An Introduction to Discrete Minimal Surfaces via the Enneper Surface

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A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Master of Science Degree

Department of Mathematics and Statistics
in the Graduate School
Southern Illinois University Edwardsville
Edwardsville, Illinois

July, 2013

SOUTHERN ILLINOIS UNIVERSITY EDWARDSVILLE

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ABSTRACT

AN INTRODUCTION TO DISCRETE MINIMAL SURFACES VIA THE ENNEPER SURFACE

by

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Chairperson: Professor Adam G. Weyhaupt

In this paper, we are exploring how to construct a discrete minimal surface. We map the conformal curvature lines of a parameterized continuous minimal surface to a unit sphere by the Gauss map. Then, based on a circle patterns we create, the Koebe polyhedron can be obtained. By dualizing the Koebe polyhedron, we are able to get the discrete minimal surface. Moreover, instead of only developing the method theoretically, we also show concrete procedures visually by Mathematica for Enneper with arbitrary domain. This is an expository project mainly based on the paper "Minimal surface from circle patterns: geometry from combinatorics" by Alexander I. Bobenko, Tim Hoffmann and Boris A. Springborn.

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CHAPTER 1

INTRODUCTION

In this thesis, discrete minimal surfaces are studied; they are in the field of discrete differential geometry which builds discrete equivalent objects of classic smooth geometric shapes in three dimensions. The former uses typically finitely many polyhedra to approach the latter.

By using different methods, one can discretize the given geometric object differently but make the discrete equivalents converge to it. However, which method of discretizing is the best? The following answers the key question: "The goal of discrete differential geometry is to find a discretization which inherits as many essential properties of the smooth geometry as possible."[BHS06]

Instead of using traditional triangle meshes, which are very intuitive, quadrilateral meshes are applied in this paper because conformal properties of surface can be respected. Why are conformal properties are so important? Because conformality is a very important property of minimal surfaces.

Then, what is the method to obtain the discrete analogs from the corresponding continuous case? A continuous minimal surface is the Christoffel dual of its Gauss map. By copying the same idea, a discrete minimal surface is the discrete Christoffel transformation of its Gauss map. The role of the Gauss map is played by a Koebe polyhedron.

Based on the method how to construct discrete minimal surfaces from the paper "Minimal surfaces from circle patterns: geometry from combinatorics", a discrete Enneper with arbitrary domain is created by Mathematica. A picture of discrete Enneper as shown in Figure 1.1.

In Chapter 2, we explain the basic results of continuous minimal surfaces, then

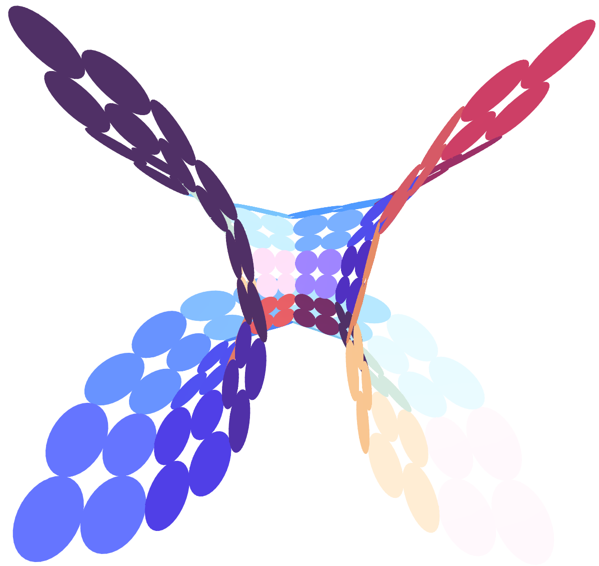


Figure 1.1: Discrete Enneper

discretize them and apply them to create discrete minimal surfaces. In Chapter 3, we investigate Enneper as an example and follow the method from Chapter 2 to construct the discrete analogous step by step. We construct Enneper by setting up a program in Mathematica. The Mathematica code can be found in Appendix A.

The general idea to construct a continuous minimal surface is shown as the following,

Conformal curvature lines parametrization

$\Downarrow$

Gauss map

$\Downarrow$

Christoffel dual

$\Downarrow$

continuous minimal surface

In a similar way, the diagram below shows the idea how to create a discrete minimal surface,

cell decomposition of the sphere

$\Downarrow$

orthogonal circle pattern

$\Downarrow$

Koebe polyhedron

$\Downarrow$

discrete Christoffel dual

$\Downarrow$

discrete minimal surface

CHAPTER 2

PRIMARY CONCEPT,THEOREM AND CONCLUSION

2.1 Continuous minimal surfaces

We start by understanding smooth minimal surfaces since we will use the counterpart to construct discrete minimal surfaces later.

Definition 2.1. [BHS06] A smooth immersed surface in $\mathbb{R}^3$ is called *isothermic* if it admits a conformal curvature line parametrization in a neighborhood of every nonumbilic point.

What is a curvature line parametrization? The curvature line parametrization is parametrized by those curves on the surface whose tangent lines are always principle directions; its derivative has no zero there. There are two curvature lines passing through each point and they will cross at right angles. But why exclude umbilic points? Because at an umbilic point, both principal curvatures are equal, and every tangent vector is a principal direction, hence, an umbilic point should not be considered in the curvature lines parametrization.

An isothermic parametrization is also defined to be conformal. What does this mean? It means the angles are preserved, but not necessarily the lengths. Therefore, an isothermic surface which is parametrized by conformal curvature lines can be divided into "squares" as small as possible by increasing the number of curvature lines. By choosing how close of those curvatures lines, one can get different refinement of a discrete Enneper.

Theorem 2.2. [BHS06] *An isothermic immersion (a surface patch in conformal curvature line parameters) $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, is characterized by the properties, $\| f_x \| = \| f_y \|, f_x \perp f_y$ and $f_{xy} \in \mathrm{Span}\{f_x, f_y\}$.*

Proof. First we show that $\| f_x \| = \| f_y \|$. Since the parametrization is conformal, then there exists a function $\lambda : D \to R$, $\lambda \neq 0$, such that

$$1 = \langle \underline{e}_1, \underline{e}_1 \rangle = \frac{1}{\lambda^2} \langle f_x, f_x \rangle$$

and

$$1 = \langle \underline{e}_2, \underline{e}_2 \rangle = \frac{1}{\lambda^2} \langle f_y, f_y \rangle.$$

Thus $\| f_x \| = \| f_y \|$.

Then we show that $f_x \perp f_y$. This is because two principal directions will cross at a right angle, so far all $e_1, e_2 \in T_p$, we have

$$\langle f_x, f_y \rangle = \lambda^2 \langle \underline{e}_1, \underline{e}_2 \rangle = 0$$

Last but not least, we show $f_{xy} \in Span\{f_x, f_y\}$. The matrix of the second fundamental form $S = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$ is a diagonal matrix, hence, $L_{21} = 0$, where L_{xy} are the coefficients of the second fundamental form and defined by

$$L_{xy} = \langle f_{xy}, n \rangle$$

This implies $\langle f_{xy}, n \rangle = 0$. By Gauss's formula[MP77], $f_{xy} = L_{ij} \cdot n + \sum \Gamma^k_{xy} f_k$, when k is either x or y, so we know f_{xy} lies in the tangent plane T_p, and hence it is a linear combination of f_x and f_y.

□

Definition 2.3. The transformation

$$\omega = \frac{az + b}{cz + d}, (ad - bc \neq 0)$$

where a, b, c and d are complex constants, is called a *linear fractional transformation*, or *Möbius transformation.*

Because a Möbius transformation is a conformal map, that means the conformal curvature lines parametrization would not be changed into a non-conformal parametrization by application of a Möbius transformation. In other words, the surfaces obtained after a Möbius transformation still admits a conformal curvature line parametrization in a neighborhood of every nonumbilic point. Hence, an isothermic surface stays isothermic after Möbius transformation.

We now define the stereographic projection that will be used later. Consider $\mathbb{C}$ as the xy-plane in $\mathbb{R}^3 = \{(x, y, z)|x, y, z \in R\}, \mathbb{C} = (x, y, 0)$. Consider the unit sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Then the sphere and the complex plane intersect in the set $\{(x, y, 0) \mid x^2 + y^2 = 1\}$, corresponding to the equator on the sphere and the unit circle on the complex plane. This setting is shown in Figure 2.1. Let N denote the North Pole (0,0,1) of S^2. Given any finite point $p \in \mathbb{C}$, the line segment from N to p intersects S^2 in a unique point (other than N); this point is the *stereographic projection* of p.

Theorem 2.4. *Stereographic projection is given algebraically by*

$$g\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{2\xi}{\xi^2+\eta^2+1}\\ \frac{2\eta}{\xi^2+\eta^2+1}\\ \frac{2(\xi+\eta)}{\xi^2+\eta^2+1}\end{pmatrix}.$$

Proof. Assume D is an arbitrary point in the extended complex plane and connect this point and the North Pole N, then the line intersects the unit sphere at $E = (a, b, c)$. $\underline{r}(t)$ is the equation for the line connecting N and E, So we have,

$$\underline{r}(t) = (0, 0, 1) + t(a, b, c - 1);$$

after projection, the third component should be gone, that is

$$t = \frac{1}{1 - c},$$

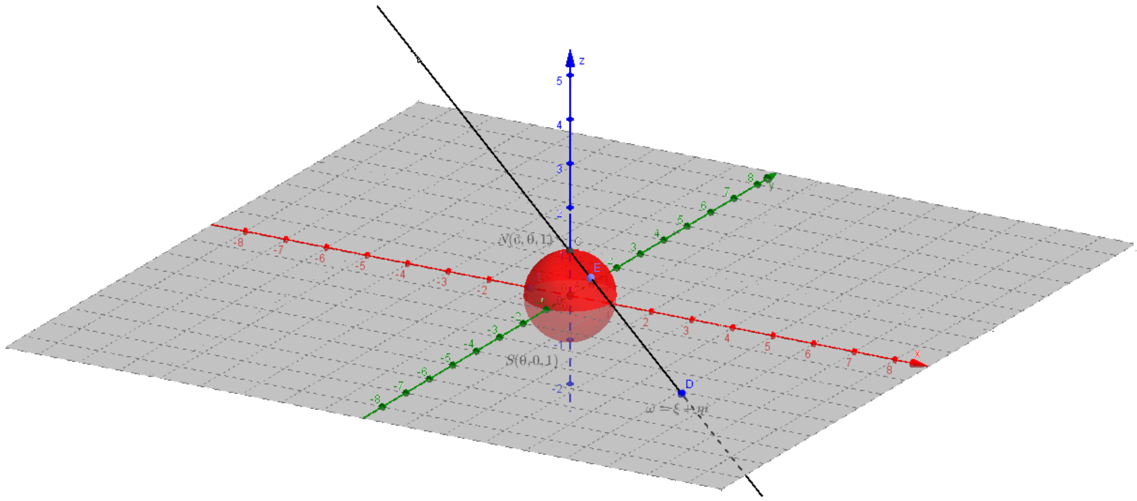


Figure 2.1: Stereographic projection

$$D = \underline{r}\left(\frac{1}{1-c}\right) = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right),$$

$$\frac{a}{1-c} + \frac{b}{1-c}i \in \mathbb{C}.$$

Assume the map from $E(a, b, c)$ to the extended complex plane is g^{-1}.

