

The moduli space of embedded triply periodic minimal surfaces and the construction of some new examples

Adam G. Weyhaupt

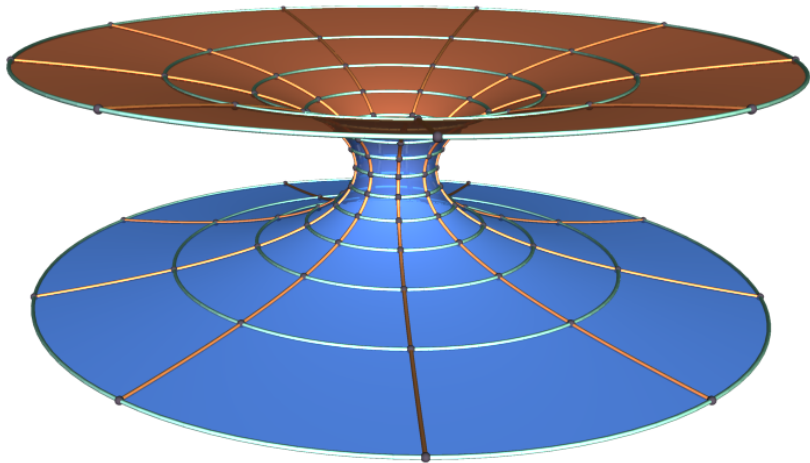
(and also some joint work with Casey Douglas & Matthias Weber)

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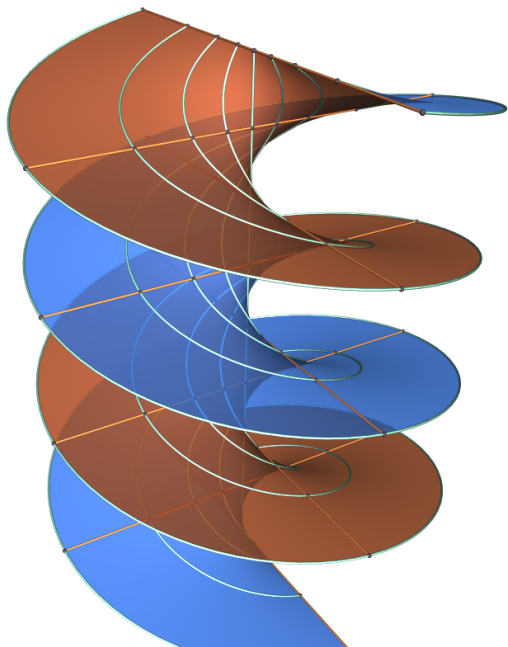
Definition of a minimal surface

A **minimal surface** is a 2-dimensional surface in \mathbb{R}^3 with constant mean curvature $H \equiv 0$.



Catenoid

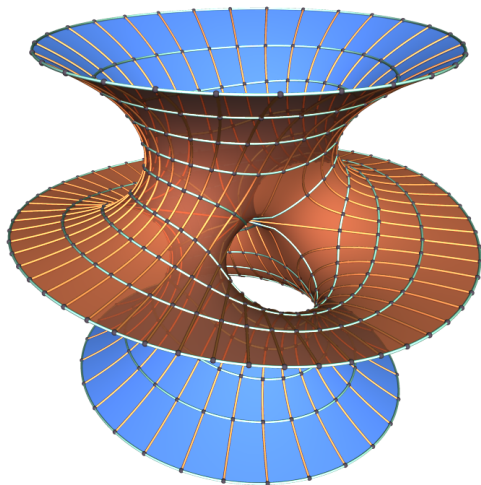
Examples - Helicoid



- ▶ Meusnier (1776)
- ▶ Only ruled minimal surface
- ▶ Plane and helicoid are only complete, embedded, simply connected minimal surfaces in \mathbb{R}^3 (Colding, Minicozzi, Meeks, Rosenberg).

