On the moduli space of triply periodic minimal surfaces

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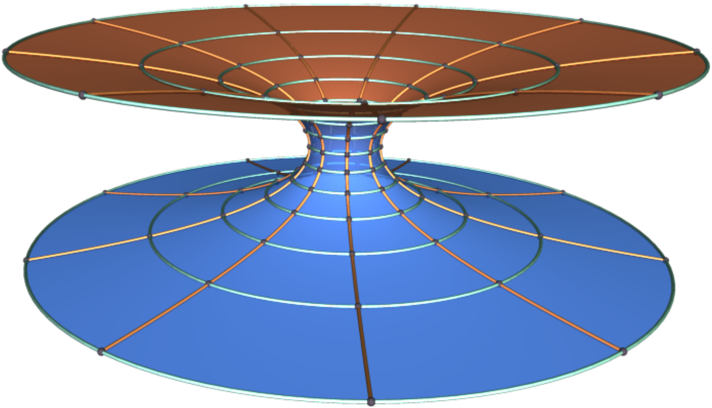
Definition of a minimal surface

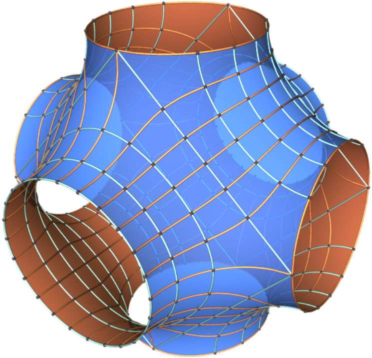
A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with constant mean curvature $H \equiv 0$.



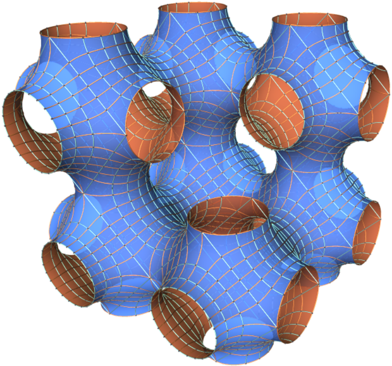
Catenoid

Definition of triply periodic minimal surface

A triply periodic minimal surface is a minimal surface M immersed in $\mathbb{R}^3$ that is invariant under the action of a rank 3 lattice Λ.



Schwarz P Surface



If M is embedded, the quotient $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact.

Large family of examples

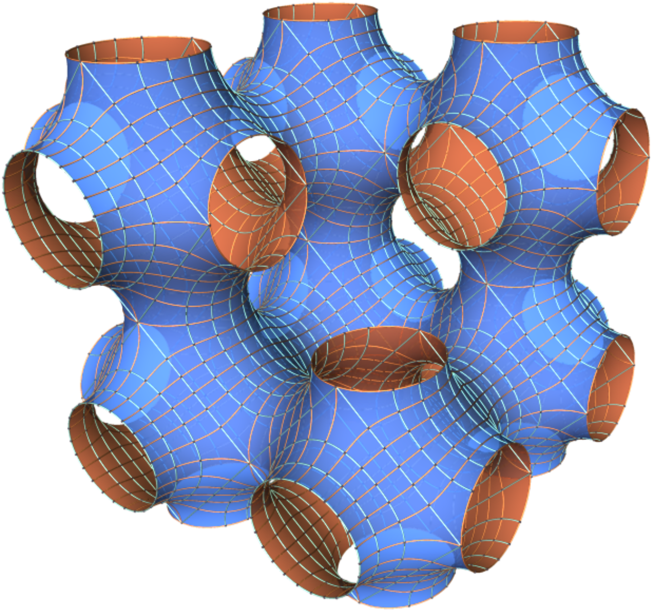
Since M/Λ is compact (for an embedded TPMS M), it has a well-defined (finite) genus.

Theorem (Meeks ’75)

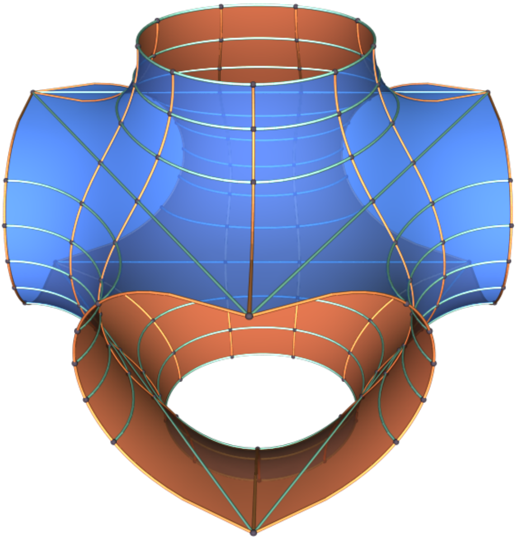
Every TPMS in $\mathbb{R}^3$ has genus at least 3.