$$g^{-1}(a, b, c) = \xi + \eta i, \frac{a+bi}{1-c} = \xi + \eta i, \frac{a-bi}{1-c} = \xi - \eta i,$$

$$\xi^2 + \eta^2 = \left(\frac{a+ib}{1-c}\right)\left(\frac{a-ib}{1-c}\right) = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c}, a^2+b^2+c^2 = 1,$$

then we have

$$\frac{2}{1+\xi^2+\eta^2} = 1-c, c = \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1},$$

$$a + bi = (\xi + i\eta)\left(\frac{2}{\xi^2+\eta^2+1}\right) = \frac{2(\xi+i\eta)}{\xi^2+\eta^2+1},$$

Finally, we get

$$a = \frac{2\xi}{\xi^2+\eta^2+1}, b = \frac{2\eta}{\xi^2+\eta^2+1}, c = \frac{2(\xi+\eta)}{\xi^2+\eta^2+1}.$$

Hence, the stereographic projection can be represented by those equations above, which is

$$g\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}\frac{2\xi}{\xi^2+\eta^2+1}\\ \frac{2\eta}{\xi^2+\eta^2+1}\\ \frac{2(\xi+\eta)}{\xi^2+\eta^2+1}\end{pmatrix}.$$

□

It is well-known that stereographic projection is conformal; one can prove it by a direct but tedious calculation. In fact, it is also an isothermic immersion.

Proposition 2.5. *Stereographic projection is an isothermic immersion in* $\mathbb{R}^3$.

Proof. Compute the norm of derivatives with respect to ξ and η, to get

$$\| f_\xi \|=\| f_\eta \|=\frac{\sqrt{8\xi^2\eta^2+4\xi^4+4\eta^4+8\xi^2+8\eta^2+4}}{(\xi^2+\eta^2+1)^2}.$$

The inner product is

$$f_\xi \cdot f_\eta = 0.$$

In order to prove $f_{\xi\eta}$ is an element of the tangent plane we compute the determinant.

$$f_{\xi\eta}=\left(\frac{16\xi\eta}{(1+\xi^2+\eta^2)^3}-\frac{4\eta}{(1+\xi^2+\eta^2)^2},\frac{16\xi\eta^2}{(1+\xi^2+\eta^2)^3}-\frac{4\xi}{(1+\xi^2+\eta^2)^2},-\frac{16\xi\eta}{(1+\xi^2+\eta^2)^3}\right)$$

$$f_\xi=\left(\frac{-2\xi^2+2+2\eta^2}{(1+\xi^2+\eta^2)^2},-\frac{4\xi\eta}{(1+\xi^2+\eta^2)^2},\frac{4\xi}{(1+\xi^2+\eta^2)^2}\right)$$

$$f_\eta=\left(\frac{-4\xi\eta}{(1+\xi^2+\eta^2)^2},\frac{2\xi^2+2-2\eta^2}{(1+\xi^2+\eta^2)^2},\frac{4\eta}{(1+\xi^2+\eta^2)^2}\right)$$

$$\begin{vmatrix}\frac{-2\xi^2+2+2\eta^2}{(1+\xi^2+\eta^2)^2} & -\frac{4\xi\eta}{(1+\xi^2+\eta^2)^2} & \frac{16\xi\eta}{(1+\xi^2+\eta^2)^3}-\frac{4\eta}{(1+\xi^2+\eta^2)^2}\\ -\frac{4\xi\eta}{(1+\xi^2+\eta^2)^2} & \frac{2\xi^2+2-2\eta^2}{(1+\xi^2+\eta^2)^2} & \frac{16\xi\eta^2}{(1+\xi^2+\eta^2)^3}-\frac{4\xi}{(1+\xi^2+\eta^2)^2}\\ \frac{4\xi}{(1+\xi^2+\eta^2)^2} & \frac{4\eta}{(1+\xi^2+\eta^2)^2} & -\frac{16\xi\eta}{(1+\xi^2+\eta^2)^3}\end{vmatrix}=0$$

This proves f_ξ, f_η, $f_{\xi\eta}$ lie on the same plane since the value of determinant is 0. □

Theorem 2.6. [BMPS09]

The stereographic projection g takes the set of circles in $\mathbb{S}^2$ bijectively to the set of circles in $\mathbb{C} \cup \infty$, where for a circle $\gamma \in \mathbb{S}^2$ we have that $\infty \in G(\gamma)$, if and only if $N \in \gamma$ and $G(\gamma)$ is a line in $\mathbb{C}$.

Proof. A circle in $\mathbb{S}^2$ is the intersection of $\mathbb{S}^2$ with some plane P. If we have a normal vector(a_0, b_0, c_0) to P,then there is a unique real number k so that the plane P is given by

$$P = \{(a, b, c) \in \mathbb{R}^3 | (a, b, c) \cdot (a_0, b_0, c_0) = k\} = \{(a, b, c) \in \mathbb{R}^3 | aa_0 + bb_0 + cc_0 = k\}.$$

Without loss of generality, we can assume that $(a_0, b_0, c_0) \in \mathbb{S}^2$ by possibly changing k. We may also assume without loss of generality that $0 \leq k \leq 1$,since if $k < 0$ we can replace(a_0, b_0, c_0) with $-(a_0, b_0, c_0)$ and if $k > 1$ then $P \cap \mathbb{S}^2 = \emptyset$.

Consider the circle of intersection $P \cap \mathbb{S}^2$. A point $(\xi, \eta, 0)$ in the complex plane lies on the image of this circle under g if and only if $g^{-1}(\xi, \eta, 0)$ satisfies the defining equation for P. Using the equations we just obtained from above for $g^{-1}(\xi, \eta, 0)$, we see that

$$(c_0 - k)\xi^2 + (2a_0)\xi + (c_0 - k)\eta^2 + (2b_0)\eta = c_0 + k.$$

If $c_0 - k = 0$, this is a straight line in the ω plane. Moreover, every line in the ω plane can be obtained in this way. Notice that $c_0 = k$ if and only if $N \in P$, which is if and only if the image under g^{-1} is a straight line. If $c_0 - k \neq 0$, then completing the square yields

$$\left(\xi + \frac{a_0}{c_0 - k}\right)^2 + \left(\eta + \frac{b_0}{c_0 - k}\right)^2 = \frac{1 - k^2}{(c_0 - k)^2}$$

Depending on whether the right hand side of this equation is positive, 0, or negative, this is the equation of a circle, point, or the empty set in the ω plane, respectively. These three cases happen when $k < 1, k = 1$, and $k > 1$ respectively. Only the first case corresponds to a circle in $\mathbb{S}^2$. □

This tells us, geometrically, the stereographic projection takes "circles" to "circles". Here "circles" may be lines, because a line can be treated as a circle with infinite radius.

The Christoffel dual plays a important role in this construction. And the following formula allows us to calculate the dual after the Gauss map.

Theorem 2.7. [BHS06] *Let $D \subset \mathbb{R}^2$ be convex. Let $f : D \to \mathbb{R}^3$ be an isothermic immersion. If*

$$f_x^* = \frac{f_x}{\parallel f_x \parallel^2}, f_y^* = -\frac{f_y}{\parallel f_y \parallel^2}$$

then there is map $f^* : D \to \mathbb{R}^3$ such that

$$df^* = f_x^* dx + f_y^* dy$$

To prove the theorem, we must show that we can integrate $f_x^* dx + f_y^* dy$ to obtain a function. To do that, we use Poincare's lemma.

Definition 2.8. [Rud76] Let ω be a $k-$form in an open set $E \in \mathbb{R}$. If there is a $(k-1)-$form λ in E such that $\omega = d\lambda$, then ω is said to be *exact* in E.

Theorem 2.9. [Rud76] *(Poincare's lemma) If $E \subset \mathbb{R}^n$ is convex and open, $k \geq 1$, ω is a $k-$form with continuous first partial derivatives in E, and if $d\omega = 0$, then there is a $(k-1)-$form λ in E such that $\omega = d\lambda$*

This theorem means closed forms are exact in a convex set.

Proof (of Theorem 2.7). In order to use Poincare's lemma to prove there is a f^* as described in Theorem by the dual formula, we first need to show that the 1-form $\alpha = f_x^* dx + f_y^* dy$ is closed.[BHS06] From Theorem 2.2, we have $f_{xy} = af_x + bf_y$, where a and b are functions of x and y. Taking the derivative of the formula from Theorem 2.7 with respect to y and x, respectively, we obtain

$$\begin{aligned} f_{xy}^* &= \frac{1}{||f_x||^2}(-af_x + bf_y) \\ &= -\frac{1}{||f_x||^2}(af_x - bf_y) \\ &= f_{yx}^*. \end{aligned}$$

Now we compute

$$\begin{aligned} d(\alpha) &= d(f_x^* dx + f_y^* dy) \\ &= f_{xx}^* dx \wedge dx + f_{xy}^* dx \wedge dy + f_{yx}^* dy \wedge dx + f_{yy}^* dy \wedge dy \\ &= 0 + f_{xy}^* dx \wedge dy - f_{yx} dx \wedge dy + 0 = 0. \end{aligned}$$

Therefore α is closed. By Poincare's lemma, it is exact and so there exists a function f^*.

□

Definition 2.10. Let $D \subset \mathbb{R}^2$. Let $f : D \to \mathbb{R}^3$ be an isothermic immersion, where D is convex. We call f^* from Theorem 2.7 the *Christoffel dual* of f.

There are many equivalent definitions of minimal surfaces, from variational definitions that emphasize surface area, to ones involving partial differential equations, to ones that are purely geometric. To emphasize the connection to the discrete setting, we choose the following definition of a minimal surface:

Definition 2.11. [HJ03] A minimal surface is the Christoffel dual of its Gauss map.

This is the central idea we use to construct discrete minimal surface by finding out the corresponding part of Gauss map and Christoffel dual in discrete case. A more standard definition of minimal surfaces would use holomorphic functions; in fact, what we state here as a theorem can be used to define minimal surfaces.

Theorem 2.12. [Web06] *A minimal surface has the form $F : U \to \mathbb{R}^3$ is of the form $F = \mathrm{Re} H$ with H a holomorphic map $H : U \to C^3$ such that $\sum_{k=1}^3 (H_k')^2 = 0$ and $\sum_{k=1}^3 |H_k^2| \neq 0$. Vice versa, for each such H, $F = \mathrm{Re} H : U \to \mathbb{R}^3$ is a minimal map.*

This is the *Weierstrass representation* if we choose the conformal Gauss map G of the minimal surface, so G is exactly the function on the complex plane before the spherical image.