Costa

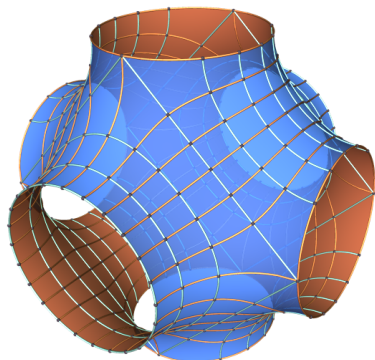
From 1700 - 1984, the only known minimal surfaces either were the catenoid, helicoid, or plane; or they had infinite topology.



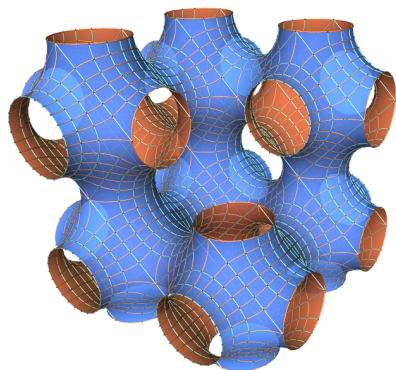
- ▶ Discovered in 1984 by Costa (a graduate student)
- ▶ Conformally is a thrice-punctured torus
- ▶ First example of an embedded torus
- ▶ Shows sharpness of Hoffman / Meeks conjecture

Definition of triply periodic minimal surface

A **triply periodic minimal surface** is a minimal surface M immersed in \mathbb{R}^3 that is invariant under the action of a rank 3 lattice Λ .

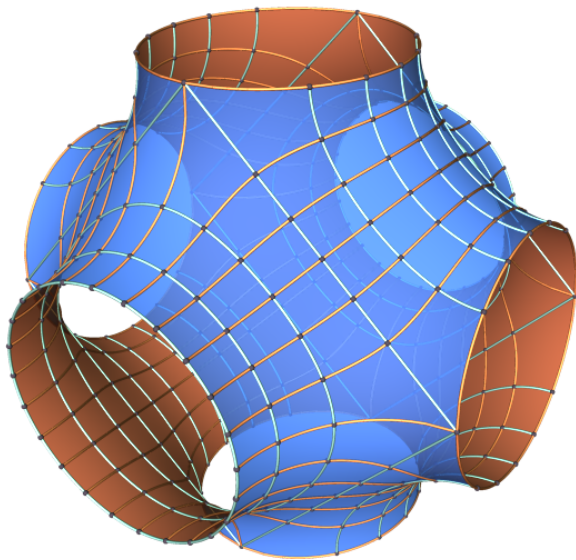


Schwarz P Surface



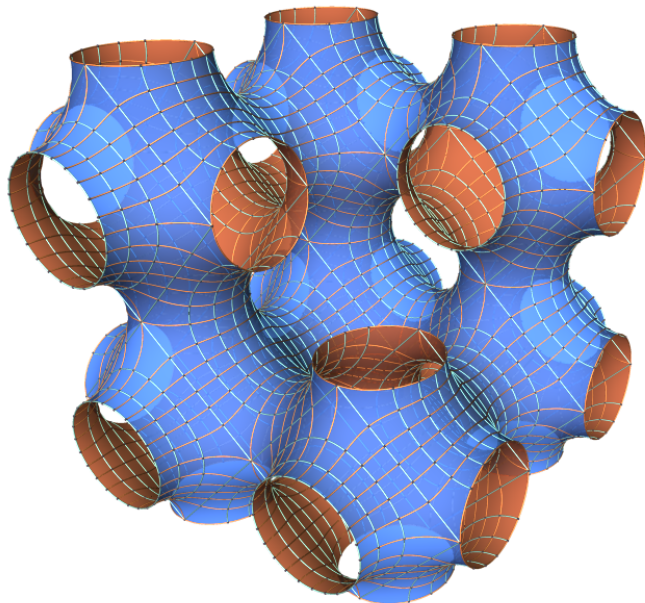
If M is embedded, the quotient $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact.

