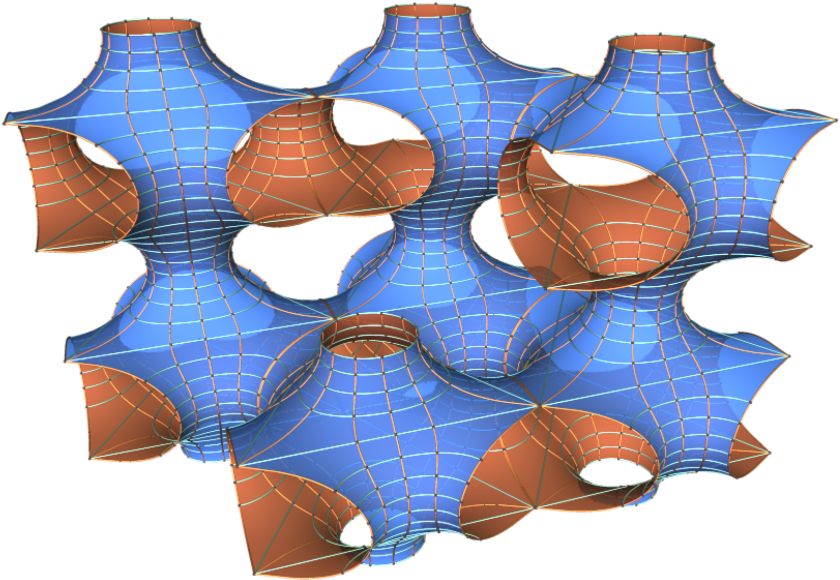


# Examples - H Surface

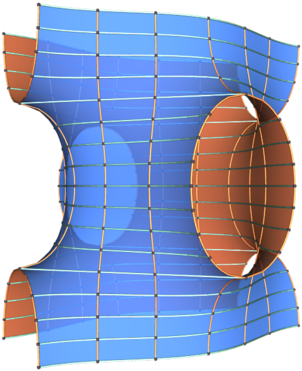


- Schwarz (1865)
- Triply periodic surface; hexagonal lattice
- Lots of straight lines, planar symmetries

# Examples - H Surface

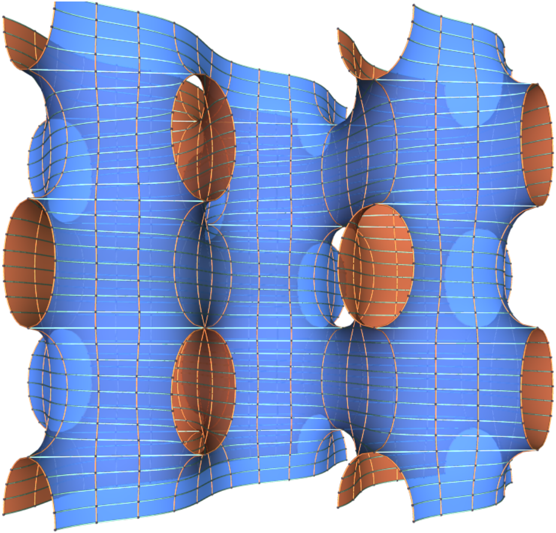


# Examples - CLP Surface



- Schwarz (1865)
- Triply periodic surface; cubical lattice
- Lots of straight lines, planar symmetries

# Examples - CLP Surface



# Definition of Triply Periodic Minimal Surface

## Definition

- A surface $M \subset \mathbb{R}^3$ is called closed if for every point $p$ not in $M$, there is a small ball centered at $p$ that does not intersect $M$.
- A surface $M \subset \mathbb{R}^3$ is called compact if it is closed and bounded.

## Definition

A triply periodic minimal surface M is a minimal surface in $\mathbb{R}^3$ that is invariant under translations in three independent directions. (The three translations generate a lattice $\Lambda$ in $\mathbb{R}^3$.) The quotient surface $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact, has no boundary, and is minimal.