Theorem 2.13. [Web06] *Let $H : U \to C^3$ be a homomorphic map with $\sum (H_k')^2 = 0$, then there is a meromorphic function G and a holomorphic 1-form dh so that*

$$dH = \left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1\right) dh$$

Furthermore, dh(the height differential) is the holomorphic extension of the third coordinate function of the minimal map $F = \mathrm{Re}H$, and G is the stereographic projection of the Gauss map,

$$N = \frac{1}{|G^2| + 1}(2\mathrm{Re}G, 2\mathrm{Im}G, |G|^2 - 1)$$

Proof. Given H, the meromorphic function G can be defined by

$$G = -\frac{h_1 + ih_2}{h_3}.$$

Clearly, any choice of G and h_3 defines an H with $\langle dH, dH \rangle = 0$. It remains that G is related to the Gauss map as claimed. For this it is sufficient to show that

$$\frac{1}{|G^2| + 1}(2\mathrm{Re}G, 2\mathrm{Im}G, |G|^2 - 1)$$

is orthogonal to the tangent space since this is the stereographic projection of the Gauss map. The tangent space is spanned by the real and imaginary parts of

$$\left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1\right)$$

which are $dH(1)$ and $dH(i)$, so we compute

$$\left\langle \left(1/2\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1\right), \frac{1}{|G^2| + 1}(2\mathrm{Re}G, 2\mathrm{Im}G, |G|^2 - 1) \right\rangle$$

$$= \frac{1}{|G^2| + 1}\left(\mathrm{Re}G\left(\frac{1}{G} - G\right) + i\mathrm{Im}G\left(\frac{1}{G} + G\right) + |G|^2 - 1\right)$$

$$= \frac{1}{|G^2| + 1}\left(\frac{1}{G}(\mathrm{Re}G + i\mathrm{Im}G) - G(\mathrm{Re}G - i\mathrm{Im}G) + |G|^2 - 1\right)$$

$$= \frac{1}{|G^2| + 1} \left(G/G - \bar{G}G + |G|^2 - 1 \right)$$

$$= 0.$$

The inner product is "0" which shows that N is orthogonal to the tangent plane.

□

2.2 Discrete minimal surfaces

We start to develop discrete analogues of a minimal surface. A cell decomposition builds the connection between smooth surfaces and discrete polyhedra and leads to the discrete level.

To start, we need some fundamental notations of conformal geometry.

Definition 2.14. [BHS06] The *cross-ratio* of four points z_1, z_2, z_3, z_4 in the Riemann sphere is $Cr(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$.

Definition 2.15. [BHS06] The *cross-ratio of four points in* $\mathbb{R}^3$ can be defined as follows: Let S be a sphere (plane) containing the four points. S is unique except when the four points lie on a circle (line); if the sphere or plane is not unique then the cross-ratio defined below does not depend of the choice of sphere. If those four points lie on a circle, there are infinitely many S; the situation is the same as when these four points lie on a line. However, this does not change the cross ratio. Choose an orientation on S and an orientation preserving conformal map from S to the Riemann sphere. The cross-ratio of the four points in $\mathbb{R}^3$ is defined as the cross-ratio of the four images in the Riemann sphere.

By this definition and the inverse of stereographic projection as the orientation preserving conformal map, we transfer into a two dimensional cross ratio from $\mathbb{R}^3$.

Definition 2.16. [BHS06] Four points in $\mathbb{R}^3$ form a *conformal square*, if their cross-ratio is -1.

For example, choose $z_1 = -1+i$, $z_2 = 1+i$, $z_3 = -1-i$, $z_4 = 1-i$ and a Möbius transformation $\omega = \frac{-z+i}{z+i}$, plugging those points into the Möbius transformation, we get $\omega_1 = (-1/5, -2/5)$, $\omega_2 = (-1/5, 2/5)$, $\omega = (-1,-2)$ and $(-1,2)$. Those four points form a conformal square.

Based on "cross ratio" and "conformal square", we can get the following definition.

Definition 2.17. [BHS06] A cell decomposition of an oriented two-dimensional manifold (possibly with boundary) is called a *quad-graph* if all its faces are quadrilaterals, that is, if they have four edges.

Definition 2.18. [BHS06] Let D be a quad-graph such that the degree of every interior vertex is even. (That is, every vertex has an even number of edges.) Let $V(D)$ be the set of vertices of D. A function $f : V(D) \to \mathbb{R}^3$ is called a *discrete isothermic surface* f, if for every face of D with vertices v_1, v_2, v_3, v_4 in cyclic order, the points $f(v_1), f(v_2), f(v_3), f(v_4)$ form a conformal square.

This is a good definition because the quad-graph and conformal square are just like the conformal curvature lines parametrization of a isothermic surface in the continuous case.

The folowing lemma leads to the definition which gives the formula to calculate the duality.

Lemma 2.19. *Suppose a,b, $a^{'}$, $b^{'}$* $\in \mathbb{C} \setminus 0$ *with*

$$a + b + a' + b' = 0,$$

$$\frac{aa^{'}}{bb^{'}} = 1.$$

and let

$a^* = \frac{1}{\overline{a}}$, $a^{'*} = \frac{1}{\overline{a'}}$, $b^* = -\frac{1}{\overline{b}}$, $b^{'*} = -\frac{1}{\overline{b'}}$, *where* $\overline{z}$ *denotes the complex conjugate of* z*.*

Then

$$a^* + b^* + a^{'*} + b^{'*} = 0,$$

$$\frac{a^* a^{'*}}{b^* b^{'*}} = 1.$$

Proof.

$$\frac{a' + a}{b' + b} = -1$$

$$\frac{a' + a}{b' + b} = \frac{aa'}{bb'},$$

take conjugate on both sides,

$$\overline{\frac{a' + a}{b' + b}} = \overline{\frac{aa'}{bb'}},$$

this is

$$\frac{\overline{a'} + \overline{a'}}{\overline{b'} + \overline{b}} = \frac{\overline{a} \cdot \overline{a'}}{\overline{b} \cdot \overline{b'}},$$

$$\frac{\overline{a'} + \overline{a}}{\overline{a} \cdot \overline{a'}} = \frac{\overline{b'} + \overline{b}}{\overline{b} \cdot \overline{b'}},$$

$$\frac{1}{\overline{a}} + \frac{1}{\overline{a'}} = \frac{1}{\overline{b}} + \frac{1}{\overline{b'}},$$

$$\frac{1}{\overline{a}} - \frac{1}{\overline{b}} + \frac{1}{\overline{a'}} - \frac{1}{\overline{b'}} = 0,$$

$$a^* + b^* + a^{'*} + b^{'*} = 0.$$

For another one,

$$\begin{aligned} a^* &= \frac{1}{\bar{a}} \\ &= \overline{\left(\frac{1}{a}\right)} \\ &= (-1)\left(\overline{-\frac{1}{a}}\right) \\ &= (-1)\overline{\left(\frac{a'}{bb'}\right)} \\ &= (-1)\frac{\overline{a'}}{\bar{b}\cdot\overline{b'}} \\ &(-1)\frac{b^* b'^*}{a'^*}, \end{aligned}$$

this is,

$$\frac{aa'}{bb'} = 1.$$

□

From the result $a^* + b^* = -a'^* - b'^*$, we know the dual method is path independent, since traversing the square along a^* and then b^* gives the same result as traversing along b'^* and a'^* (note that the direction of travel must be reversed from the natural orientation of the square).

Definition 2.20. [BHS06]

Let $f : V(D) \to \mathbb{R}^3$ be a discrete isothermic surface, where the quad-graph D is simply connected. Label the edges of D "+" and "-" such that each quadrilateral has two opposite edges labeled "+" and the other two opposite edges labeled"-". The *dual discrete isothermic surface* is defined by the formula

$$\Delta f^* = \pm \frac{\Delta f}{||\Delta f||^2}$$

where Δf denotes the difference of neighboring vertices and the sign is chosen according to the edge label.

This formula is for a discrete isothermic surface. In other words, those vertices come from conformal squares.

By giving an arbitrary vertex in D the position $(0,0,0)$ in the dual space, all other vertices in dual space corresponding to $V(D)$ can be obtained by Theorem 2.20, which plays the role of an iteration formula. Theorem 2.20 is essential in dualization because it provides a practical way to calculate the dual. We will use it as the method to calculate the dual of Enneper.

Definition 2.21. [BHS06] An *S-quad-graph* is a quad-graph D with interior vertices of even degree as in the definition of quad-graph and the following additional properties: (1)The 1-skeleton of D is bipartite and the vertices are bi-colored "black" and "white". (Then each quadrilateral has two black vertices and two white vertices). (2)Interior black vertices have degree 4. (3)The white vertices are labeled "c" and "s" in such a way that each quadrilateral has one white vertex labeled "c" and one white vertex labeled "s".

Definition 2.22. [BHS06] Let D be an *S-quad-graph*, and let V_b be the set of black vertiecs. A *discrete S-isothermic surface* is a map

$$f_b : V_b(D) \to \mathbb{R}^3$$

with the following properties: (i)If $v_1, ..., v_{2n} \in V_b(D)$ are the neighbors of a "c" labeled vertex in cyclic order, then $f_b(v_1), ..., f_b(v_{2n})$ lie on a circle in $\mathbb{R}^3$ in the same cyclic order. This defines a map from the "c" labeled vertices to the set of circles in $\mathbb{R}^3$. (ii)If $v_1, ..., v_{2n} \in V_b(D)$ are the neighbors of a "s" labeled vertex, then $f_b(v_1), ..., f_b(v_{2n})$ lie on a sphere in $\mathbb{R}^3$.This defines a map from the "s" labeled vertices to the set of spheres in $\mathbb{R}^3$. (iii)If v_c and v_s are the "c" labeled and "s" labeled vertices of a quadrilateral of D, then the circle corresponding to v_c intersects the sphere corresponding to v_s orthogonally.

"Discrete S-isothermic surfaces are therefore composed of touching spheres and touching circles with spheres and circles intersecting orthogonally." [BHS06]

Definition 2.23. [BHS06] Let $f_b : V_b(D) \to \mathbb{R}^3$ be a discrete S-isothermic surface. The *central extension* of f_b is the discrete isothermic surface $f : V \to \mathbb{R}^3$ defined by $f(v) = f_b(v)$, if $v\in V_b$ and otherwise by defining $f(v)$ to be the center of the circle or sphere corresponding to v.

In order to be able to get discrete minimal surfaces, someone has to dualize a Koebe polyhedron.

Definition 2.24. [BHS06] A *Koebe polyhedron* is a special discrete S-isothermic surfaces, which is a polyhedron whose edges are tangent to a sphere.

Here are some properties of a Koebe polyhedron.