Examples - P Surface

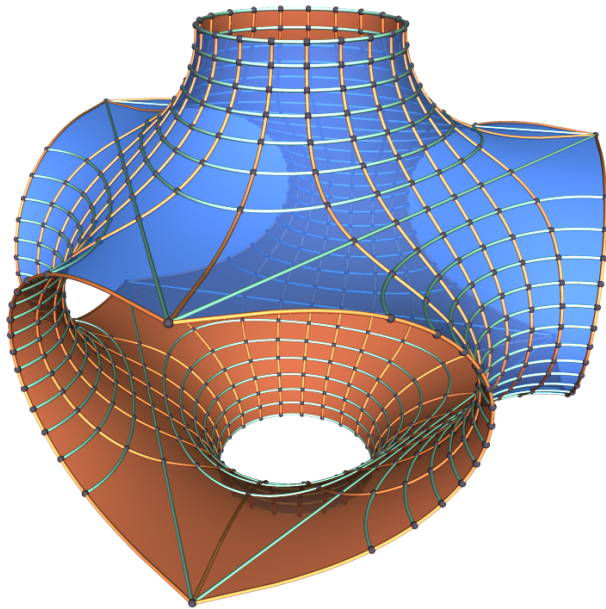


- ▶ Schwarz (1865)
- ▶ Triply periodic surface; cubical lattice
- ▶ Tiled by right angled hexagons

Examples - P Surface

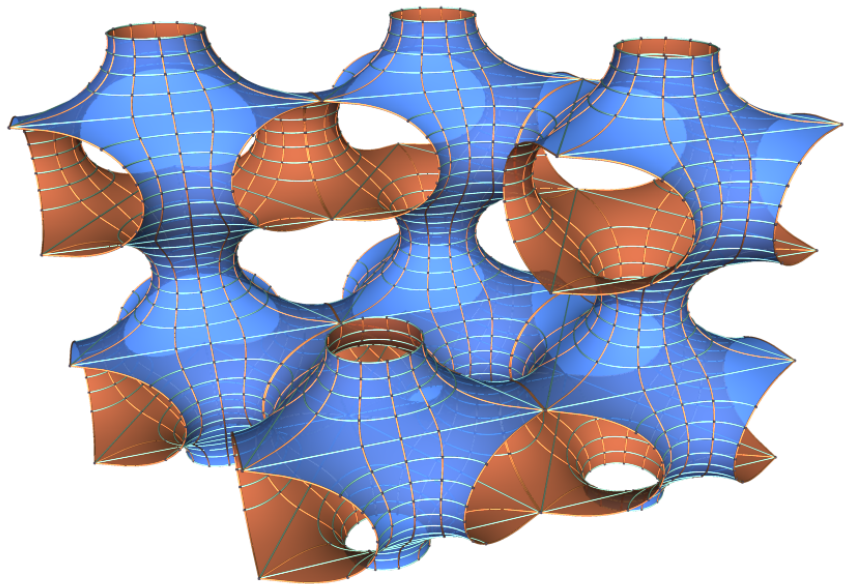


Examples - H Surface



- ▶ Schwarz (1865)
- ▶ Triply periodic surface; hexagonal lattice
- ▶ Lots of straight lines, planar symmetries

Examples - H Surface



Other classifications?

Many triply periodic surfaces are known to come in a continuous family (or deformation).

Theorem

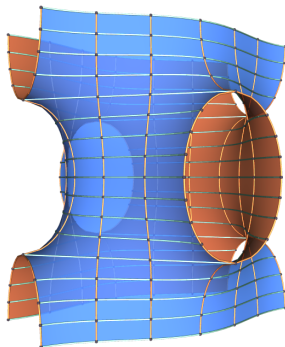
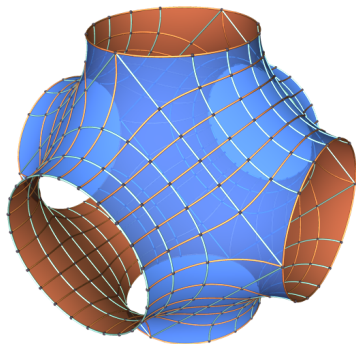
(Meeks, 1975) *There is a **five-dimensional continuous family** of embedded triply periodic minimal surfaces of genus 3.*

