Gyroids, Lidinoids, and the moduli space of embedded triply periodic minimal surfaces of genus 3

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Outline of Part I: Generalities

1. Quick intro - ETPMS definition
2. Description of the gyroid and Lidinoid
3. Explain results
4. Conjectured map of moduli space
5. Some qualitative remarks about limits

Outline of Part II: Technicalities

6. Sketch of existence of rG and rL

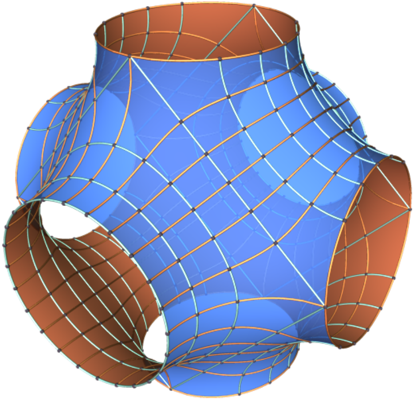
7. rG = rL

Part I

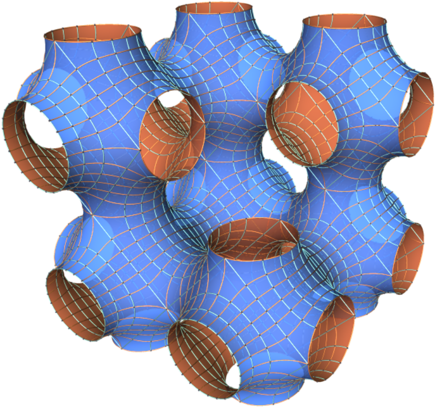
Generalities

Definition of triply periodic minimal surface

A triply periodic minimal surface is a minimal surface M immersed in $\mathbb{R}^3$ that is invariant under the action of a rank 3 lattice Λ.



Schwarz P Surface



If M is embedded, the quotient $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact.

Theorem (Osserman, 1969)

A complete minimal surface with finite total curvature is conformally equivalent to a compact Riemann surface with finitely many punctures.

If M is a TPMS, then M/Λ is compact, so M/Λ has finite total curvature and is conformally equivalent to a Riemann surface X. (Note that X is complete — punctures correspond to ends — which M/Λ does not have.) M is completely determined by X, a meromorphic function M defined on X (the Gauss map), and a holomorphic 1-form dh (the height differential).

We now specialize to the case where the TPMS has rotational symmetry. This will allow us to explicitly characterize X.

Parameterizing surfaces with tori

Let M/Λ be an embedded genus 3 TPMS invariant under an order 3 rotation ρ. The map

$$\rho : M/\Lambda \to M/\Lambda/\rho$$

is a 3-fold branched covering map.

By Riemann-Hurwitz there are exactly 2 branch points and $M/\Lambda/\rho$ has genus 1; these fixed points of ρ are exactly where the surface normal is vertical. Thus, X is a 3-fold branched cover of a torus.

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Height differential and Gauss map

dh is invariant under ρ (we have oriented the surface so that the axis of ρ is vertical). Therefore, *dh* descends as a holomorphic 1-form to the torus. Since up to a constant there is only one holomorphic 1-form on a torus,

$$dh = re^{i\theta}dz$$

Changing *r* simply dilates the surface in space. θ is the "angle of association" and is used to solve the period problem.

In the second part of the talk, we will give an explicit formula for the Gauss map. For the moment, simply assume that the Gauss map exists. It does not descend to the torus, but G^3 does.

Description of the gyroid

The P surface (Schwarz, 1865) is an ETPMS. There are many ways to describe it: it is the solution to the Plateau problem for two squares in close parallel planes, conjugate of D surface, etc. We of course use a Weierstrass representation that emphasizes the rotational symmetry which we desire to study. For instance, there are parametrization that emphasize the order 2 (4) symmetry, order 3 symmetry, etc.

We typically call "the P surface" the surface in the P family that exhibits both order 3 and order 2 symmetry.