Many classical examples have genus 3.

Schwarz P surface



Schwarz H surface



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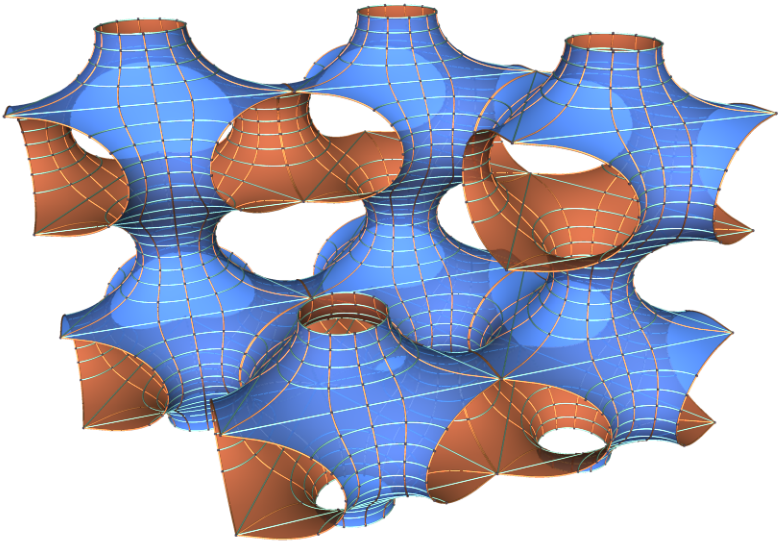

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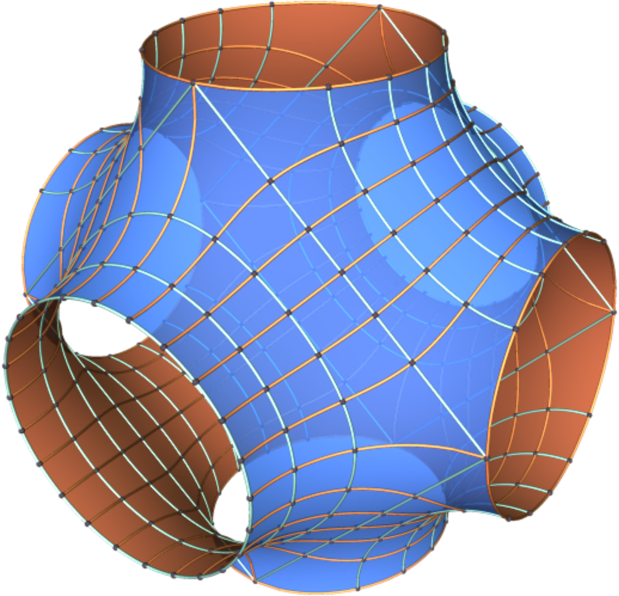
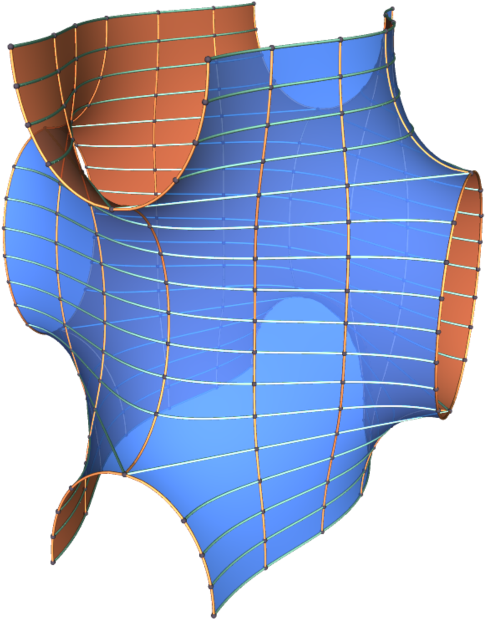


CLP surface



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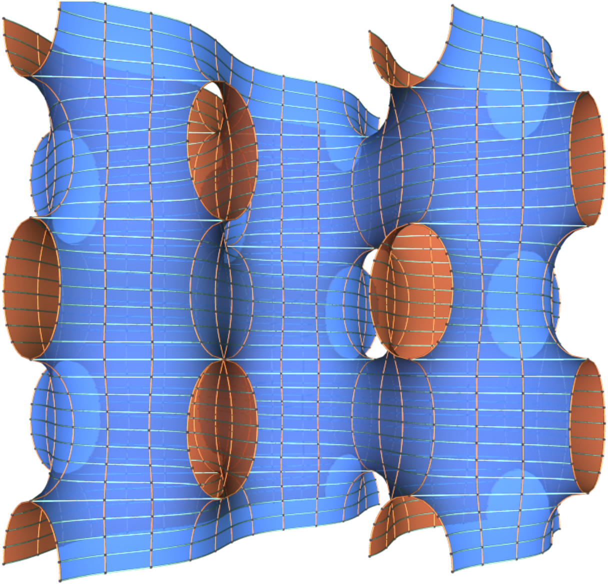

Previous moduli space results

Theorem (Meeks '75)

There is a continuous 5-parameter family of embedded triply periodic minimal surfaces of genus 3 in $\mathbb{R}^3$.

