An Illustrated Stroll Through the Forest of Minimal Surfaces

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Definition of a Minimal Surface

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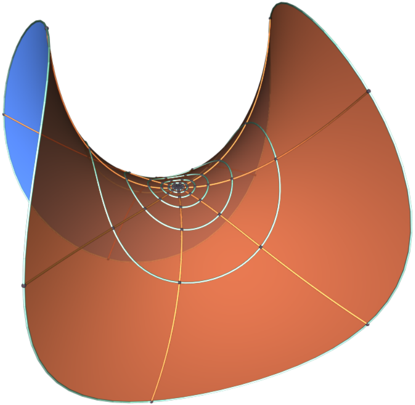

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A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with mean curvature $H \equiv 0$.

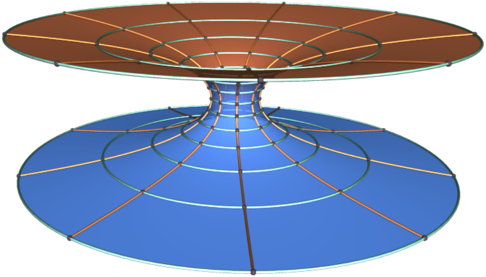
Where does the name minimal come from?

The condition $H \equiv 0$ is equivalent to the condition that a small, local deformation will *increase* the area.

The intersection of a minimal surface with sufficiently small balls is a surface patch which minimizes area with respect to the boundary.

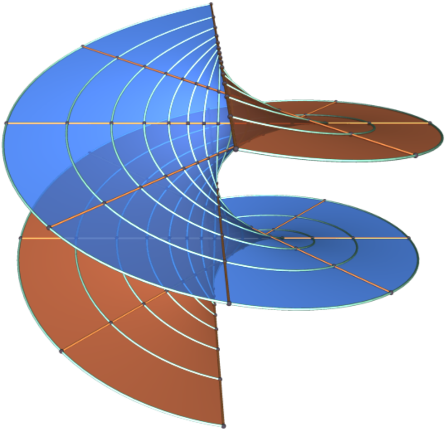
Examples - Plane

Examples - Catenoid



- Euler (1741), Meusnier (1776)
- Only minimal surface of revolution
- "Physically" formed from two congruent circles translated a fixed perpendicular distance

Helicoid



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- Only “ruled” minimal surface
- Is simply connected (all loops are contractible)

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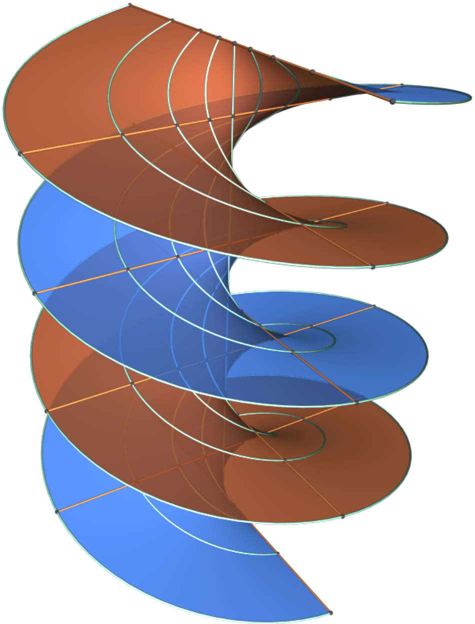

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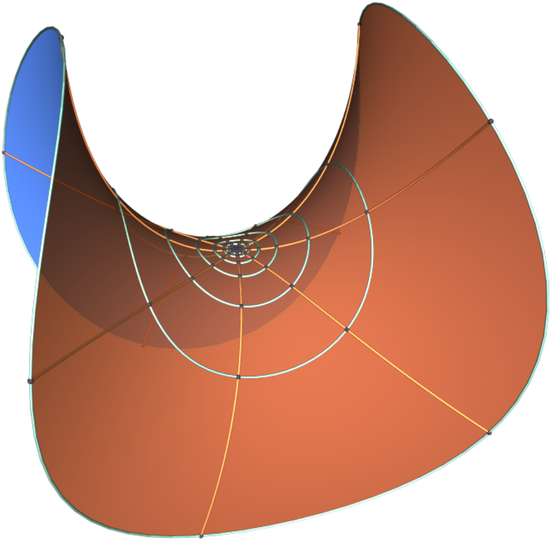
Example of a surface that is not embedded - Enneper's surface



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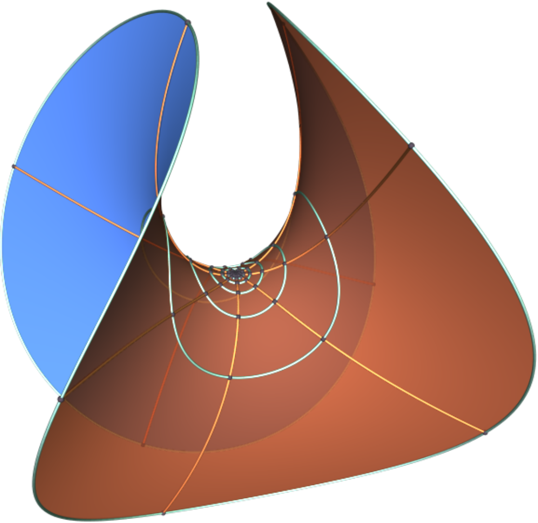

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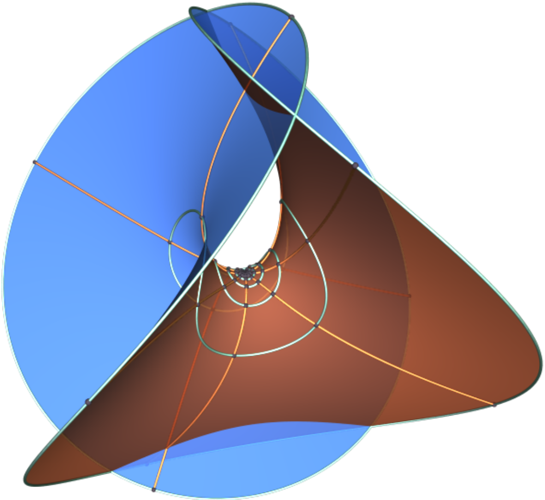