Theorem 2.25. [BHS06] *Every polytopal cell decomposition of the sphere can be realized by a polyhedron with edges tangent to the sphere. This realization is unique up to projective transformations which fix the sphere. There is a simultaneous realization of the dual polyhedron, such that corresponding edges of the dual and the original polyhedron touch the sphere in the same points and intersect orthogonally.*

Theorem 2.26. [BHS06] *For every polytopal cellular decomposition of the sphere, there exists a pattern of circles in the sphere with the following properties. There is a circle corresponding to each face and to each vertex.The circles form a packing with two circles touching if and only if the corresponding faces are adjacent. For each edge,there is a pair of touching vertex circles and a pair of touching face circles.These pairs touch in the same point, intersecting each other orthogonally. This circle pattern is unique up to Möbius transformations.*

As for circle patterns, it depends on which concrete discrete minimal surface to construct. In the simplest case, like the discrete Enneper, we can construct it from $\mathbb{Z}^2$ by making it a S-quad-graph. Generally speaking, the construction of circle patterns in the sphere can be very hard. But they all come from the S-quad-graph: two black points become intersection points on the unit sphere; "s" becomes orthogonal spheres on the unit sphere; and "c" becomes orthogonal circle on the unit sphere. Based on circle patterns, one will be able to construct Koebe Polyhedron.

Here is the procedure to construct Koebe Polyhedron from circle patterns on the plane,[BHS06]

(1)Mark the center of the circles with white dots and mark the intersection points, where two touching pairs of circles intersect each other orthogonally, with black dots.

(2)Draw edges from the center of each circle to the intersection points on its periphery.

(3)Label "s" and "c" to make the quad-graph an S-quad-graph.

(4)Construct the spheres intersecting S^2 orthogonally along the circles marked by "s".

(5)Connecting the centers of touching spheres, one contains a Koebe polyhedron.

An example is explained in detail in Chapter 3.

By the definition, the Koebe polyhedron is a convex polyhedron with all edges tangent to a unit sphere. It means the lines connecting those touching spheres from (5) above are tangent to the unit sphere. Why is that? Let's assume s_1 and s_2 are centers of touching spheres and l is the line through the intersection point, p, of those two spheres which is perpendicular to the tangent plane T_pS of the unit sphere S. Since those two spheres are orthogonal to the unit sphere, then $ps_1 \perp l$, $ps_2 \perp l$, so ps_1 and ps_2 lie in the same plane T_pS in $\mathbb{R}^3$. Let l' pass through p in the tangent plane and have no other intersection points with those two spheres. Connecting s_1p and s_2p, we have $ps_1 \perp l'$, $ps_2 \perp l'$. Since they lie in a plane and are perpendicular to l', ps_1 and ps_2 are parallel. Also those two segments share the same point p, so ps_1s_2 must lie on the same line. This shows that edges of Koebe polyhedron are tangent to a unit sphere.

The following procedure gives a brief review of the method how to construct a discrete minimal surface.

Step 1: Map conformal curvature lines of a given continuous minimal surface to the complex plane via Gauss map.

Step 2: Map the combinatoric conformal curvature lines onto Riemann sphere by stereographic projection to get a cell decomposition of the sphere; each cell is a conformal square (because each cell is an isothermic surface) and each inscribed circles within the cell is touching the others inscribed circles with in cells which share common edge with it.

Step 3: Construct circle patterns based on vertices and faces. And those which are built on faces are touching with other 4 face circles orthogonally. For those vertex-centered circles, they are touching with each other, too. Finally, vertex-centered circles turn into spheres, face circles stay circles.

Step 4: Build Koebe Polyhedron by connecting the centers of touching sphere which intersect S^2 orthogonally in these face circles.

Step 5: Dualize the Koebe Polyhedron to obtain the corresponding discrete minimal surface.

CHAPTER 3

ENNEPER AS AN EXAMPLE

3.1 Enneper as a continuous minimal surface

Let's start with the continous Enneper minimal surface. It is a very typical minimal surface that is self-intersecting and has no umbilic points. It is named after the German mathematician Alfred Enneper who introduced it. The function for Enneper is [dC83]

$$u(x,y) = \left(x - \frac{x^3}{3} + xy^2, y - \frac{y^3}{3} + yx^2, x^2 - y^2 \right).$$

First, let's show that the Enneper minimal surface has no umbilic point. In the first fundamental form, f is 0, e is $\frac{2\left(x^2+y^2\right)\left(x^2+y^2+2\right)}{2x^2y^2+3x^2+3y^2+1}$, g is $-\frac{2\left(x^2+y^2\right)\left(x^2+y^2+2\right)}{2x^2y^2+3x^2+3y^2+1}$, and $EG - F^2 = E^2$ or G^2. Compute *Gaussian curvature*,

$$K = \frac{eg}{EG} = -\frac{\left[2\left(x^2+y^2\right)\left(x^2+y^2+2\right)\right]^2}{\left(2x^2y^2+3x^2+3y^2+1\right)^2\left[\left(1-x^2+y^2\right)^2+\left(2xy\right)^2+\left(2x\right)^2\right]} < 0$$

The *Gaussian curvature* $K = k_1 k_2 < 0$, then we have, k_1, k_2 can not be zero. Therefore, there is no umbilic point in Enneper minimal surface.

The Enneper surface is given by Weierstrass data $G(z) = z$ and $dh = zdz$, and this map is given by $f : \mathbb{C} \to \mathbb{R}^3$. Thus

$$\begin{aligned} H(z) &= \mathrm{Re} \int \frac{1}{2}\left(\frac{1}{z} - z, i\left(z + \frac{1}{z}\right), 1\right) zdz \\ &= \mathrm{Re} \frac{1}{2}\left(z - \frac{1}{3}z^3, \frac{i}{3}z^3 + iz, z^2\right). \end{aligned}$$

use $x + iy$ to take the place of z, we get

$$z - \frac{1}{3}z^3 = \left(x - \frac{1}{3}x^3 + xy^2\right) + i\left(y + \frac{1}{3}y - x^2 y\right)$$

$$\mathrm{Re}\left(z - \frac{1}{3}z^3\right) = x - \frac{1}{3}x^3 + xy^2.$$

In a same manner, we have

$$\mathrm{Re}\left(\frac{i}{3}z^3 + iz\right) = y - \frac{y^3}{3} + yx^2.$$

and

$$\mathrm{Re}\left(z^2\right) = x^2 - y^2$$

This is exactly the Enneper minimal surface.

We show that in fact Enneper is the Christoffel dual of its Gauss map. By Theorem 2.2 and its proof, and Theorem 2.1, we have

$$f_x^* = \left(\frac{1}{2} - \frac{1}{2}x^2 + \frac{1}{2}y^2, -xy, x\right)$$

$$f_y^* = \left(xy, -\frac{1}{2}x^2 - \frac{1}{2} + \frac{1}{2}y^2, -y\right)$$

Integrating the first formula with respect to x, we get

$$\left(\frac{x}{2} - \frac{x^3}{6} + \frac{xy^2}{2}, -\frac{x^2y}{2}, \frac{x^2}{2}\right) + g(y),$$

where $g : R \to \mathbb{R}^3$.

Take the derivative with respect to y to get

$$\left(xy, -\frac{x^2}{2}, 0\right) + g'(y).$$

Compare with the equation of f_y^*, we have

$$f^*(x,y) = \left(\frac{x}{2} - \frac{x^3}{6} + \frac{xy^2}{2}, -\frac{x^2y}{2}, \frac{x^2}{2}\right) + \left(0, -\frac{1}{2}y + \frac{1}{6}y^3, -\frac{1}{2}y^2\right)$$

Simplify it to get,

$$\left(-\frac{1}{6}x\left(x^2 - 3\left(1 + y^2\right)\right), \frac{1}{6}y\left(-3 - 3x^2 + y^2\right), \frac{1}{2}\left(x - y\right)\left(x + y\right)\right)$$

$$= \left(-\frac{1}{6}x^3 + \frac{1}{2}x + \frac{1}{2}xy^2, -\frac{1}{2}y + \frac{1}{2}yx^2 + \frac{1}{6}y^3, \frac{1}{2}\left(x^2 - y^2\right)\right)$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + xy^2, y - \frac{y^3}{3} + yx^2, x^2 - y^2\right).$$

This is Enneper's Minimal Surface. The picture of Enneper whose domain is $(-2, 2) \times (-2, 2)$ is shown in Figure 3.1, which is self-intersecting and forms three covers of the plane.

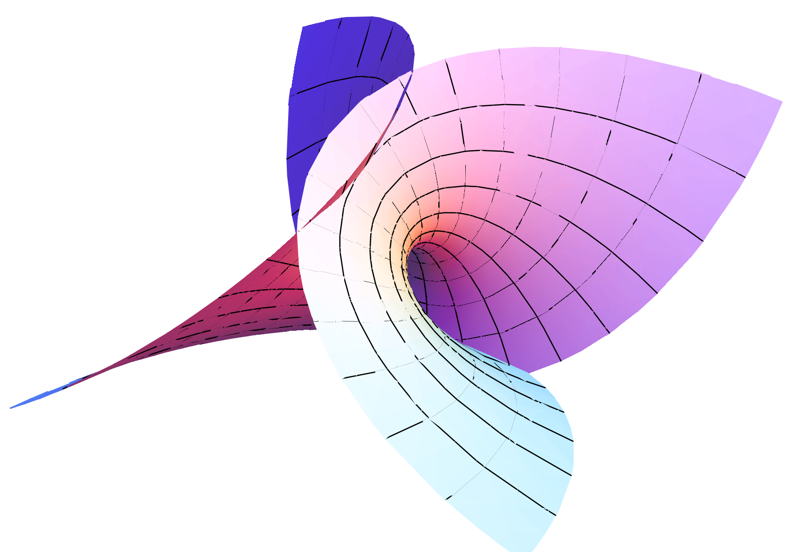


Figure 3.1: Enneper on $(-2, 2) \times (-2, 2)$

3.2 Image of curvature lines of Enneper under Gauss-map

By the Weierstrass Representation, the function for Enneper is $G(z) = z$. Hence, those combinatoric curvature lines are parallels to the real and imaginary axes in the complex plane under the Gauss map. For an example, let's choose $x = -3, x = -1, x = 1, x = 3$ as vertical lines; $y = -3, y = -1, y = 1, y = 3$ as parallel lines. Hence, the cell decomposition is the grid consists of 9 squares on the plane. Each square consists of four conformal squares which are planar. Within every single conformal square, two opposite angles add to 180 degrees and the other two angles are right angles – it looks like a "kite" – because two opposite angles come from stereographic projection of a conformal map of the conformal curvature lines of parametrization, and angles are preserved. If we draw edges from the center of each circle to the intersection points on its periphery, we will have 36 squares; each square forms a "kite" after the stereographic projection in $\mathbb{R}^3$.