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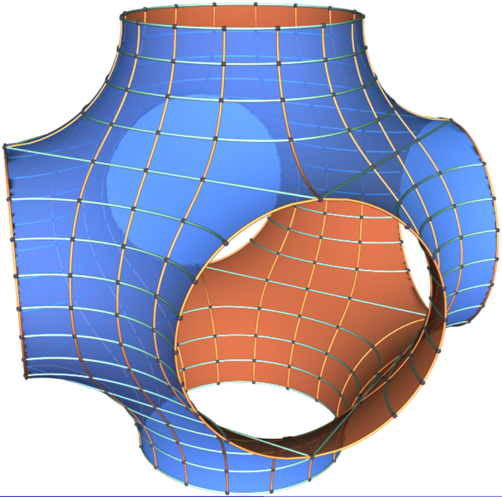

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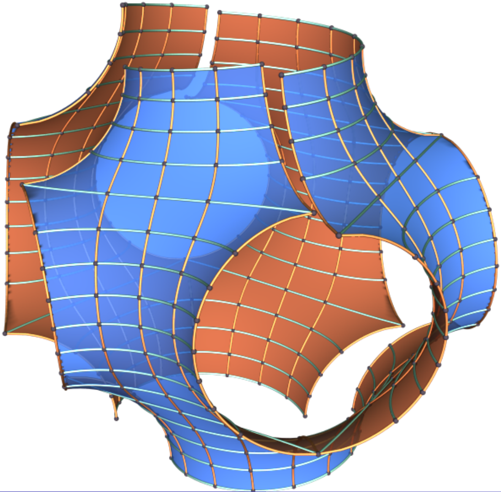

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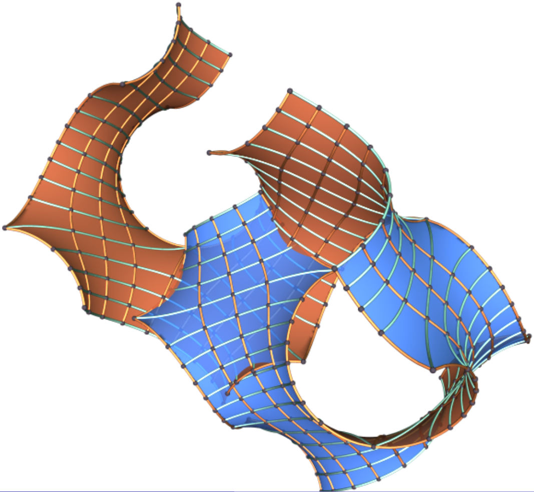

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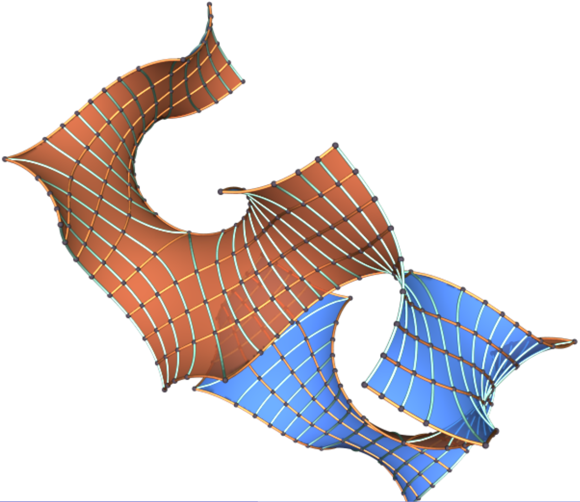

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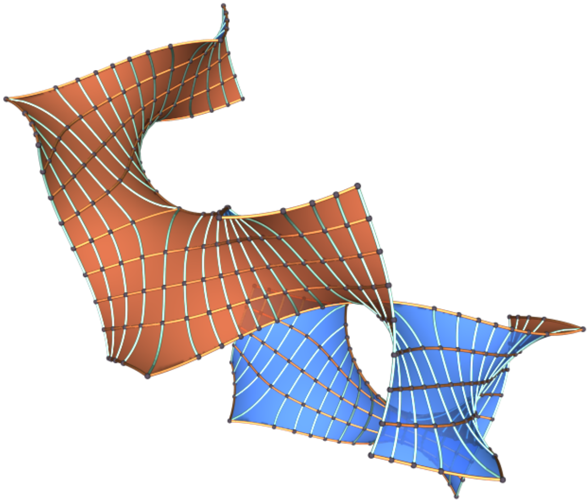