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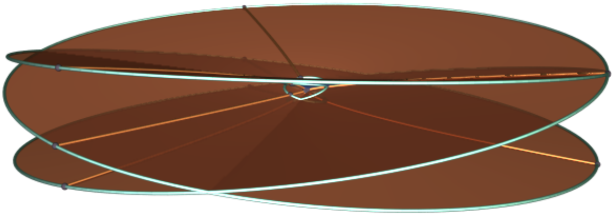

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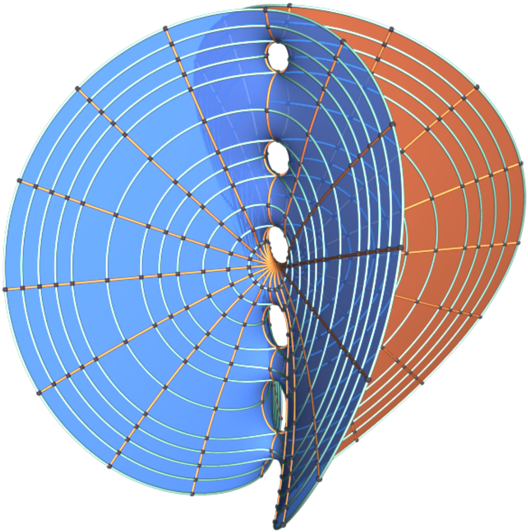

2000 International Snow Sculpture Championships

Typical Topology / Geometry Interplay

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Classify all complete, embedded, simply connected minimal surfaces.

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This problem was open until 2005, when Meeks/Rosenberg (using Colding/Minicozzi) proved:

Theorem

The only complete, embedded, simply connected minimal surfaces in $\mathbb{R}^3$ *are the plane and the helicoid.*

This is, in some sense, the most basic classification question!

The Hoffman / Meeks conjecture

A relationship between genus and ends

Hoffman and Meeks conjectured that for any embedded minimal surface (with “finite topology”)

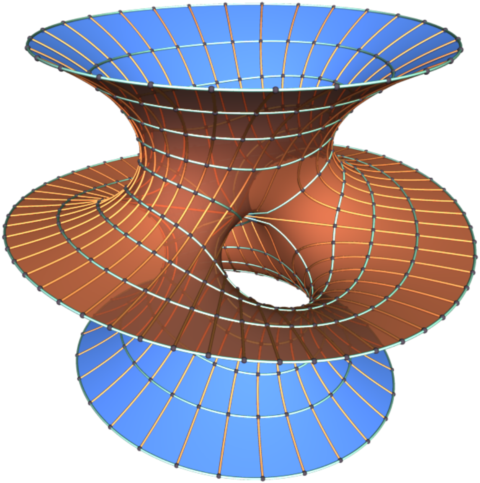
$$\text{genus} + 2 \geq \text{number of ends}$$

This is widely believed to be true, but we have essentially no progress on this conjecture.

(Genus refers to the genus of the underlying Riemann surface (ignoring punctures for the ends).)

Costa (sharpness of Hoffman / Meeks)

From 1700 - 1984, the only known minimal surfaces either were the catenoid, helicoid, or plane; or they had infinite topology.



- Discovered in 1984 by Costa (a graduate student)
- Conformally is a thrice-punctured torus
- First example of an embedded torus
- Shows sharpness of Hoffman / Meeks conjecture

Creating new surfaces from old (associate family)

Harmonic Functions and Minimal Surfaces

Minimal surfaces are described by harmonic functions

$H \equiv 0$ implies that if a map $f = (f_1, f_2, f_3) : U \to \mathbb{R}^3$ is a conformal parameterization of a minimal surface, then f_i is harmonic. All (smooth) surfaces admit a conformal parameterization.

Harmonic Functions and Minimal Surfaces

Better yet - complex analysis!

If M is a minimal surface, there exists a (meromorphic) function G and a holomorphic 1-form dh such that

$$f(w) = \left(\mathrm{Real} \frac{1}{2} \int_{\cdot}^{w} (1/G - G)\, dh, \mathrm{Real} \frac{i}{2} \int_{\cdot}^{w} (1/G + G)\, dh, \mathrm{Real} \int_{\cdot}^{w} dh \right)$$

(Here the domain is a Riemann surface with punctures.)

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(Here the domain is a Riemann surface with punctures.)

Who cares? G is the Gauss map (stereographic projection of the normal). Even better, given any meromorphic function G and a dh (subject to very mild compatibility conditions), the above formula generates a minimal surface!

The associate family construction

Associate family

This cut and twist procedure we saw above can be very easily parameterized as follows. If

$$f(w) = \left(\mathrm{Real} \int_{\cdot}^{w} \omega_1 \, dh, \mathrm{Real} \int_{\cdot}^{w} \omega_2 \, dh, \mathrm{Real} \int_{\cdot}^{w} dh \right)$$

is a minimal surface, then

$$f_\theta(w) = \left(\mathrm{Real}\ e^{i\theta} \int_{\cdot}^{w} \omega_1 \, dh, \mathrm{Real}\ e^{i\theta} \int_{\cdot}^{w} \omega_2 \, dh, \mathrm{Real}\ e^{i\theta} \int_{\cdot}^{w} dh \right)$$

is also a minimal surface (in fact, they are isometric).

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Adjoint surface

If $\theta = \frac{\pi}{2}$, then f_θ becomes

$$f_{\frac{\pi}{2}} = \left(-\mathrm{Imag} \int \omega_1 dh, -\mathrm{Imag} \int \omega_2 dh, -\mathrm{Imag} \int dh\right)$$

Parameterization of Catenoid / Helicoid Family

The catenoid

The catenoid can be described by $G = z$, $dh = \frac{1}{z}dz$ with domain $\mathbb{C} - \{0\}$. Thus

$$f(w) = \mathrm{Real}\left(\int_{.}^{w} 1 - \frac{1}{z^2} dz, \mathrm{Real}\, i \int_{.}^{w} 1 + \frac{1}{z^2} dz, \mathrm{Real} \int_{.}^{w} 1/z dz \right)$$

f is well-defined, since the integral of any non-trivial loop has zero real part.