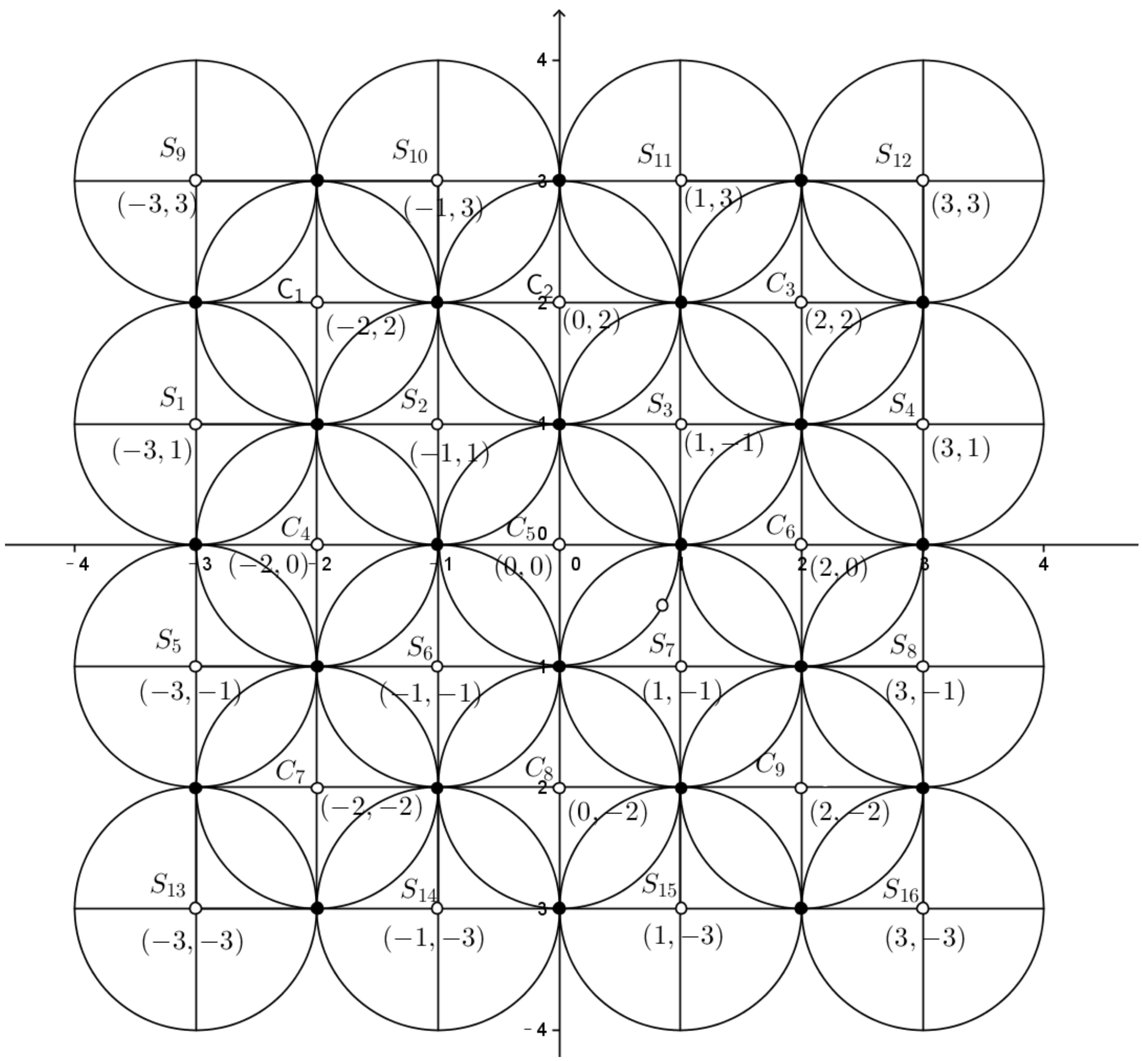


Figure 3.2: Circle patterns for Enneper

3.3 From orthogonal circle pattern to Koebe polyhedron of Enneper

A parametrization of the circle centered at p with radius r on the complex plane in exponential notation is $\gamma(p, r, t) = p + re^{it}, 0 \leq t \leq 2\pi$.

For Enneper, we know $r = 1$ because of its circle pattern. In fact, there are two different types of circle patterns on the plane. One type are those circles labeled by "c", which will become orthogonal circles on the unit sphere; type two are those circles labeled by "s", which will become orthogonal spheres to the unit sphere.

Let's generate type one circle patterns first, and here is the method how to make them become orthogonal circles on the unit sphere:

1) Select 3 points from the circle corresponding to a point labeled"c".

2) Apply stereographic projection to them.

We use Mathematica to draw the graph. The graph of type one circle patterns created by those circles labeled "c" from Figure 3.2 is shown above,

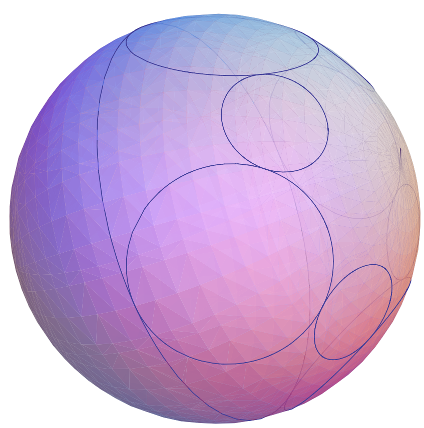


Figure 3.3: Circles on the unit sphere corresponds to "c" after stereographic projection

What about their centers? The method is as follows,

1) Pick up three black points which are intersection points of a circle, say $p_1^{'}$, $p_2^{'}$, $p_3^{'}$. By using the stereographic projection, we get p_1, p_2, p_3 three points on the unit sphere.

2) Let $n = (p_2 - p_1) \times (p_3 - p_1)$, n is a vector which is orthogonal to the plane generated by p_1, p_2, p_3.

3) Let $(\rho n - p_1) \cdot n = 0, \rho \in \mathbb{R}^3$

Then $c_0 = \rho \cdot n$ is the center of circle on the unit sphere.

The situation of type two circle patterns is much more complicated than type one. To create orthogonal spheres, the plan is as follows:

First, create touching spheres from the orthogonal circle pattern from the plane; these spheres are not unique, although their centers all lie on a line. Then extend the center along the line and adjust the radius in order to make those spheres also orthogonal to the unit sphere (central extension).

Our specific steps to construct orthogonal spheres which is also orthogonal to the unit sphere is,

The first three steps are exactly the same as for constructing circles on the unit sphere, and let's continue the steps from above after we get c_0,

4) Extend the center by multiplying by a scalar k to get a new center $k\underline{c_0}$.

5) Let $(p_1 - (0, 0, 0)) \cdot (p_1 - kc_0) = 0$. This means the vector from p_1 to the center of the unit sphere is orthogonal to the vector from p_1 to the center of the sphere we are looking for because it should be orthogonal to the unit sphere.

6) Solve for k, we get k_0; and $k_0 c_0$ is the center of the sphere.

The next step is to get the center of the sphere as well,

7) The radius can be obtained by $r_0 = |k_0 c_0 - p_1|$. Finally, we obtain the orthogonal sphere denoted by $S(k_0 c_0, r_0)$.

Figure 3.4 shows touching circles and orthogonal spheres on the unit sphere.

We get a Koebe polyhedron by connecting the centers of every four touching spheres

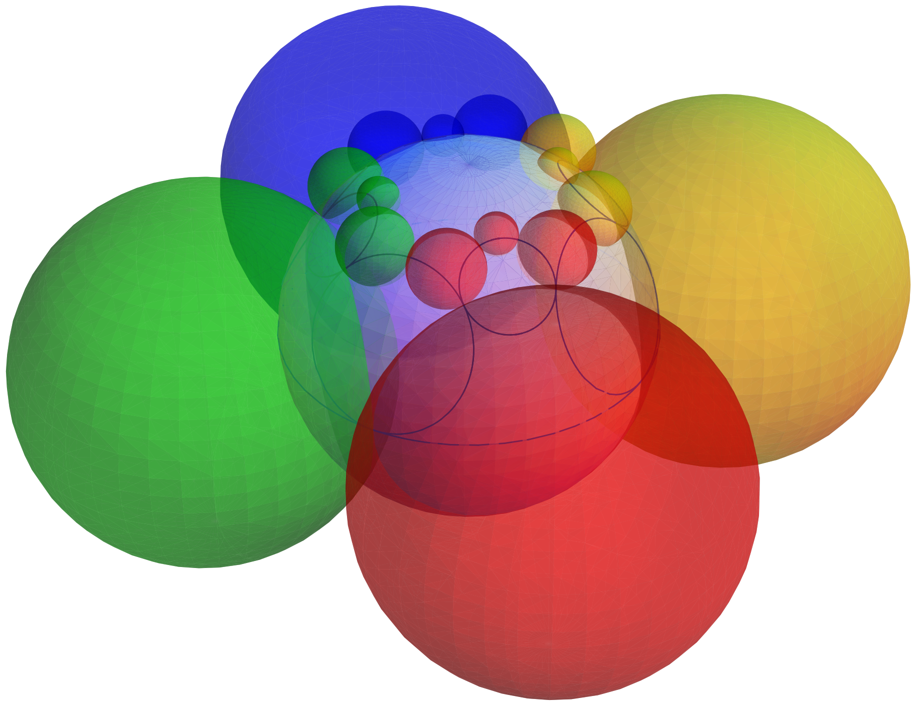


Figure 3.4: Orthogonal spheres after center extension

which is orthogonal to the unit sphere, which is shown in Figure 3.5.

Figure 3.6 shows a combination of orthogonal circles, Koebe polyhedron and the unit sphere.

3.4 Discrete Enneper surface

Now we give an explicit example. Choose the circular net from the circle pattern on the plane, say circle 1, 2, 10, 9 and the circle C inscribed within them (see Figure 3.6). The

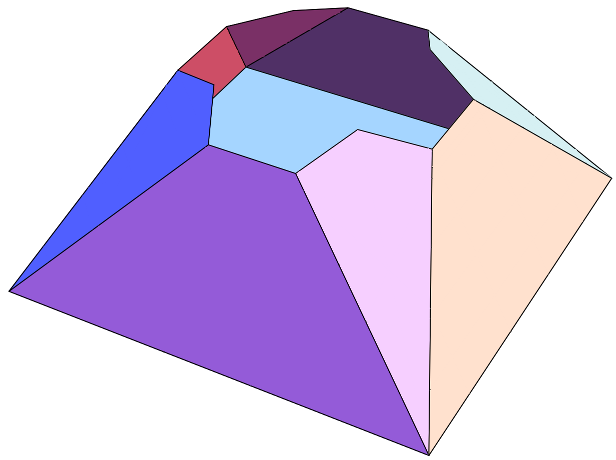


Figure 3.5: Koebe polyhedron

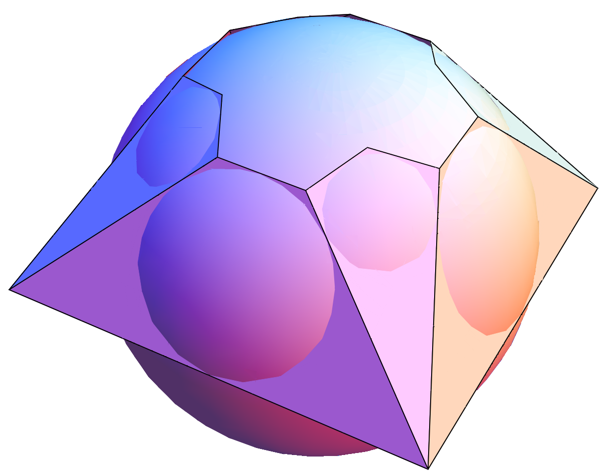


Figure 3.6: Orthogonal circles, Koebe polyhedron and the unit sphere

coordinates for B, S_{10}, D, S_9, E, S_1, A, S_2, C in $\mathbb{R}^3$ are $(-1/3, 2/3, 2/3)$, $(-1/5, 3/5, 4/5)$, $(-2/7, 3/7, 6/7)$, $(-1/3, 1/3, 8/9)$, $(-3/7, 2/7, 6/7)$, $(-3/5, 1/5, 4/5)$, $(-2/3, 1/3, 2/3)$, $(-1, 1, 0)$, $(-8/17, 8/17, 12/17)$.