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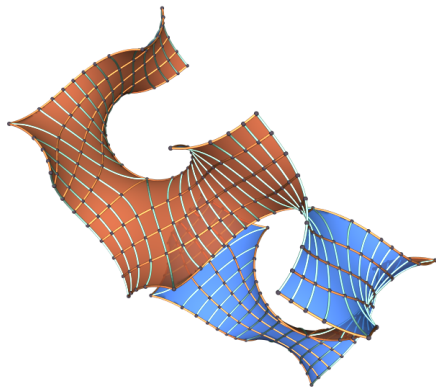
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What's in the Meeks' family?

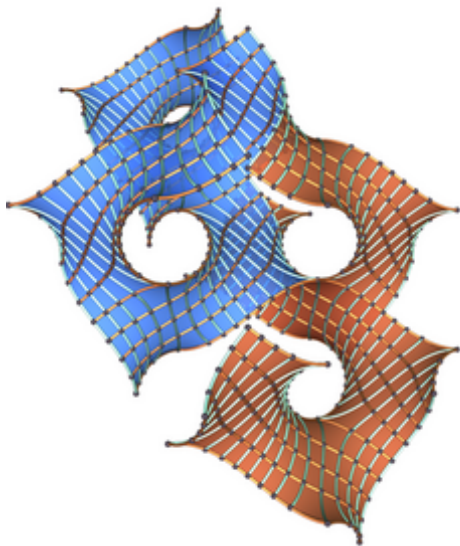
The gyroid and the Lidinoid are not in Meeks' family (neither is the H surface).

Schoen's Gyroid Minimal Surface



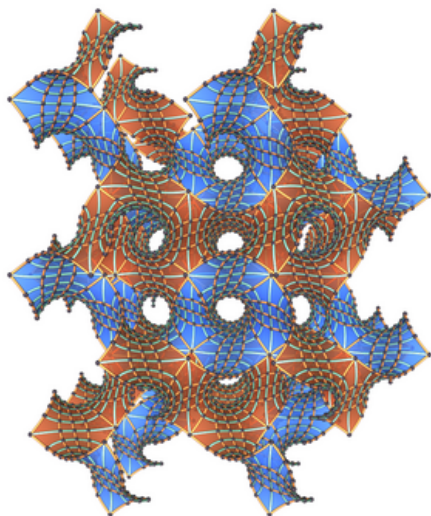
- ▶ The associate family of the P and D surface contains **exactly one** embedded member - the gyroid
- ▶ Discovered 1970 by Alan Schoen (NASA scientist)
- ▶ Contains no straight lines and no planar symmetries (first example)
- ▶ Lattice is rectangular
- ▶ Quotient by lattice is genus 3, compact