# Who Cares?

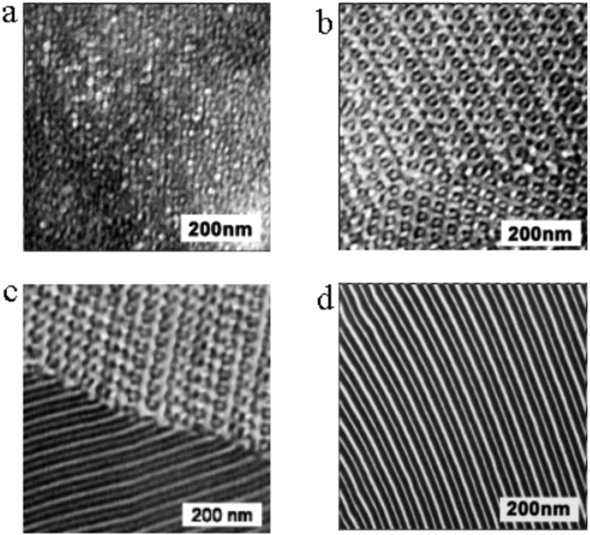
Mathematicians are interested in these surfaces because:

- Solution to a differential equation that is neither too easy nor too hard
- Finding the minima of something is a common problem, and area is a natural thing to want to minimize
- They are at the intersection of tons of beautiful mathematics, like complex analysis, topology, geometry, differential equations, and algebraic geometry.

Physical scientists are interested in these surfaces:

- Interface in polymers
- Physical assembly during chemical reactions
- Microcellular membrane structures

# TEM of Polymers Showing Periodic Structure



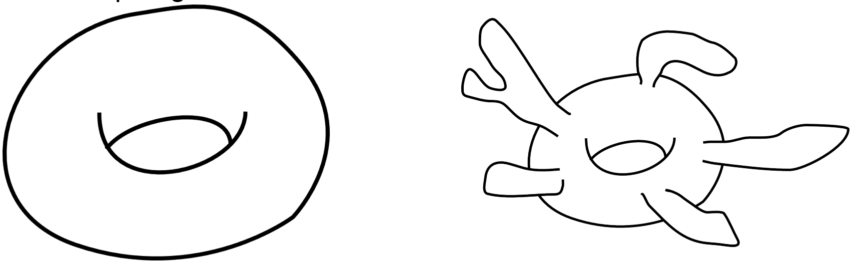

Novel Morphologies of Block Copolymer Blends via Hydrogen Bonding. Jiang, S., Gopfert, A., and Abetz, V.

Macromolecules, 36, 16, 6171 - 6177, 2003, 10.1021/ma0342933

# Topology and genus

Topologists study (among *lots* of other things) surfaces and spaces in the following way: two surfaces are equivalent if one can be (continuously) deformed into the other.

To a topologist, these two are the same surface:



# Topology and genus

Topologists study (among *lots* of other things) surfaces and spaces in the following way: two surfaces are equivalent if one can be (continuously) deformed into the other.

## How can we classify closed, oriented surfaces?

By genus. This is, loosely, the number of “holes” in the surface. To be more precise, we use the concept of fundamental group. For these surfaces, the fundamental group is always

$$\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{2 \cdot g}$$

The $g$ in the formula is called the genus.

- The sphere has genus 0.
- The torus has genus 1.
- The two-holed torus has genus 2.

# Classification of TPMS

Rough classification by the genus of $M/\Lambda$:

## Theorem

*(Meeks, 1975) Let $M$ be a triply periodic minimal surface of genus $g$. The Gauss map of $M/\Lambda$ is a conformal branched covering map of the sphere of degree $g-1$.*

## Corollary

*The smallest possible genus of $M/\Lambda$ is 3.*

## Theorem

*(Traizet, January 9, 2006 (!)) There exists an embedded triply periodic minimal surface of every genus at least 3.*

# Other classifications?

Many triply periodic surfaces are known to come in a continuous family (or deformation).