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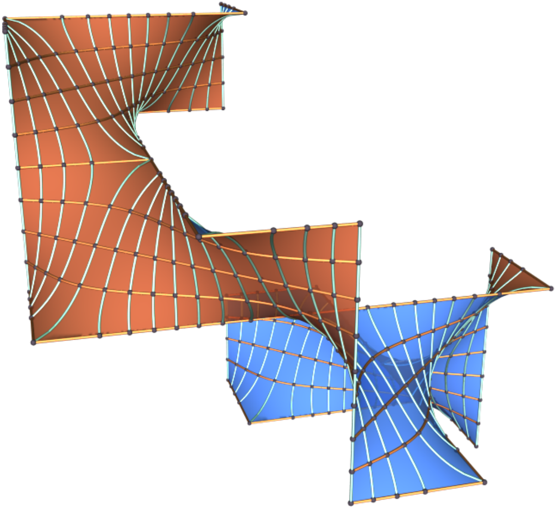
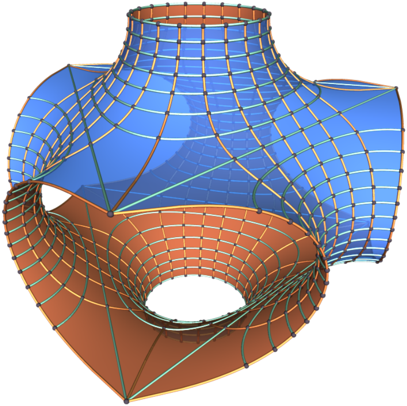


Description of the Lidinoid

The H surface (Schwarz, 1865) is similar to the P surface, where squares are replaced by triangles. The H surface exhibits both order 3 and order 2 symmetry. (Order 2 is fascinating - work by Matthias).



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The situation is a little different. We can't distinguish "an" H surface as the H surface with the most symmetry. But, for each H surface, we can examine the associate family. For exactly one H surface, there is exactly one embedded member of its associate family. We call this surface the Lidinoid.

Previous moduli space results

Theorem (Meeks '75)

There is a continuous 5-parameter family of embedded triply periodic minimal surfaces of genus 3 in $\mathbb{R}^3$.

There are many surfaces not in Meeks' family: for example, the H surface family, the gyroid, and the Lidinoid.

Theorem (W. '06)

There is a one parameter family of embedded TPMS of genus 3 that contains the Lidinoid and a one parameter family of embedded TPMS of genus 3 that contains the Lidinoid. None of these surfaces are in Meeks' family.

Theorem (Weber '07)

There is a 2 parameter family of embedded surfaces that contains the H surface.

Moduli space results

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Moduli space results

Theorem (Weber, W. '07)

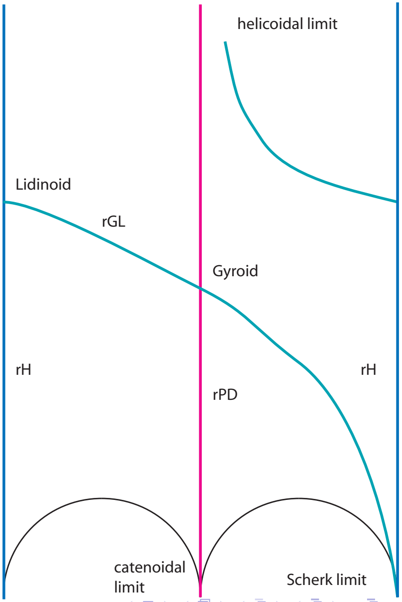
$rG = rL$, that is, the gyroid and Lidinoid families coincide.

Conjectured map of moduli space

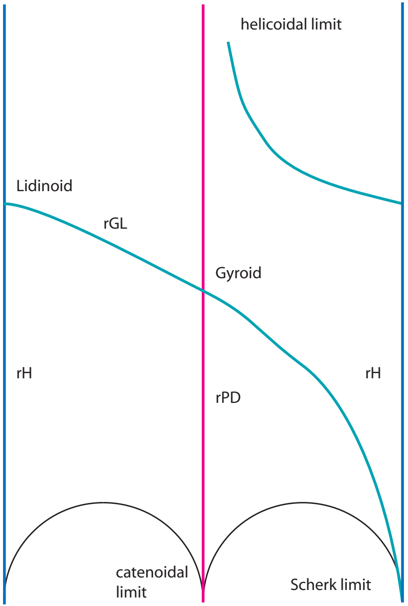
Conjecture

Let M be an embedded, genus 3, TPMS in $\mathbb{R}^3$ that is invariant under an order 3 rotational symmetry. Then M is a surface in one of the rH, rPD, or rGL families. (In particular, the rG and rL families coincide.)

The picture shows moduli space, parameterized using marked tori and the upper half-plane.