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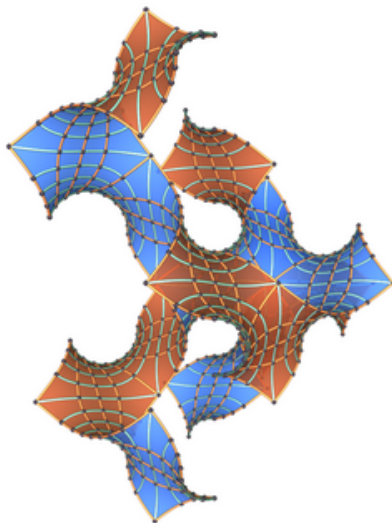
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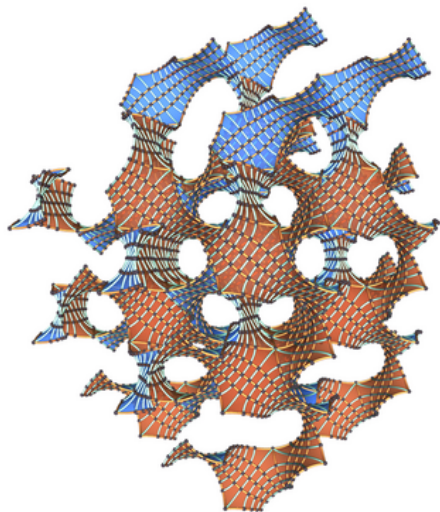
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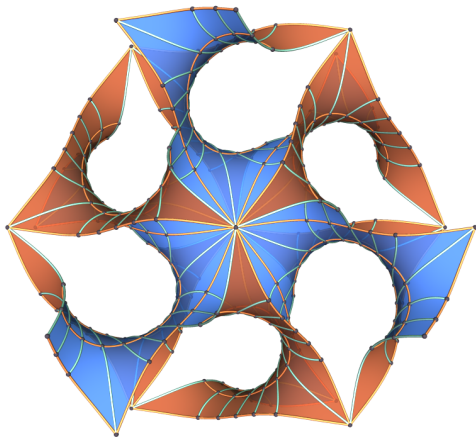
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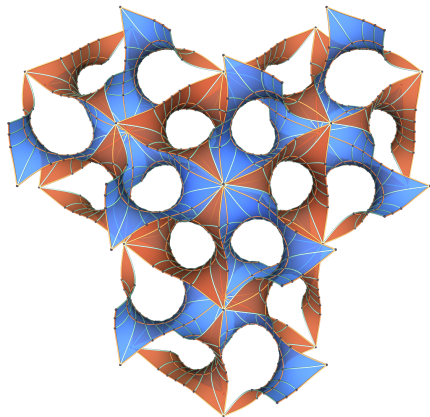
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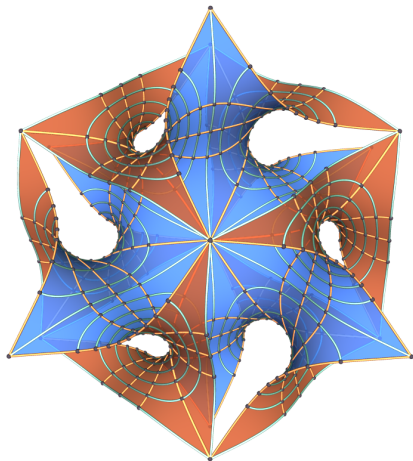
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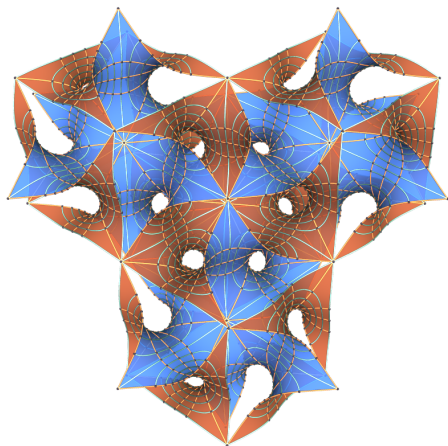
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The Lidinoid