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The helicoid

The helicoid is thus

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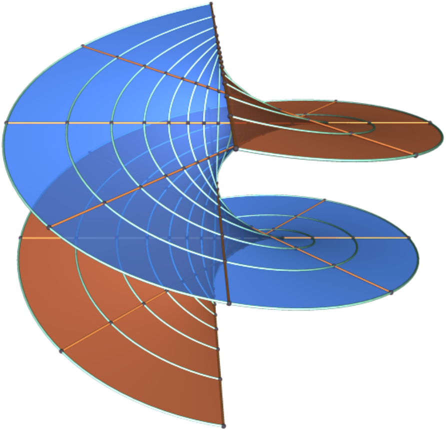
Integration of a closed loop around the origin has non-zero real part!

Helicoid



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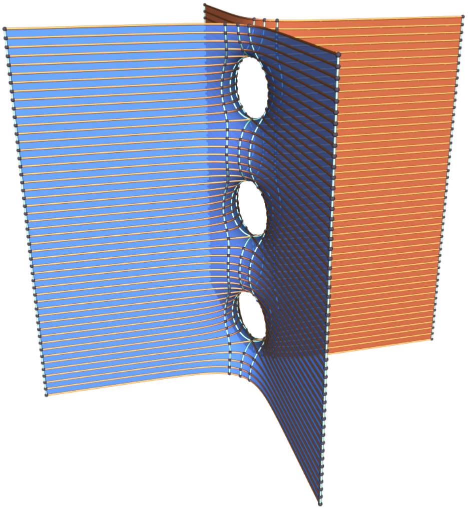
Sherk's Singly Periodic Surface



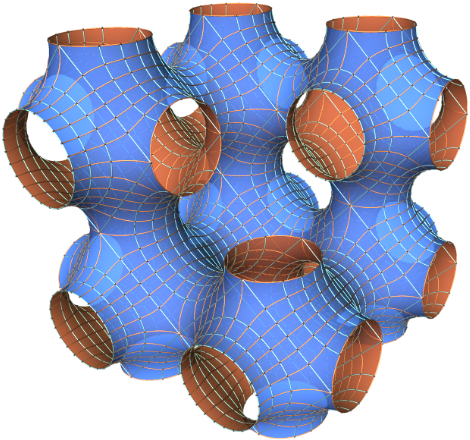
- Classical example from Sherk
- There are examples with any even number of "wings"
- Karcher discovered a screw motion invariant example

Triply periodic minimal surfaces

A triply periodic minimal surface is a minimal surface that is invariant under a translation in space by 3 independent vectors.

These 3 vectors generate a lattice Λ in $\mathbb{R}^3$. Another way of saying this is that all non-trivial loops in the domain must have periods that form a rank-3 lattice.

Triply periodic minimal surfaces



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- Has a cubical lattice
- Quotient of surface by lattice is genus 3, compact
- Physically – "square catenoids"

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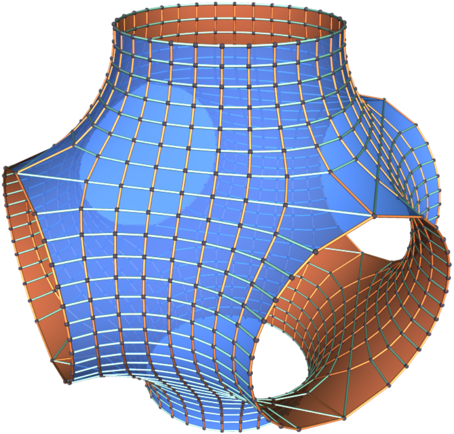

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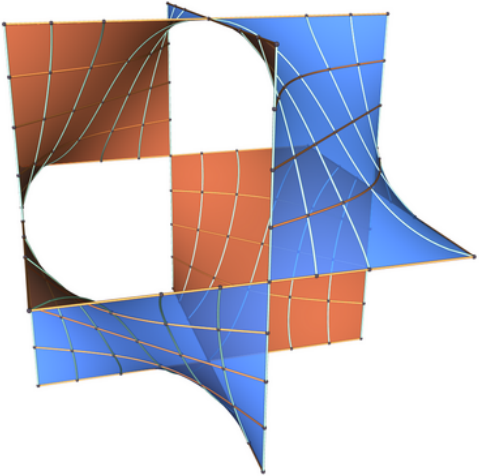

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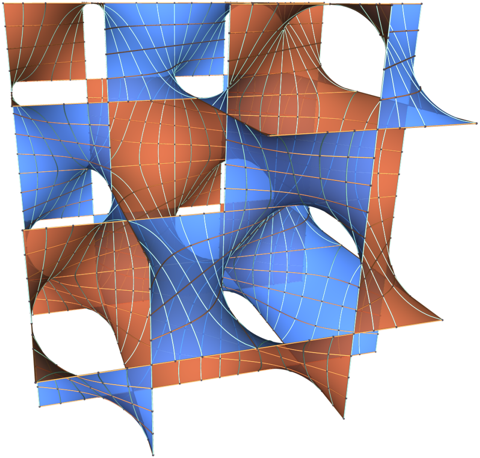

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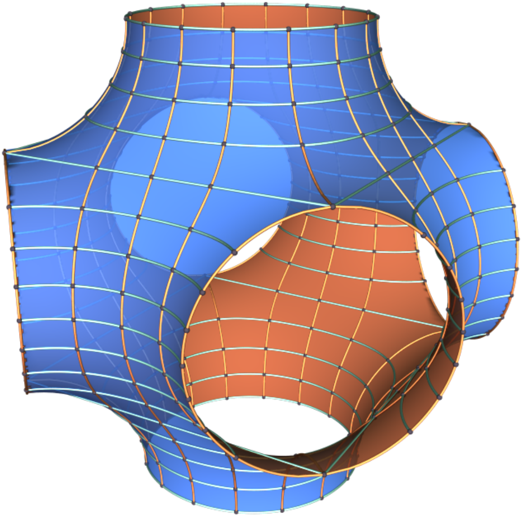