The rPD family



The rPD family

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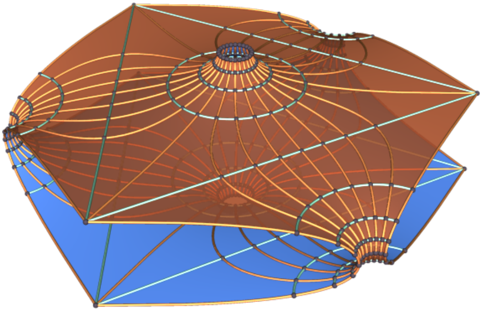
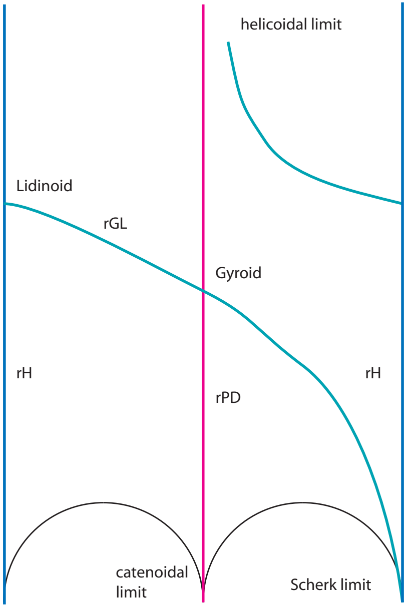
Regenerating minimal surfaces from limits that are planes joined by catenoidal necks (Traizet) or helicoids (Traizet / Weber) has been a productive method.

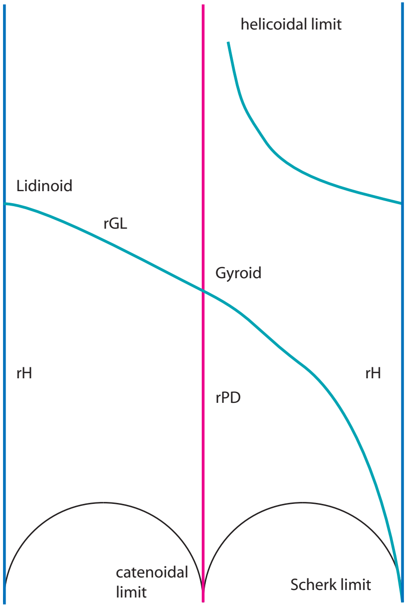


The rH family



The rH family

The rGL family



The rGL family

Part II

Technicalities

Reminders

Recall:

- We are considering only EPTMS of genus 3 that are invariant under an order 3 rotational symmetry.
- Under this restriction, X is a 3-fold branched cover of a torus (branched over two points).
- Since dh descends to the torus,

$$dh = re^{i\theta} dz$$

- We seek a family of surfaces that contain the gyroid and Lidinoid.

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Gauss map

G is not invariant, but G^3 is invariant under ρ. G^3 has double order zeros (poles) at the branch points. Also, G^3 is well-defined on the torus (so is doubly periodic). By Liouville's theorem,

$$G^3 = \lambda e^{i\varphi} \frac{\theta_{11}^2(z,\tau)}{\theta_{11}^2(z-w,\tau)}$$

θ_{11} is the Jacobi theta function. $\theta_{11}(0,\tau) = 0$ and $\theta_{11}(z+1,\tau) = \theta_{11}(z,\tau)$.

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Location of the branch points

The torus can be described as $\mathbb{C}/\langle 1, \tau \rangle$ where $\tau = a + bi, b > 0$. The branch points are the zeros and poles of G.

One branch point can be placed at 0. The other branch point, w, is $\frac{1}{2}$, $\frac{\tau}{2}$, or $\frac{1+\tau}{2}$ by Abel's theorem.

In each of these cases, the torus and branch points are fixed under the involution $-id$. The quotient $\mathbb{C}/\langle 1, \tau \rangle / -id$ is topologically a sphere.

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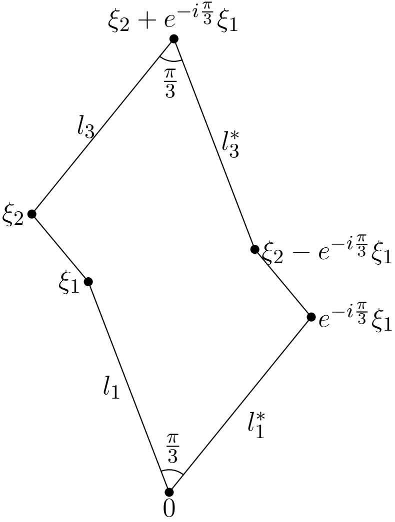
Flat structures - Gdh

Every holomorphic 1-form on a Riemann surface defines a flat structure. This flat structure gives rise to a cone-metric on the torus which descends to a sphere.