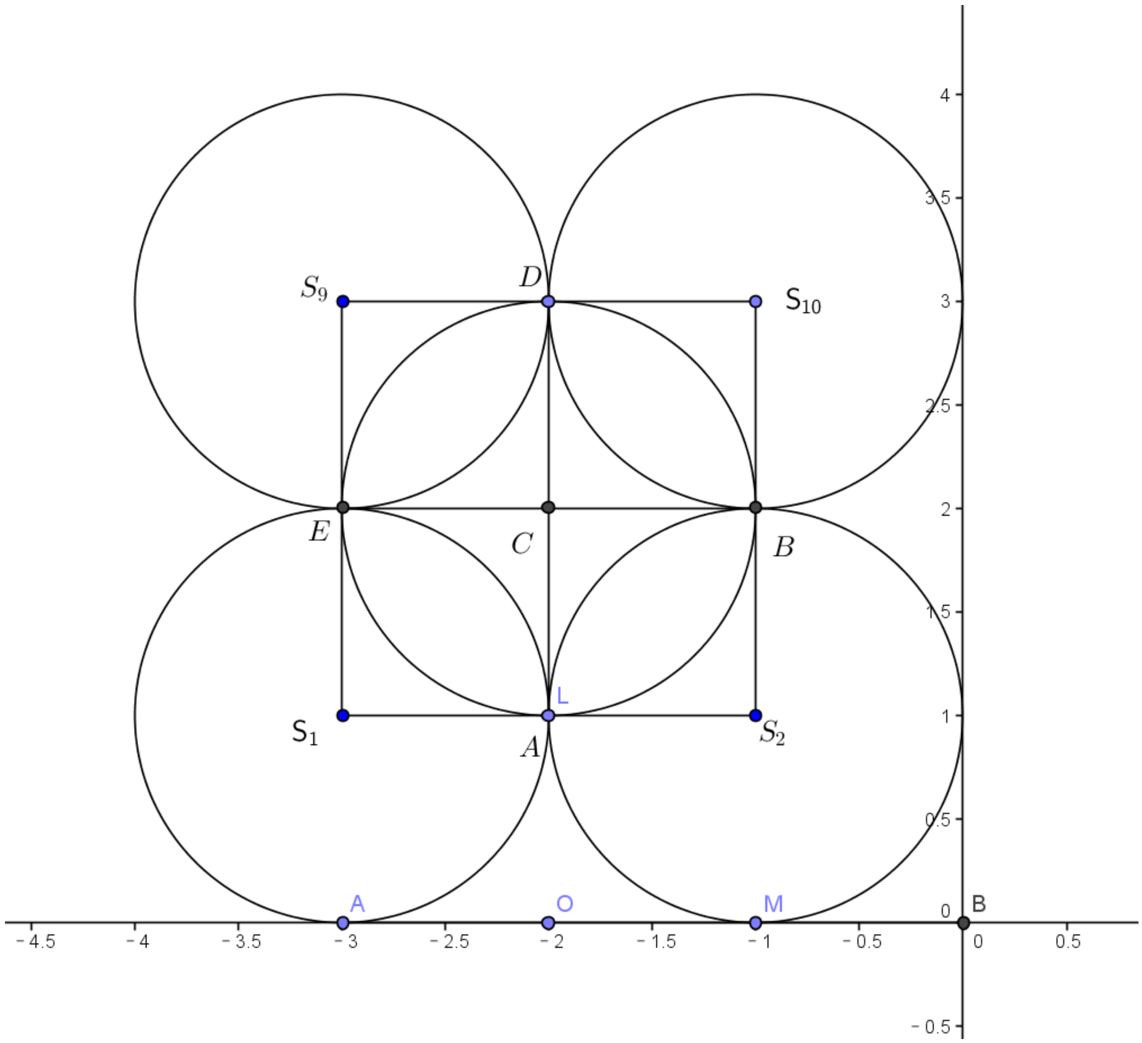


Figure 3.7: Circular nets of circle patterns on the complex plane

S_2ACB, AS_1EC, $CDS_{10}B, CES_9D$ are four conformal squares, which is shown in Figure 3.6. S_2ACB is a discrete isothermic surface. Let's compute the cross ratio of one quadrilateral on this isothermic surface,

$$q(S_2, A, C, B) = \frac{(S_2 - A)(C - B)}{(A - C)(B - S_2)} = -\frac{\frac{1}{9}}{\frac{1}{9}} = -1.$$

In the same manner, we will find other quadrilaterals' cross ratio is also -1. Thus, every single quadrilateral of this isothermic surface is a conformal square.

Use Theorem 2.20 to calculate the dual discrete isothermic surfaces of S_2ACB, AS_1EC,$CDS_{10}B$,CES_9D in $\mathbb{R}^3$.

Let the counterclockwise direction be positive in each "kite", and then assign "+" and "-" to each side alternatively.

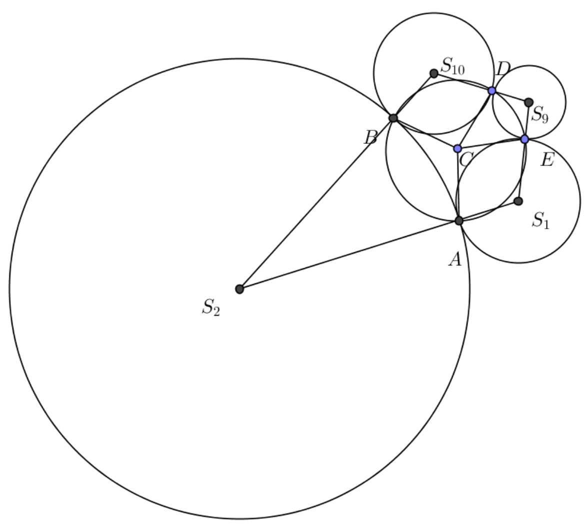


Figure 3.8: Tangential quadrilaterals of an isothermic surface consists of four kites; the notation is the same as in the complex plane.

For an example, let's choose S_2ACB,

$$f^*_{S_2A} = \frac{S_2A}{|S_2A|^2} = (1/3, -2/3, 2/3)$$

$$f^*_{AC} = -\frac{AC}{|AC|^2} = (-10/3, -7/3, -2/3)$$

$$f^*_{CB} = -\frac{CB}{|CB|^2} = (7/3, 10/3, -2/3)$$

$$f^*_{BS_2} = -\frac{BS_2}{|BS_2|^2} = (2/3, -1/3, 2/3)$$

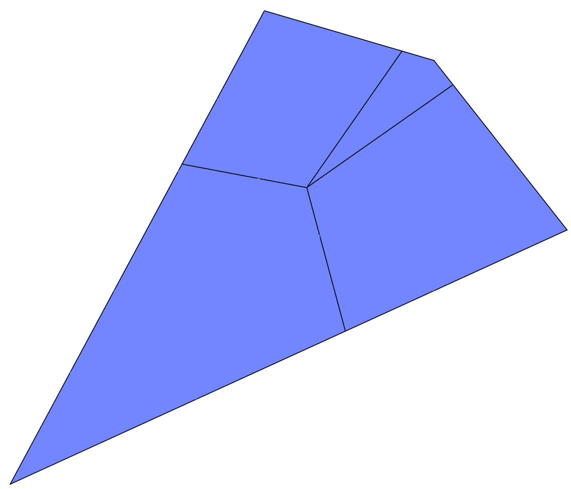


Figure 3.9: A part of dual of Koebe polyhedron of Enneper from $S_1S_2S_{10}S_9$

By Mathematica, one can generate the whole graph of discrete Enneper with arbitrary domain based on the circle patterns above. See Figure 3.10. Also, discrete Enneper can be refined by increasing conformal squares of the circle pattern as shown in Appendix. Given the step and a parameter k, the circle pattern can be constructed in an arbitrary domain. Follow the exact same method until the Koebe polyhedon is obtained. As for its dual, we treat all vertices on the Koebe polyhedron in three different cases. If the vertex is at the upper left corner, it is the original point; if the vertex is at the first row, but not first column, we get the dual of the vertex from its adjacent left-hand side one; if the vertex is not located at the first row, then its dual comes from the vertex right above it. Based on Theorem 2.20, we can dualize the Koebe and get the discrete

Enneper finally. Moreover, we can see the discrete Enneper minimal surface shares some similarities of smooth Enneper minimal surface. From Figure 3.10, it is obvious to see that it self-intersecting and it covers the plane three times.

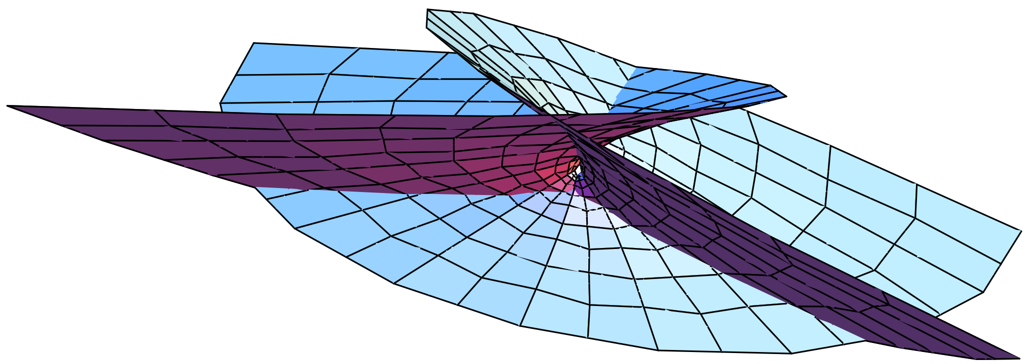


Figure 3.10: Discrete Enneper from the simple circle pattern

CHAPTER 4

CONCLUSION

So far we have already created two typical minimal surfaces; one is the Enneper minimal surface and the other one is the catenoid minimal surface by following the method from the paper [BHS06]. However, why can one not use the same method to create more minimal surfaces? The reason is because the circle pattern is very difficult to create in general. For example, to construct the circle patterns for the Schwarz P-surface and the Scherk tower one must know the circle pattern corresponding to the Gauss maps, and these are difficult. The circle patterns for Enneper surfaces and for the catenoid are z and e^z. The circle pattern for z is obvious. The circle pattern for e^z is not obvious, but the locations of the centers and radius was given in the paper [BHS06]. As for Mathematica program, Enneper and Catenoid have very similar codes but also those are slightly different. For Enneper, one can get the black intersection points directly from the circle pattern which are midpoints of adjacent vertices, but for Catenoid, one has to calculate these intersections of circles since those circles have different sizes on the complex plane. To construct the Schwarz P-surface, the Scherk tower, one must deal with umbilic points since Enneper surface and Catenoid surface have no umbilic points and their circle patterns are hard to construct.