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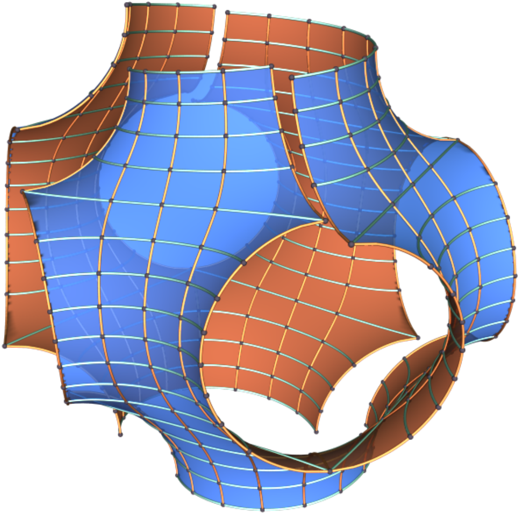

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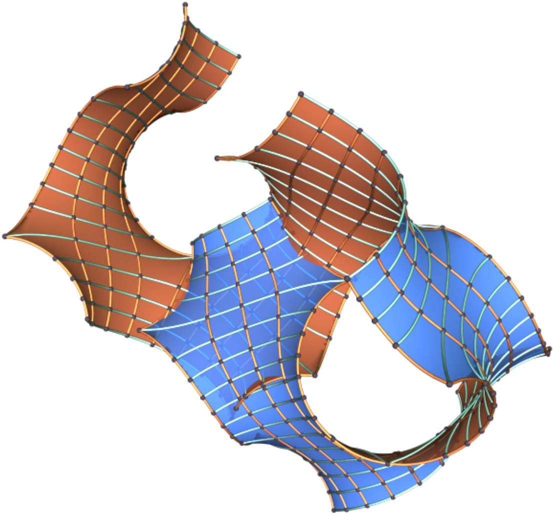

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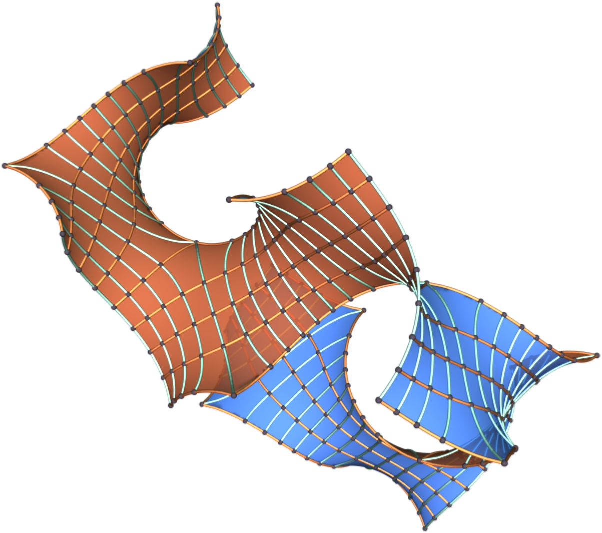

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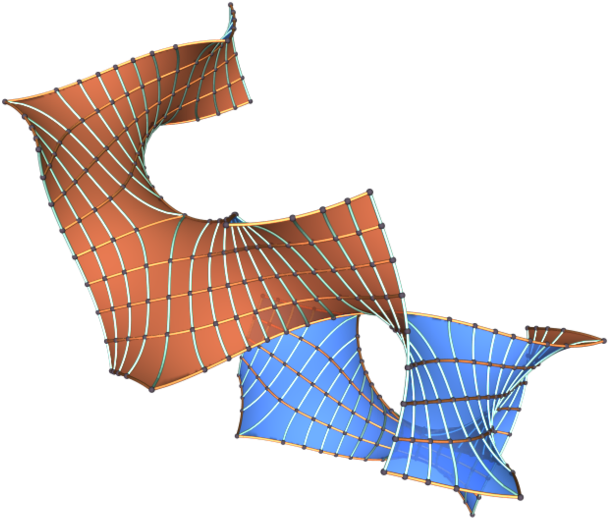

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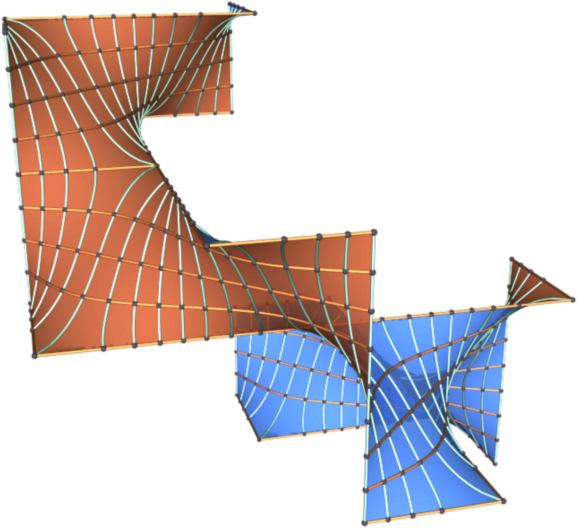
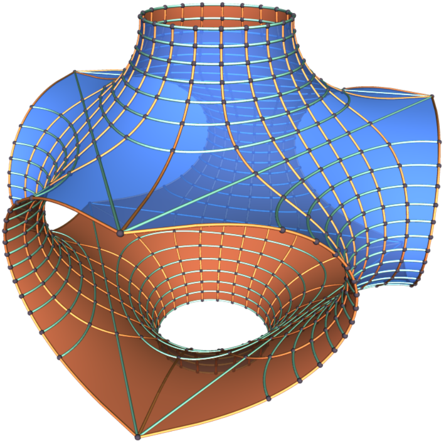


H Surface



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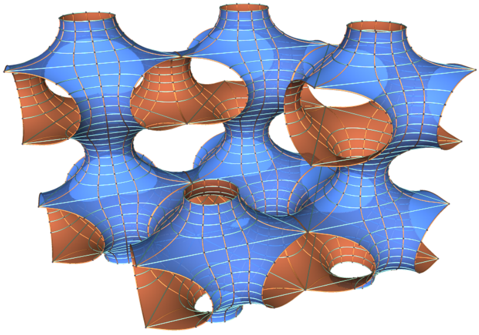