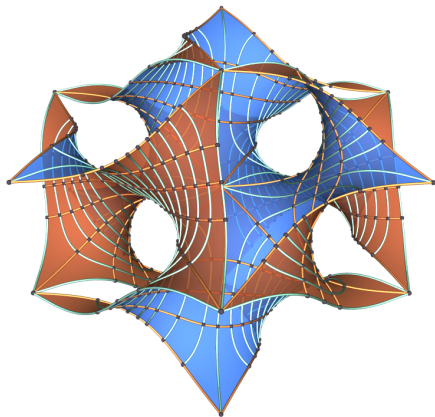
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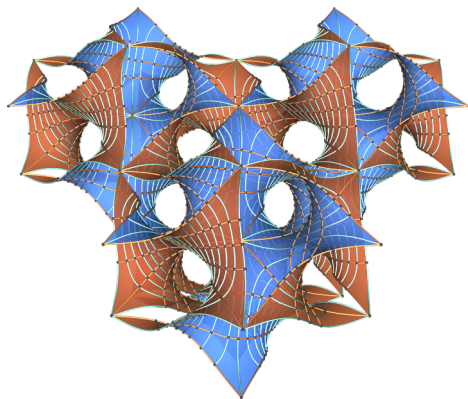
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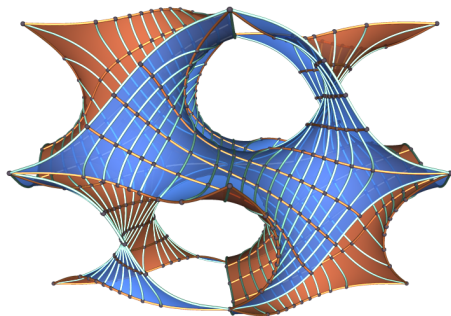
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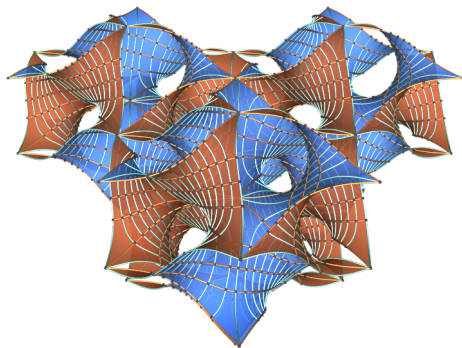
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Results

Theorem

(W., 2006) There is a continuous, one-parameter family of embedded triply periodic minimal surfaces of genus 3 that contains the gyroid. Each surface admits an order 2 rotational symmetry.

A slight extension gives:

- ▶ Another 1-parameter gyroid family (preserves order 3 rotation)
- ▶ A (single!) 1-parameter family of Lidinoids preserving the order 3 rotation

As a consequence, **all known examples** of triply periodic minimal surfaces of genus 3 **are deformable**.

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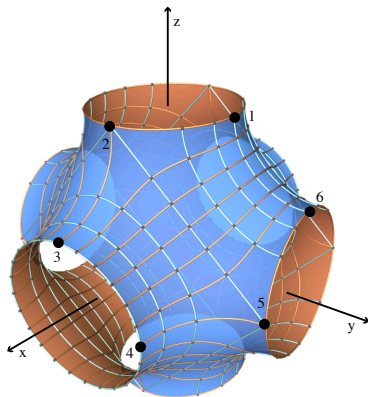
Overview of the Construction

To construct the gyroid deformation, we'll:

- ▶ Describe gyroid with certain (complex analytic) data
- ▶ Modify this data to construct new surfaces that are still minimal and still embedded
- ▶ Use **flat structures**, a technique that transfers a difficult analysis problem to one involving Euclidean polygons.

As a toy problem and to get some intuition, we'll construct the P surface and study its “periods” using flat structures.

The P Surface

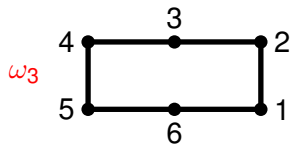
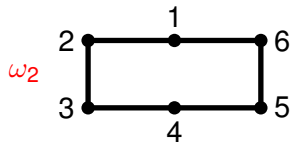
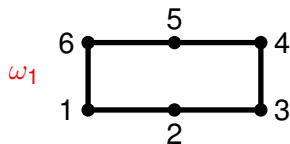
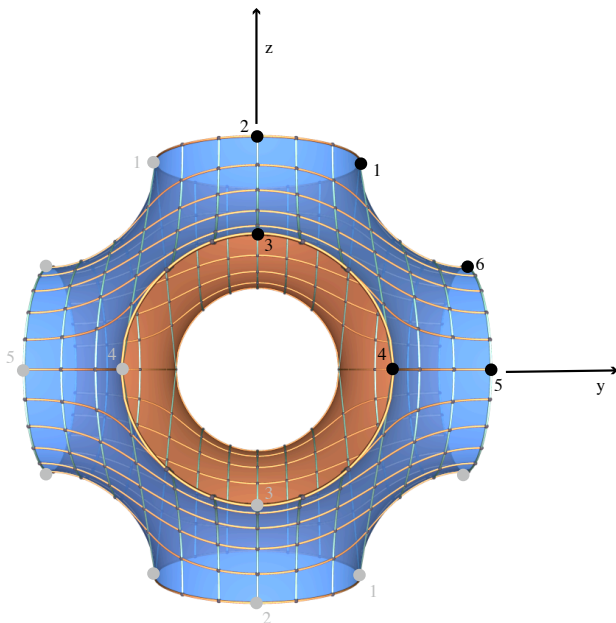