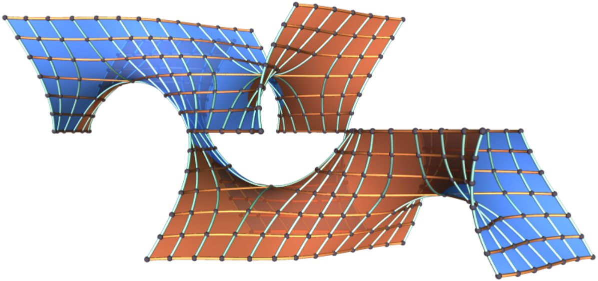
## Theorem

*(Meeks, 1975) There is a five-dimensional continuous family of embedded triply periodic minimal surfaces of genus 3.*

## What's in the Meeks' family?

All proven examples of genus 3 triply periodic surfaces are in the Meeks' family, with two exceptions, the gyroid and the Lidinoid. Do these surfaces admit no deformations?

# The Gyroid



- Schoen, 1970 (while trying to find strong / light structures for NASA)
- Triply periodic surface
- Contains no straight lines or planar symmetry curves

# What is the Gyroid?

## The Associate Family

There is a way of deforming minimal surfaces by bending them. This new surface patch is locally isometric to the original (which is mathspeak for: if you could make your minimal surface out of paper, you could get to the new surface without tearing the paper).

## The Gyroid

In general, this transformation does not yield patches that fit together to give an embedded (or even immersed) surface.

For exactly one value of $\theta$, $\theta \approx 51.9852^\circ$, (an index for the deformation) so that this transformation applied to the P surface gives an embedded, triply periodic minimal surface, called the gyroid.

# Results

### Theorem

*(W., 2005) There is a continuous, one-parameter family of embedded triply periodic minimal surfaces of genus 3 that contains the gyroid. Each surface admits an order 2 rotational symmetry.*

A slight extension gives:

- Another 1-parameter gyroid family (preserves order 3 rotation)
- Analogous results for the Lidinoid

These show that all known examples of triply periodic minimal surfaces (with genus 3) are not isolated.

# Homework Problems to Work On

## Describe a 5-parameter family of gyroids

- The gyroid is v.p.-stable; this suggests that a 5-parameter family (like Meeks' should exist).
- Meeks' methods do not seem immediately adaptable
- Neither do these methods (not enough symmetries)
- Maybe exploit a "hidden symmetry"

# Homework Problems to Work On

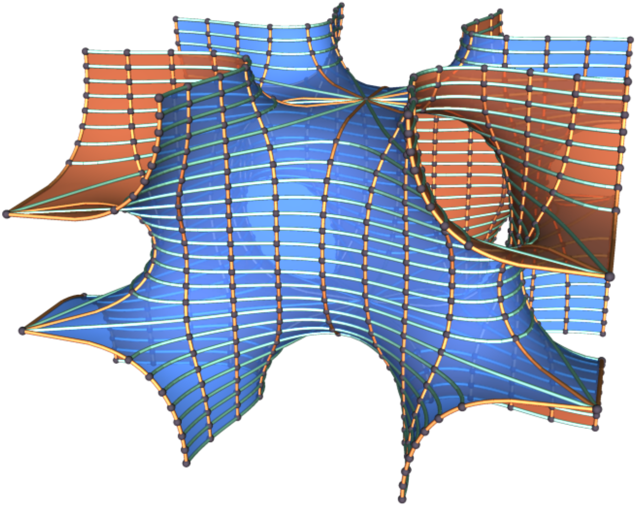
## Understand limits of this family

- Traizet works “backward” from proposed limit surfaces to construct minimal surface examples
- An understanding of the limits of this and other families may help to discover more examples
- Beautiful stuff!

## Attack genus four surfaces

- Precious little is known about genus 4 surfaces
- Method generalizes nicely, even though these surfaces are no longer hyperelliptic
- Would be nice to find examples, deformations, start of a classification?

# Last Picture - A Genus Four Surface - Schoen's I-WP



- Alan Schoen (1970), same NASA scientist
- Has genus four

# For More Information and Pictures

The details I omitted, and more (in progress):

A. Weyhaupt. *Deformations of triply periodic minimal surfaces*. Preprint.

The definitive introduction to applying flat structures to minimal surfaces:

M. Weber and M. Wolf. *Teichmüller theory and handle addition for minimal surfaces*. Ann. of Math. (2), 156(3):713-795, 2002.

The Virtual Museum of Minimal Surfaces:

`http://www.indiana.edu/~minimal/`

(click on “Archive”)