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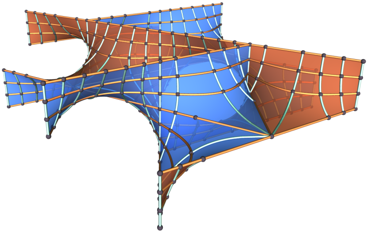
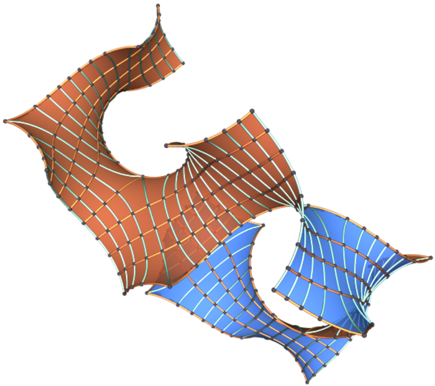
Adjoint of H Surface - not embedded



- Not all adjoints of embedded surfaces are embedded

Schoen's Gyroid Minimal Surface



- The associate family of the P and D surface contains exactly one embedded member - the gyroid
- Discovered 1970 by Alan Schoen (NASA scientist)
- Contains no straight lines and no planar symmetries (first example)
- Lattice is rectangular
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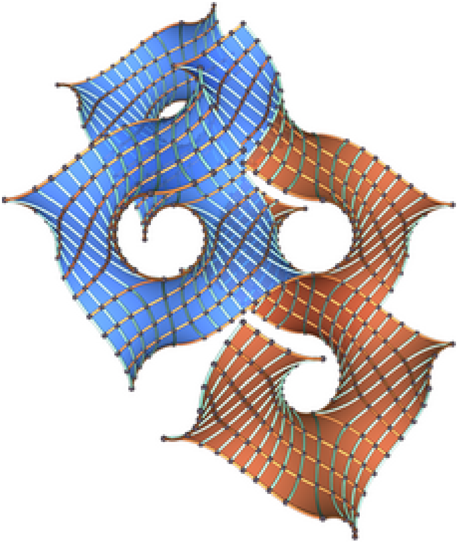

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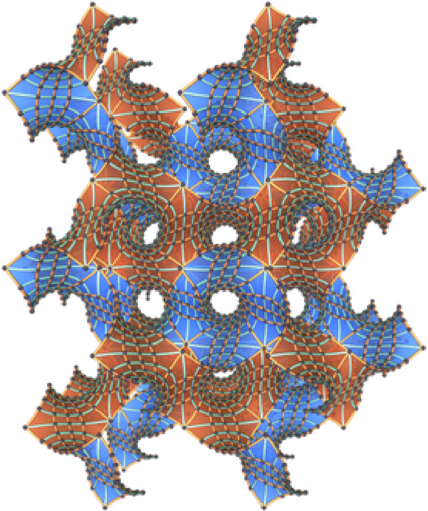

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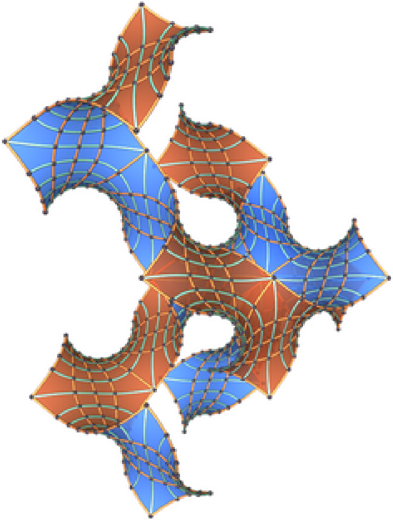

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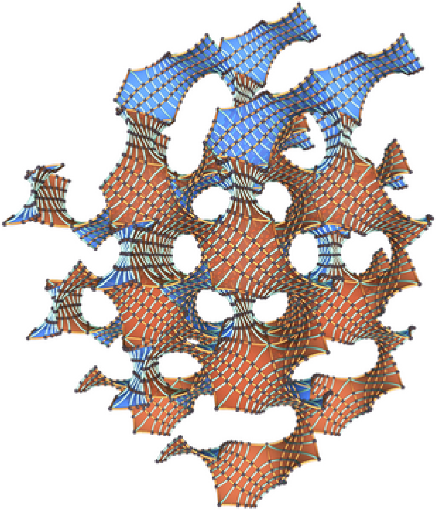

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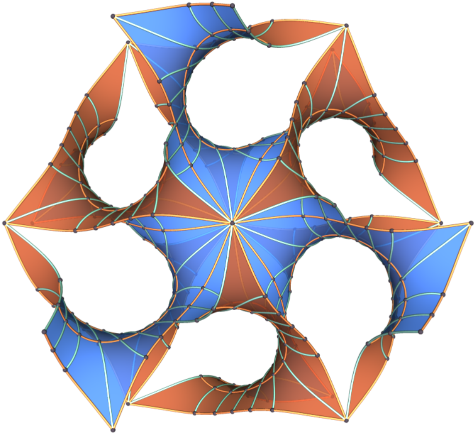

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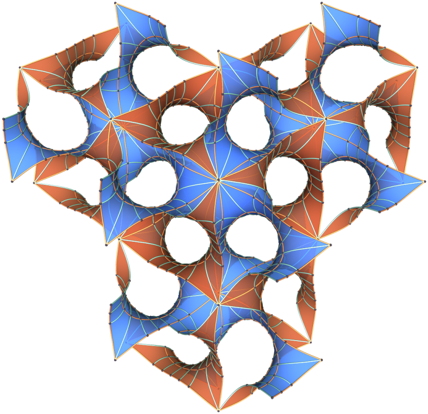