- ▶ Let $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $F(z) = (f_1(z), f_2(z), f_3(z))$ be a conformal parameterization of one of the P surface hexagons.
- ▶ The f_i are harmonic.
- ▶ In a simply connected domain, can write

$$f_i(w) = \text{Real} \int^w \omega_i$$

- ▶ We can explicitly write down ω_i using Schwarz-Christoffel maps from the upper half plane.

Analytic Continuation



What is the Gyroid?

The Associate Family

Modify coordinate functions by:

$$\text{Real} \int \omega_i \longrightarrow \text{Real} \int e^{i\theta} \omega_i$$

This new surface patch is locally **isometric** to the original.

What is the Gyroid?

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This new surface patch is locally **isometric** to the original.

The Gyroid

In general, this transformation does not yield patches that fit together to give an embedded (or even immersed) surface.

For **exactly one value of θ** ($\theta \approx 51.9852^\circ$), this transformation applied to the P surface gives an embedded, triply periodic minimal surface, called the **gyroid**. Flat structures make this curious value of θ less mysterious.

Sketch of the Proof (of the gyroid deformation)

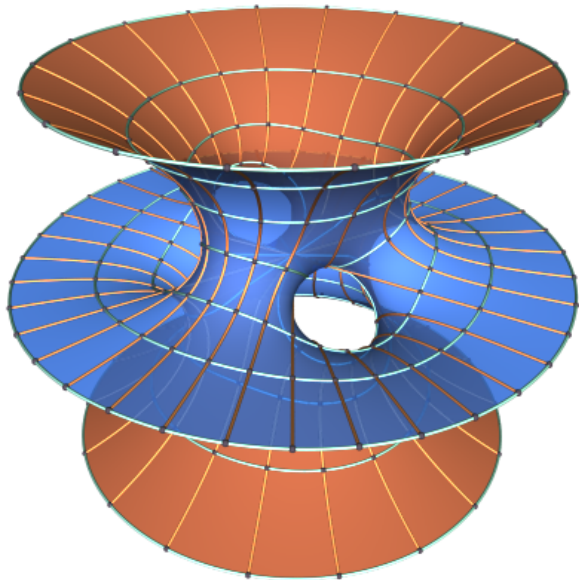
The Costa-Hoffman-Meeks Triply Periodic Minimal Surfaces

Theorem (Ramos-Batista, 2003)

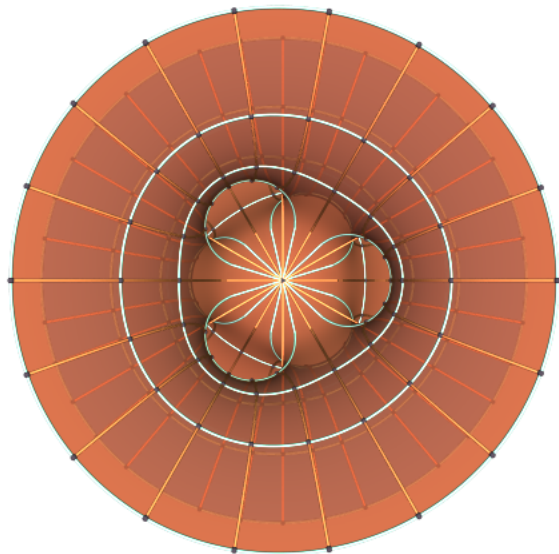
There is a continuous family of embedded, triply periodic minimal surfaces so that a translational fundamental domain has the same symmetries as the Costa minimal surface.