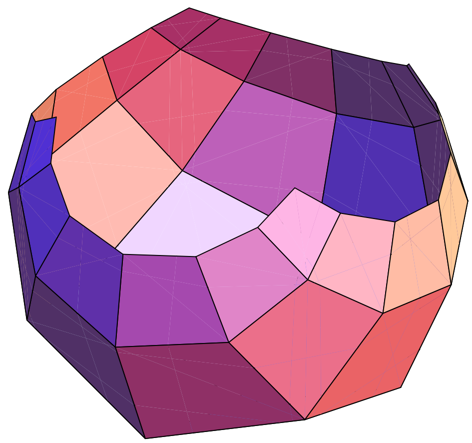
APPENDIX A

MATHEMATICA CODE FOR DISCRETE ENNEPER WITH ARBITRARY DOMAIN AND DISCRETE CATENOID

The following pages contain Mathematica code that generates the discrete Enneper surface for different step sizes and the discrete Catenoid.

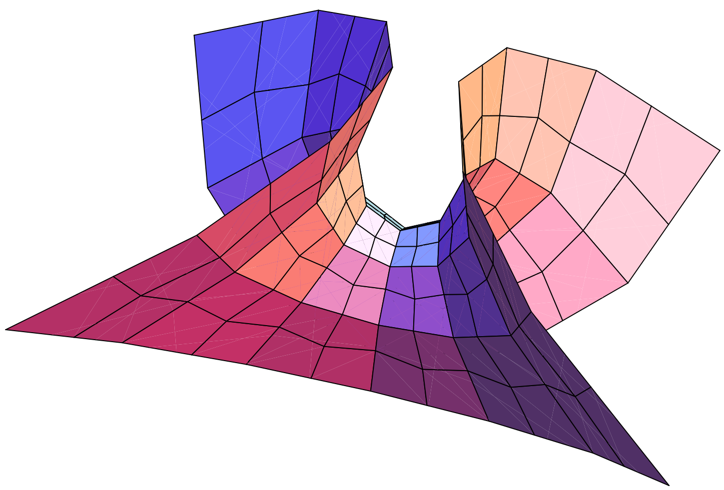
```
Stereographic[z_] := {2 Re[z] / (z Conjugate[z] + 1),
  2 Im[z] / (z Conjugate[z] + 1), (z Conjugate[z] - 1) / (z Conjugate[z] + 1)}
  (*the formula of stereographic projection from complex plane to the unit sphere*)

step = 1 / 4;(*give the length of each step*)

k = 1 / 4;(*parameter*)

s = Table[j + i I, {i, (2 k + 1), -(2 k + 1), -2 * step}, {j, -(2 k + 1), 2 k + 1, 2 * step}];
(*centers of circles on complex plan which is supposed to generate orthogonal spheres*)

γ[p_, r_ , t_] := p + r E^(I t);(*parametrized circle formula*)

computeNormal[z_] := Module[{p1, p2, p3, n, ρ, rhon, normal2, scale1},
  p1 = Stereographic[γ[z, step, 0]];
  p2 = Stereographic[γ[z, step, π / 2]];
  p3 = Stereographic[γ[z, step, π]]; (*generate a circle's three different point*)
  n = Cross[(p2 - p1), (p3 - p1)];
  (*a nomal vector to the plane consists those three points*)
  rhon = Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]];
  (*center corresponding to orthogonal circle*)
  normal2 = (p1 - scale1 * rhon); (*to make spheres be othognal to the unit sphere*)
  Flatten[scale1 * rhon /. Solve[p1.normal2 == 0, scale1]]
  (*center extension to gnerate centers of orthogonal spheres*)
 ](*the template to calculate centers of orthognal spheres*)

computeRadius[z_] := Module[{p1, p2, p3, n, ρ, rhon, normal2, scale1, center},
  p1 = Stereographic[γ[z, step, 0]];
  p2 = Stereographic[γ[z, step, π / 2]];
  p3 = Stereographic[γ[z, step, π]];
  n = Cross[(p2 - p1), (p3 - p1)];
  rhon = Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]];
  normal2 = (p1 - scale1 * rhon);
  center = Flatten[scale1 * rhon /. Solve[p1.normal2 == 0, scale1]];
  EuclideanDistance[center, p1]
 ](*the template to calculate radius of orthogonal spheres*)

centers = Table[computeNormal[s[[i]][[j]]], {i, 1, 2 * (2 k + 1) * (1 / step) / 2 + 1},
   {j, 1, 2 * (2 k + 1) * (1 / step) / 2 + 1}];(*centers of orthogonal spheres*)

radii = Table[computeRadius[s[[i]][[j]]], {i, 1, 2 * (2 k + 1) * (1 / step) / 2 + 1},
   {j, 1, 2 * (2 k + 1) * (1 / step) / 2 + 1}];(*radius of orthogonal spheres*)


generateKoebeFace[a_, b_] := Module[{},
  Graphics3D[Polygon[{centers[[a]][[b]], centers[[a + 1]][[b]], centers[[a + 1]][[b + 1]],
     centers[[a]][[b + 1]]}]]](*connect centers of orthogonal spheres*)
```

```
Show[Table[generateKoebeFace[i, j], {i, 1, 2 * (2 k + 1) * (1 / step) / 2 },
  {j, 1, 2 * (2 k + 1) * (1 / step) / 2 }], PlotRange → All, Boxed → False](*Koebe Polyhedron*)
```



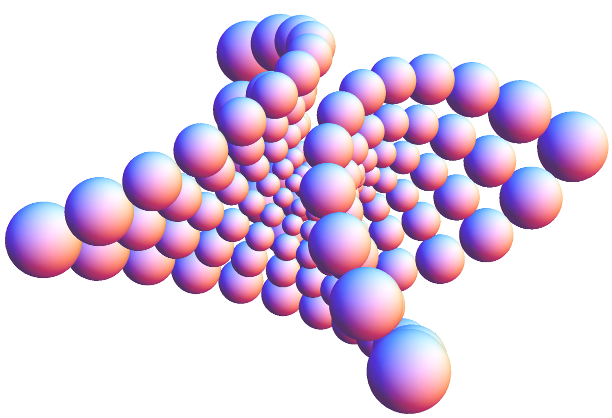
```
computeC[z_] := Module[{p1, p2, p3, n, ρ, rhon, normal2, scale1},
  p1 = Stereographic[γ[z, step, 0]];
  p2 = Stereographic[γ[z, step, π / 2]];
  p3 = Stereographic[γ[z, step, π]];
  n = Cross[(p2 - p1), (p3 - p1)];
  rhon = Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]]
 ](*the template to calculate centers of circles in R^3*)

c = Table[computeC[j + i I], {i, 2 k, -(2 k + 1), -2 * step}, {j, -(2 k), 2 k + 1, 2 * step}];
(*calculate centers of circles in R^3*)

Table[-(2 k + 1) + (j - 1) step + (2 k + 1 - (i - 1) step) I,
  {i, 1, 2 (2 k + 1) / step + 1, 1}, {j, 1, 2 (2 k + 1) / step + 1, 1}];
 (*calculate all points on the complex plane by transforming to index from coordinates*)

Table[-(2 k + 1) + (j - 1) step + (2 k + 1 - (i - 1) step) I,
   {i, 1, 2 (2 k + 1) / step + 1, 1}, {j, 1, 2 (2 k + 1) / step + 1, 1}] ==
  Table[j + i I, {i, (2 k + 1), -(2 k + 1), -step}, {j, -(2 k + 1), 2 k + 1, step}];

pts = Table[
   Which[Mod[i j, 2] == 1, computeNormal[-(2 k + 1) + (j - 1) step + (2 k + 1 - (i - 1) step) I],
    Mod[i + j , 2] == 0, computeC[-(2 k + 1) + (j - 1) step + (2 k + 1 - (i - 1) step) I],
    True, Stereographic[-(2 k + 1) + (j - 1) step + (2 k + 1 - (i - 1) step) I] ],
   {i, 1, 2 (2 k + 1) / step + 1, 1}, {j, 1, 2 (2 k + 1) / step + 1, 1}];
(*if i,j are odd numbers, it is the point to genereate centers of
 orthognal spheres in R^3; if i,j are both even numbers,
they are the points to generate centers of orthongonal circles in R^3; otherwise,
they are intersection points of orthogonal circles and spheres in R^3*)

Clear[dual]
```

```
For[j = 1, j ≤ (2 k + 1) * 2 * (1 / step) + 1 , j ++,
 For[i = 1, i ≤ (2 k + 1) * 2 * (1 / step) + 1 , i ++,
  Which[(i == 1) && (j == 1),
    dual[1, 1] = {0, 0, 0}, (i == 1) && (j ≠ 1), dual[i, j] = dual[i, j - 1] +
      (pts[[i]][[j]] - pts[[i]][[j - 1]]) / Norm[(pts[[i]][[j]] - pts[[i]][[j - 1]])]^2 ,
    i ≠ 1, dual[i, j] = dual[i - 1, j] - (pts[[i]][[j]] - pts[[i - 1]][[j]]) /
       Norm[(pts[[i]][[j]] - pts[[i - 1]][[j]])]^2 ];
 ]
](*if i=1 and j=1, then the dual is the original point;
if the point is in the first row, calculate it by the dual of its lefthand side points;
otherwise, the point is calculated by the dual of
 the point located at the previous row but same column*)
Show[Table[Graphics3D[
   Polygon[{dual[i, j], dual[i + 1, j], dual[i + 1, j + 1], dual[i, j + 1]}]],
  {i, 1, 2 * (2 k + 1) * (1 / step) }, {j, 1, 2 * (2 k + 1) * (1 / step) }], PlotRange → All,
 Boxed → False](*Generate discrete Enneper consists of combinatorics
 of "conformal squares" by taking the dual of Koebe Polyhedon, *)
```



```
Table[{Norm[dual[i, j] - dual[i + 1, j]] + Norm[dual[i, j + 1] - dual[i + 1, j + 1]] ==
     Norm[dual[i, j] - dual[i, j + 1]] + Norm[dual[i + 1, j] - dual[i + 1, j + 1]]},
   {i, 1, 2 * (2 k + 1) * (1 / step) }, {j, 1, 2 * (2 k + 1) * (1 / step)}] // N ;
 (*test: each piece is isothermic, which means has an inscribed circle*)

computenewradius[i_, j_] := Module[{r, norm1, v, w, θ, c1, l1, l2, l3, l4},
  l1 = Norm[dual[i, j] - dual[i, j + 1]];
  l2 = Norm[dual[i, j + 1] - dual[i + 1, j + 1]];
  l3 = Norm[dual[i + 1, j + 1] - dual[i + 1, j]];
  l4 = Norm[dual[i + 1, j] - dual[i, j]];
  r = Sqrt[l1 l2 l3 l4] / ((l1 + l2 + l3 + l4) / 2); (*radius=
   √Area / semiperimeter in a tangential quadrilateral with opposite angles are supplimentary*)
  r
 ]
```