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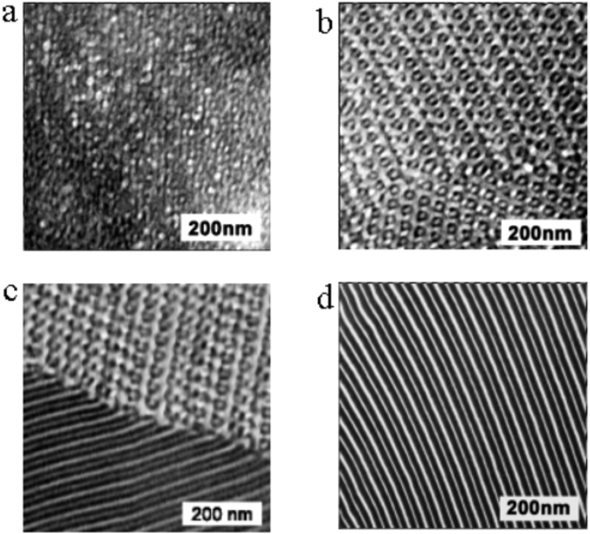
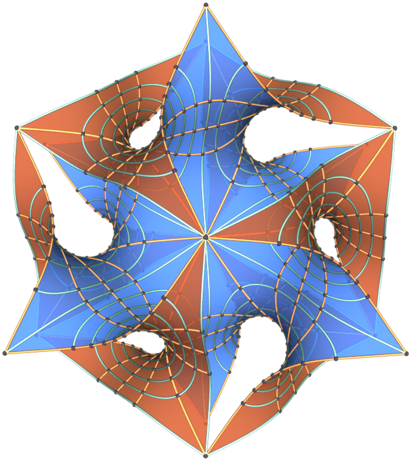
TEM of Polymers Showing Periodic Structure



Novel Morphologies of Block Copolymer Blends via Hydrogen Bonding. Jiang, S., Gopfert, A., and Abetz, V.

Macromolecules, 36, 16, 6171 - 6177, 2003, 10.1021/ma0342933

The Lidinoid



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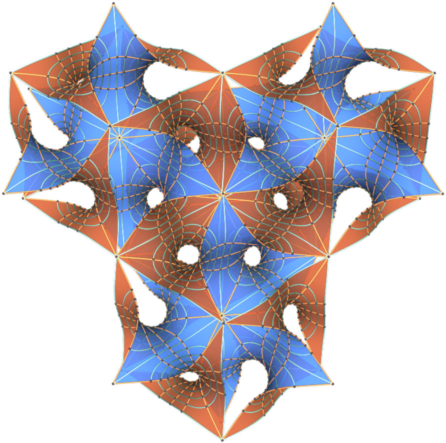

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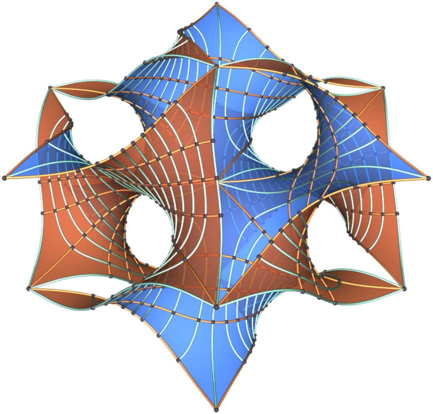

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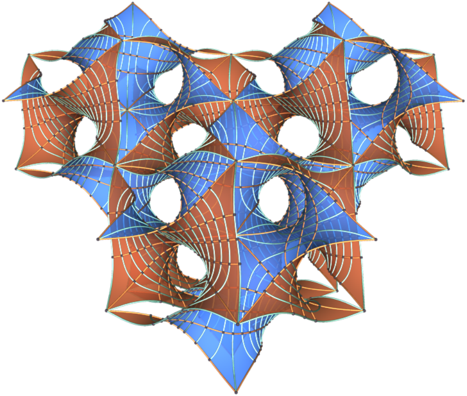

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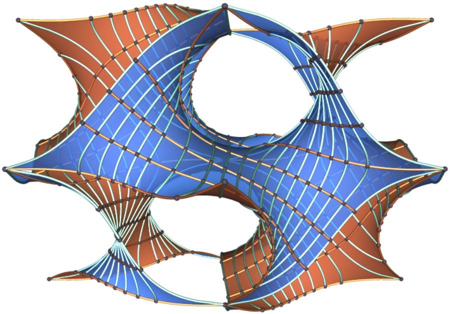

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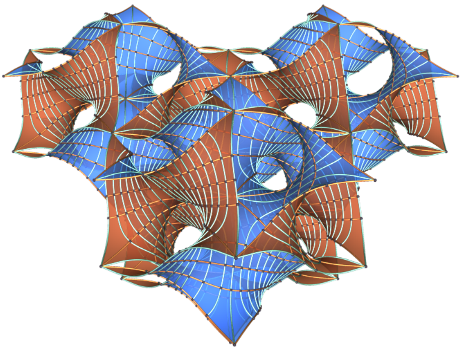

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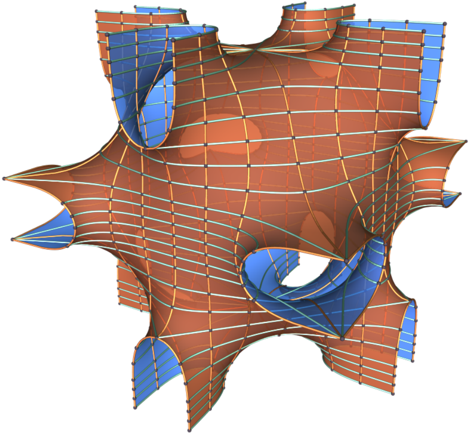
A genus 4 surface - Neovius



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- Quotient by lattice is genus 4 (first example)

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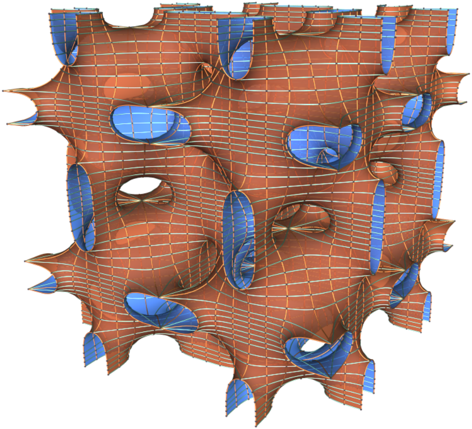

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More topological questions - genus of TPMS

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Theorem (Meeks 1975)

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$$\mathrm{genus}(M/\Lambda) \geq 3.$$

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Theorem (Meeks 1975)

Let M be an immersed triply periodic minimal surface with lattice Λ. Then

$$\mathrm{genus}(M/\Lambda) \geq 3.$$

Theorem (Traizet 2006)

Let Λ be any rank 3 lattice of $\mathbb{R}^3$. Given any integer $g \geq 3$, there exists a triply periodic minimal surface M with lattice Λ such that $\mathrm{genus}(M/\Lambda) = g$.