The proof involves some pretty tricky estimates involving elliptic integrals.

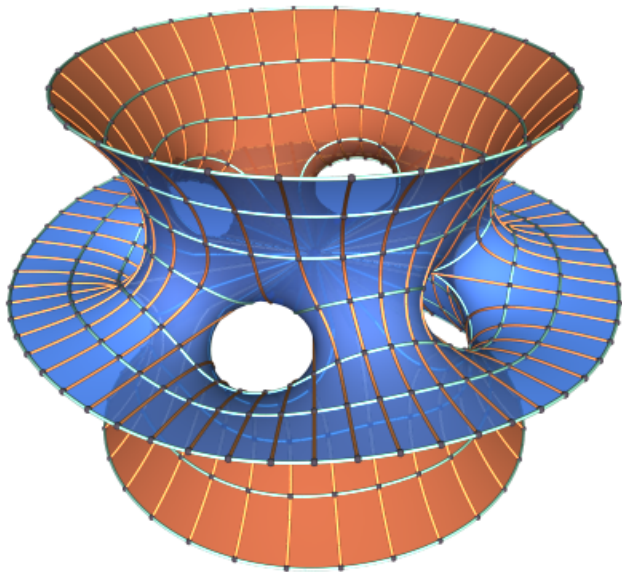
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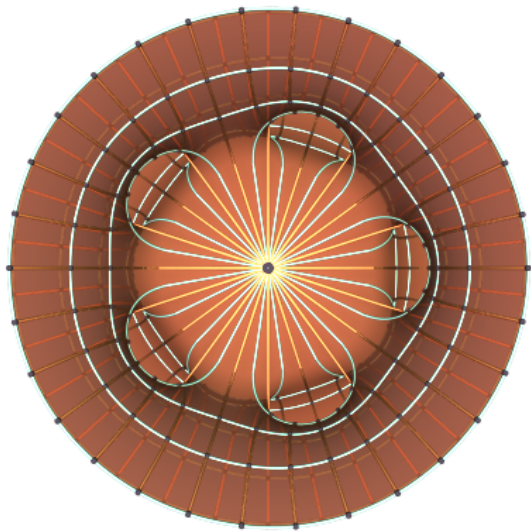
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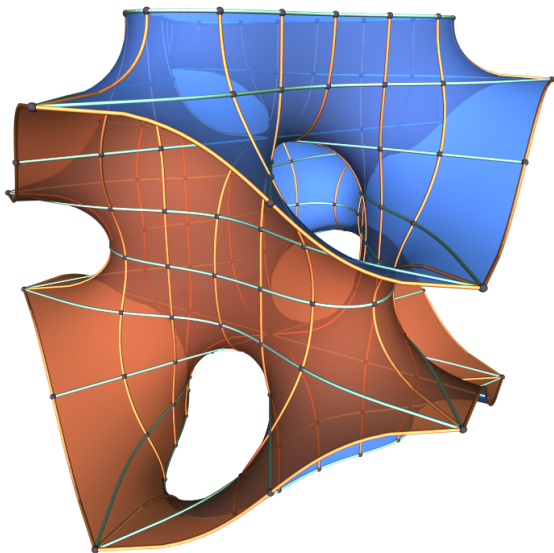
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Costa-Hoffman-Meeks Surfaces



Order 3 Triply Periodic Costa Surface



Overall Goal

Goal

Completely understand the structure of the moduli space of triply periodic minimal surfaces.

Some results lead us to believe that this structure is very bad, but there certainly is some structure. We conjecture the following as a picture for a very small part of moduli space:

Conjectured map of moduli space

Conjecture (Weber, W.)

Let M be an embedded, genus 3, TPMS in \mathbb{R}^3 that is invariant under an order 3 rotational symmetry. Then M is a surface in one of the rH, rPD, or rGL families. (In particular, the rG and rL families coincide.)

The picture shows moduli space, parameterized using marked tori and the upper half-plane.