There are many surfaces not in Meeks' family: for example, the H surface family, the gyroid, and the Lidinoid.

Theorem (W. '06)

There is a one parameter family of embedded TPMS of genus 3 that contains the Lidinoid and a one parameter family of embedded TPMS of genus 3 that contains the Lidinoid. None of these surfaces are in Meeks' family.

Theorem (Weber '07)

There is a 2 parameter family of embedded surfaces that contains the H surface.

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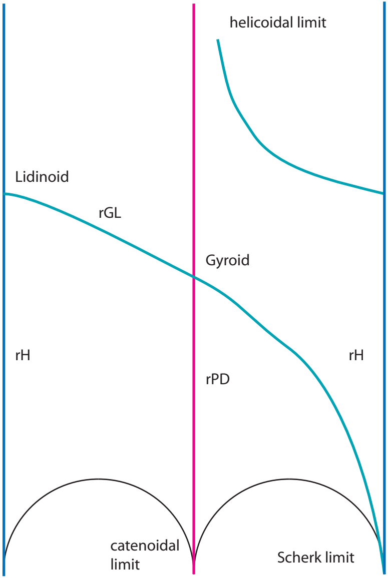
A full understanding of a (small part of) moduli space

Theorem

Let M be an embedded, genus 3, TPMS in $\mathbb{R}^3$ that is invariant under an order 3 rotational symmetry. Then M is a surface in one of the rH, rPD, or rGL families. (In particular, the rG and rL families coincide.)

The picture shows moduli space, parameterized using marked tori and the upper half-plane.



The rH family

The rPD family

The rGL family

Parameterizing surfaces with tori

Let M/Λ be an embedded genus 3 TPMS invariant under an order 3 rotation ρ. The map

$$\rho : M/\Lambda \to M/\Lambda/\rho$$

is a 3-fold branched covering map.

By Riemann-Hurwitz there are exactly 2 branch points and $M/\Lambda/\rho$ has genus 1; these fixed points of ρ are exactly where the surface normal is vertical.

All minimal surfaces are completely determined by a Riemann surface X, a Gauss map G (stereographic projection of the normal) and their “height differential” dh (a holomorphic 1-form).

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Height differential

dh is invariant under ρ (we have oriented the surface so that the axis of ρ is vertical). Therefore, *dh* descends as a holomorphic 1-form to the torus. Since up to a constant there is only one holomorphic 1-form on a torus,

$$dh = re^{i\theta} dz$$

Changing *r* simply dilates the surface in space. θ is the "angle of association" and is used to solve the period problem.

Gauss map

G is not invariant, but G^3 is invariant under ρ. One can see that G^3 has double order zeros (poles) at the branch points. Also, G^3 is well-defined on the torus (so is doubly periodic). By Liouville's theorem,

$$G^3 = \lambda e^{i\varphi} \frac{\theta_{11}(z,\tau)}{\theta_{11}(z-a,\tau)}$$

Adjusting φ simply rotates the surface in space. λ is the so-called Lopez-Ros factor and is used to solve the period problem.

Summary of the parameterization

Any genus 3, triply periodic minimal surface that is invariant under an order 3 rotation is determined by the following (real) parameters:

λ	$G^3 = \lambda e^{i\varphi} \frac{\theta_{11}(z,\tau)}{\theta_{11}(z-a,\tau)}$	used to close periods (embedded sfc)
φ	$G^3 = \lambda e^{i\varphi} \frac{\theta_{11}(z,\tau)}{\theta_{11}(z-a,\tau)}$	rotation in space
r	$dh = re^{i\theta}dz$	dilation in space
θ	$dh = re^{i\theta}dz$	used to close period (embedded sfc)
$a + bi$	gen. of torus lattice	one is used to obtain 1-param. family

Future work

1. Tie up some of the loose ends for order 3 case

2. The case of order 2 invariant surfaces is more difficult. Here, the branch points are not as tightly constrained by Abel's formula, so one gets additional parameters. Two dimensional families seem to occur.

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