(Continuous, embedded) families of TPMS

Meek's 5-parameter family

This family is not an anomaly — most triply periodic minimal surfaces (of genus 3) are deformable (come in a continuous family).

Theorem (Meeks 1975)

There is a 5-dimensional continuous family of embedded, triply periodic minimal surfaces of genus 3.

The P, D, and H surfaces are all members of this family. In fact all known surfaces except the gyroid and Lidinoid are members of the Meeks family.

What can one say about the gyroid and Lidinoid?

Deformations of the gyroid and Lidinoid

Theorem (W, 2006)

The gyroid and Lidinoid both admit (1-parameter) deformations that preserve an order 3 rotational symmetry.

A consequence is that all currently known embedded triply periodic minimal surfaces of genus 3 admit deformations.

A (moduli) space odyssey

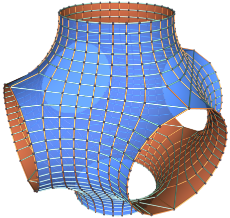
One broad goal of minimal surface work is to understand the moduli space of embedded TPMS of genus 3.

Questions to ask about the moduli space:

- Connected?
- Do all surfaces admit deformations?
- Dimension? (Conjecture: 5)
- Boundary / limits?

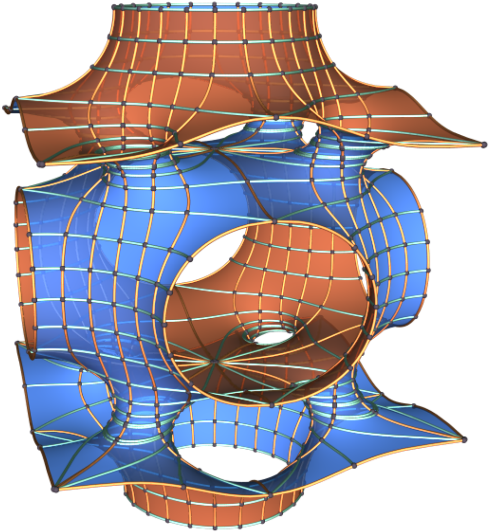
Introduction to flat structures

Every minimal surface is described by a Riemann surface X (domain), and two 1-forms Gdh and dh (G is a meromorphic map - the Gauss map - stereographic projection of the normal) Gdh and dh put a flat structure on X:

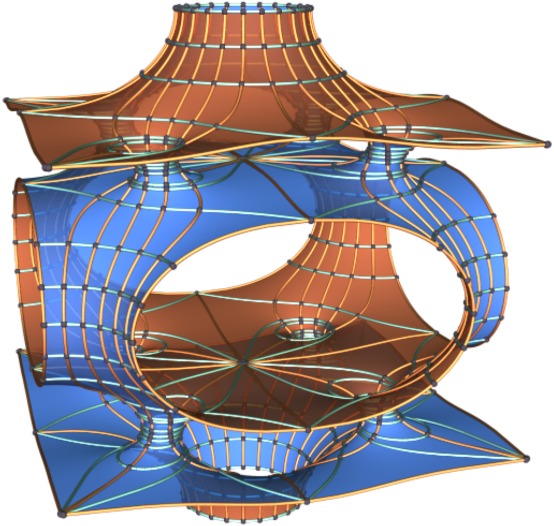


Think: unfolding the cube into the plane.
The periods can be written in terms of these polygons!

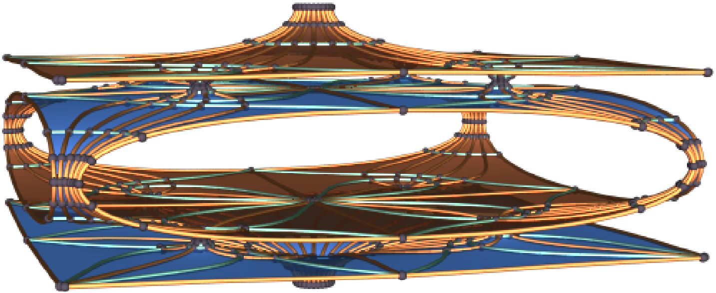
(Degenerate) limits of triply periodic minimal surfaces (Schoen's FRD surface)



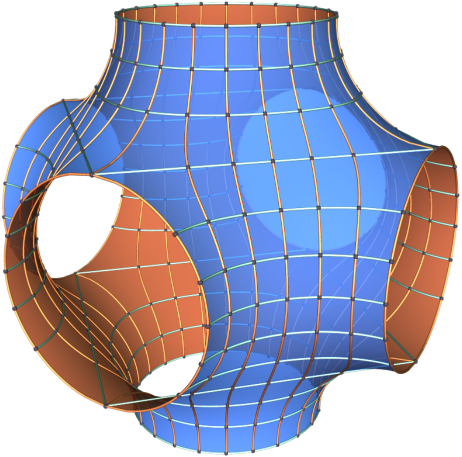
(Degenerate) limits of triply periodic minimal surfaces (Schoen's FRD surface)



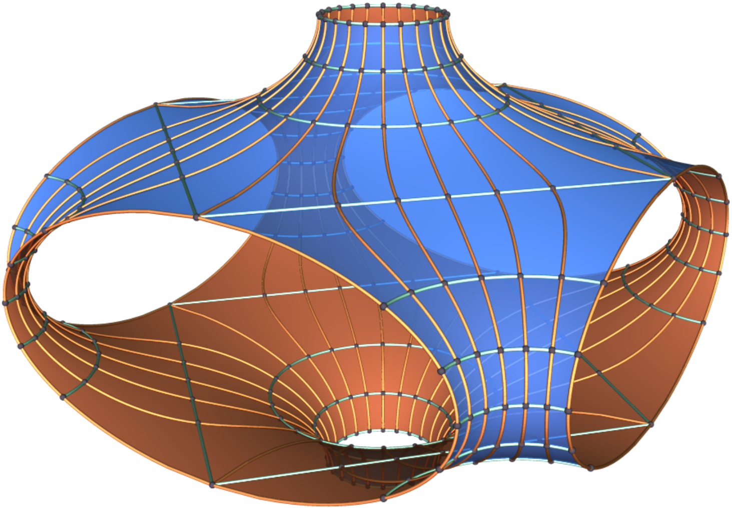
(Degenerate) limits of triply periodic minimal surfaces (Schoen's FRD surface)