For each of the 1-forms $(Gdh, \frac{1}{G}dh, dh)$ we obtain a flat structure on the sphere. Cutting along geodesics between the vertices of this tetrahedron gives (here we specialize to the case when $w = \frac{1}{2}$):

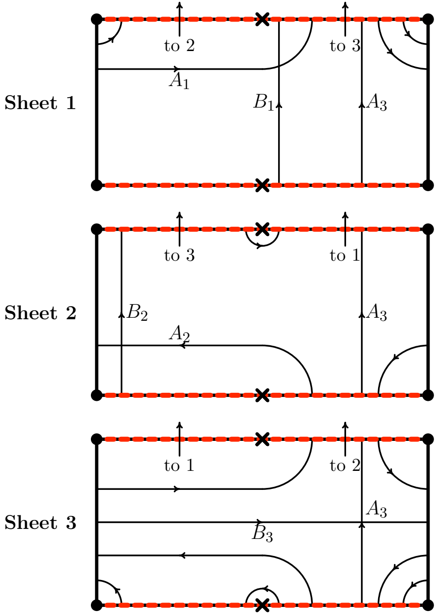


Period problem

Conformal model of the Riemann surface, showing the generators in red.

(We show a rectangular torus (H surface) but model holds for arbitrary torus.)

The surface is immersed iff the periods lie in a lattice Λ of rank 3.



Normalizations

Recall:

$$G^3 = \lambda e^{i\varphi} \frac{\theta_{11}(z,\tau)}{\theta_{11}(z-w,\tau)}$$

λ, ϕ

In general, the flat structure for $\frac{1}{G}dh$ is guaranteed to be congruent to that for Gdh up to rotation and dilation. By adjusting ϕ and λ, we guarantee that the flat structures are translates

Free parameters

Our remaining free parameters are $\tau = a + bi$ and θ (the angle of association).

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Period problem

We can compute these periods in terms of the flat structures, e.g.

$$\int_{A_1} G dh = (1 + e^{2\pi i/3}) e^{i\pi/3} \xi_1$$

$$\int_{A_1} \frac{1}{G} dh = (1 + e^{-2\pi i/3}) \xi_1$$

One sees that the periods like in a lattice iff $\mathrm{Imag}(\xi_2) = 0$ and $\mathrm{Real}(1 + \tau) = 0$.

Relative turning angle

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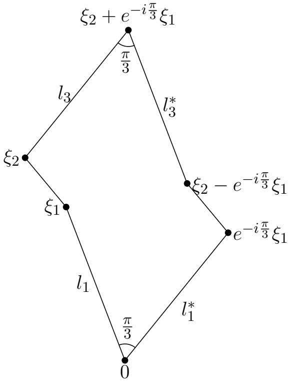
There is a choice of θ, call it $h(\tau)$, that will satisfy $\mathrm{Imag}(\xi_2) = 0$.



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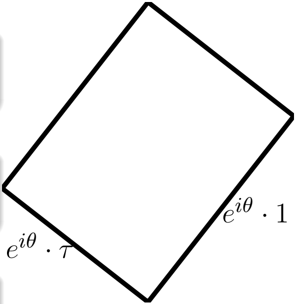

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Define $b(\tau) = h(\tau) - v(\tau)$. If $b(\tau) = 0$, then the Weierstrass data X, G, dh corresponding to $\tau, \theta = b(\tau)$ yields an immersed surface.

Sketch of rH existence

Recall that the Lidinoid is in the associate family of the H surface, which is described by $\tau_0 = w_0 i$ for a fixed $w_0 \in \mathbb{R}$. Therefore, $b(\tau_0) = 0$. For $\tau = wi$, we can compute explicitly (using the symmetries induced by the flat torus) that $b(0 + wi) > 0$ if $w > w_0$ and $b(0 + wi) < 0$ if $w < w_0$.

Therefore, there is a curve $\tau = \tau(t)$ separating the positive and negative segments of the imaginary axis.

Applying Dehn twists one can determine the values of b far to the left and right to show that as $\mathrm{Real}(\tau) \to \pm 1$, $\mathrm{Imag}(\tau) \to +\infty$ or 0.

Show rL = rG