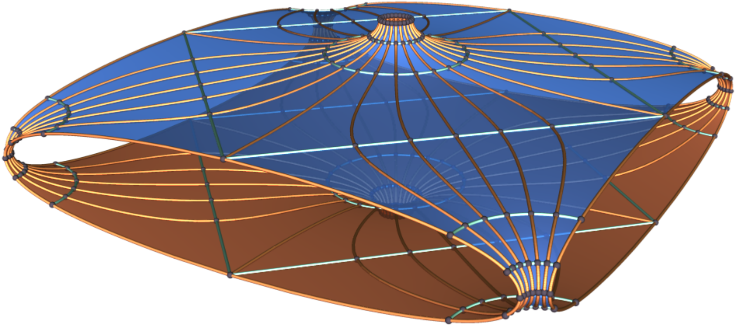
(Degenerate) limits of triply periodic minimal surfaces (P surface)



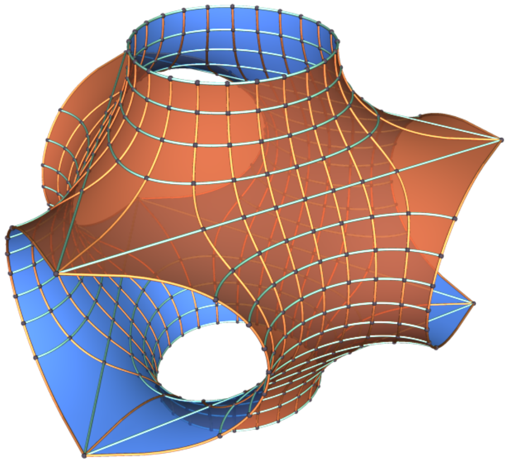
(Degenerate) limits of triply periodic minimal surfaces (P surface)



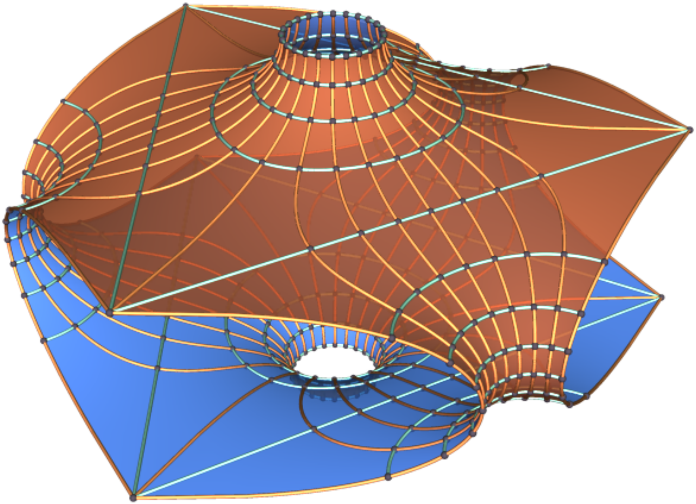
(Degenerate) limits of triply periodic minimal surfaces (P surface)



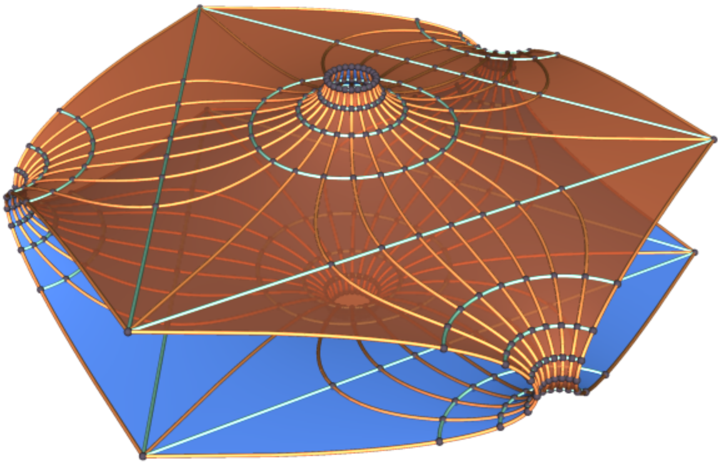
(Degenerate) limits of triply periodic minimal surfaces (H surface)



(Degenerate) limits of triply periodic minimal surfaces (H surface)



(Degenerate) limits of triply periodic minimal surfaces (H surface)



Traizet's Technique (Opening Nodes)

Restrictions on these necks

The location of these catenoidal necks must satisfy a balancing condition (which can be interpreted in terms of electrostatic forces).

To construct more surface families. . .

Any solution to Traizet's balancing condition equation (along with some "rank condition") will yield a continuous family of embedded minimal surfaces. Unfortunately solutions are hard to come by. Traizet's technique does not require any enforced symmetries. It has been used to find minimal surfaces with no symmetries!

Current project - Finding "more gyroids"

Limitations of the flat structure method

The flat structure method used to construct the gyroid families explicitly requires that all surfaces have the same rotational symmetry. This ruins any hope of finding a larger family of gyroids using flat structures!

Combination of Traizet and flat structures

Look (numerically, for instance) at the degenerate limits of the gyroid and Lidinoid families. These limits will likely be foliations of $\mathbb{R}^3$ by planes with catenoidal necks (noded surface). These necks must satisfy Traizet's balancing equation, which might provide a new solution to the balancing equation. If this solution can be perturbed a bit, we can construct a 2 or more parameter family.

Some fun and some useful references

More images (and some "art") are at:

`http://www.indiana.edu/~minimal`

Accessible to advanced undergrads:

Wolf, M. Minimal surfaces, flat cone spheres and moduli spaces of staircases. *Six themes on variation*, 79–125, Stud. Math. Libr., 26, Amer. Math. Soc., Providence, RI, 2004.

The whole volume is good, but especially:

Meeks, W. Global problems in classical minimal surface theory. *Global theory of minimal surfaces*, 453–469, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI, 2005.

Presentation (and gyroid proof) is available at:

`http://www.siue.edu/~aweyhau/`