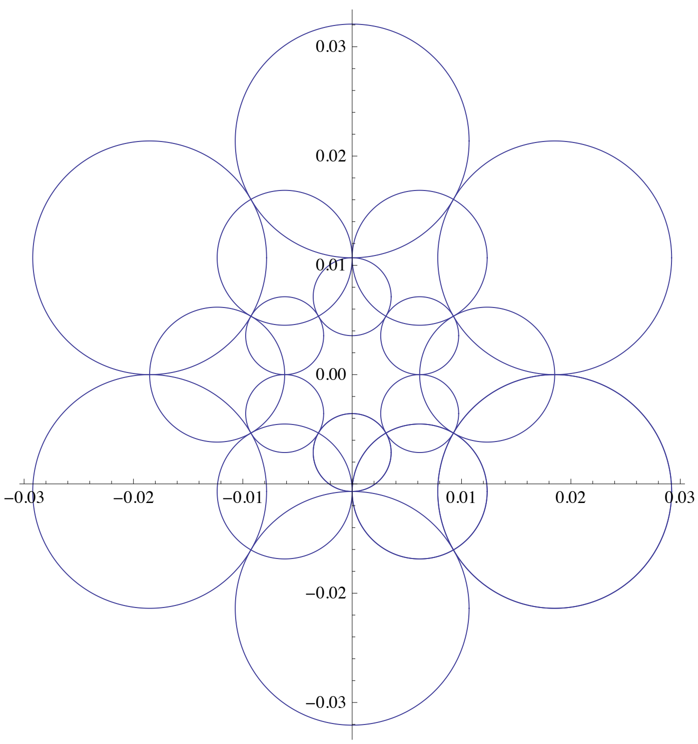
```
computenewcenter[i_, j_] := Module[{r, norm1, v, w, θ, c1, l1, l2, l3, l4},
  l1 = Norm[dual[i, j] - dual[i, j + 1]];
  l2 = Norm[dual[i, j + 1] - dual[i + 1, j + 1]];
  l3 = Norm[dual[i + 1, j + 1] - dual[i + 1, j]];
  l4 = Norm[dual[i + 1, j] - dual[i, j]];
  r = Sqrt[l1 l2 l3 l4] / ((l1 + l2 + l3 + l4) / 2);
  v = dual[i, j] - dual[i, j + 1];
  w = dual[i, j] - dual[i + 1, j];
  norm1 = r * Cross[Cross[v, w], v] / Norm[Cross[Cross[v, w], v]];
  θ = ArcCos[v.w / (Norm[v] Norm[w])] / 2;
  c1 = r / Tan[θ] (-v) / Norm[v] + dual[i, j] - norm1
 ](*the template to calculate the
  centers of inscribed circles for each isothermic piece*)

newcenters = Table[computenewcenter[i, j],
    {i, 1, 2 (2 k + 1) / step, 1}, {j, 1, 2 (2 k + 1) / step,  1}] // N;
 (*calculate centers of inscribed circles of each isotheric piece*)

normal := Cross[dual[i, j] - dual[i, j + 1], dual[i + 1, j] - dual[i, j]]
 (*a normal vector of isothermic piece*)

generateenneper[a_, b_] := Module[{},
  Graphics3D[{CapForm[Round], Tube[{newcenters[[a]][[b]] - .001 normal / Norm[normal],
      newcenters[[a]][[b]] + .001 normal / Norm[normal]}, computenewradius[i, j]]}]]

(*generate semisphere based on each inscribed cirlces*)

Show[Table[generateenneper[i, j],  {i, 1, 2 (2 k + 1) / step, 1},
  {j, 1, 2 (2 k + 1) / step,  1}], PlotRange → All, Boxed → False]
```



```
generateenneper[a_, b_] := Module[{},
  Graphics3D[{CapForm[None], Tube[{newcenters[[a]][[b]] - .001 normal / Norm[normal],
      newcenters[[a]][[b]] + .001 normal / Norm[normal]}, {0, computenewradius[i, j]}]}]]
```

```
(*generate semisphere based on each inscribed cirlces*)
Show[Table[generateenneper[i, j], {i, 1, 2 (2 k + 1) / step, 1},
  {j, 1, 2 (2 k + 1) / step, 1}], PlotRange → All, Boxed → False]
```

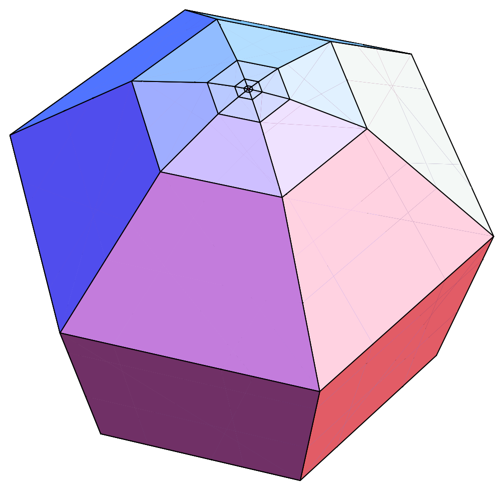
```
Stereographic[z_] := {2 Re[z] / (z Conjugate[z] + 1),
  2 Im[z] / (z Conjugate[z] + 1), (z Conjugate[z] - 1) / (z Conjugate[z] + 1)}(*the
  formula of stereographic projection from complex plane to the unit sphere*)

step = 6;

ρ = π / step;

α = ArcTanh[1 / 2 Abs[1 - Exp[2 I ρ]]];

c[i_, j_] := Exp[α i + I ρ j](*the center of circle pattern*)

r[i_, j_] := Sin[ρ] Norm[c[i, j]](*the radius of circle pattern*)

s = Table[i + I (2 j + i), {i, -3, 3}, {j, -3, 3}];
(*centers of circles on complex plan
 which is supposed to generate orthogonal spheres*)

toCartesian[z_] := {Re[z], Im[z]}

Show[Table[ParametricPlot[
   Tooltip[toCartesian[c[i, 2 j + i] + E^(I t) r[i, 2 j + i]], {i, 2 j + i}],
   {t, 0, 2 π}], {i, -9, -7, 1}, {j, -3, 3}], PlotRange → All]
```




```
γ[c_, r_ , t_] := c + r E^(I t)
```

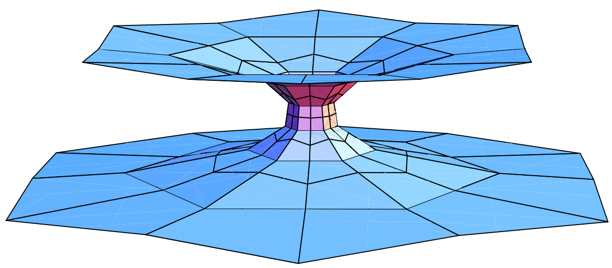
```
computeNormal2[z_] :=
 Module[{i = Re[z], j = Im[z], p1, p2, p3, n, ρ, rhon, normal2, scale1},
  p1 = N[Stereographic[γ[c[i, j], r[i, j], 0]]];
  p2 = N[Stereographic[γ[c[i, j], r[i, j], π / 2]]];
  p3 = N[Stereographic[γ[c[i, j], r[i, j], π]]];
  (*generate a circle's three different point*)
  n = Cross[(p2 - p1), (p3 - p1)];
  (*a nomal vector to the plane consists those three points*)
  rhon = Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]];
  (*center corresponding to orthogonal circle*)
  normal2 = (p1 - scale1 * rhon);
  (*to make spheres be othognal to the unit sphere*)
  Chop[N[Flatten[scale1 * rhon /. Solve[p1.normal2 == 0, scale1]]]]
  (*center extension to gnerate centers of orthogonal spheres*)
 ](*the template to calculate centers of orthognal spheres*)

computeC[z_] :=
 Module[{i = Re[z], j = Im[z], p1, p2, p3, n, ρ, rhon, normal2, scale1},
  p1 = N[Stereographic[γ[c[i, j], r[i, j], 0]]];
  p2 = N[Stereographic[γ[c[i, j], r[i, j], π / 2]]];
  p3 = N[Stereographic[γ[c[i, j], r[i, j], π]]];
  n = Cross[(p2 - p1), (p3 - p1)];
  rhon = Chop[Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]]]
 ](*the template to calculate centers of circles in R^3*)

computeRadius[z_] := Module[
  {i = Re[z], j = Im[z], p1, p2, p3, n, ρ, rhon, normal2, scale1, center},
  p1 = N[Stereographic[γ[c[i, j], r[i, j], 0]]];
  p2 = N[Stereographic[γ[c[i, j], r[i, j], π / 2]]];
  p3 = N[Stereographic[γ[c[i, j], r[i, j], π]]];
  (*generate a circle's three different point*)
  n = Cross[(p2 - p1), (p3 - p1)];
  rhon = Flatten[ρ n /. Solve[(ρ n - p1).n == 0, ρ]];
  normal2 = (p1 - scale1 * rhon);
  center = Flatten[scale1 * rhon /. Solve[p1.normal2 == 0, scale1]];
  N[EuclideanDistance[center, p1]]
 ](*the template to calculate radius of orthogonal spheres*)

centers = Table[computeNormal2[s[[i]][[j]]], {i, 1, 7, 2}, {j, 1, 7}];
(*centers of orthogonal spheres*)

radii = Table[computeRadius[s[[i]][[j]]], {i, 1, 7, 2}, {j, 1, 7}];
(*radius of orthogonal spheres*)


generateKoebeFace[a_, b_] := Module[{},
  Graphics3D[Polygon[{computeNormal2[a + b I], computeNormal2[a + (b + 2) I],
     computeNormal2[(a + 2) + (b + 2) I], computeNormal2[(a + 2) + (b + 0) I]}]]]
  (*connect centers of orthogonal spheres*)
```

```
Show[Table[generateKoebeFace[i, j], {i, -9, 9, 2}, {j, -9, 9, 2 }],
 PlotRange → All, Boxed → False](*Koebe Polyhedron of Catenoid*)
```



```
Clear[i, j]

Clear[pts];
Do[
 Do[
   pts[i, j] = computeNormal2[i + j I];
   (*Compute centers of spheres in "s" points*)
   pts[i + 1, j + 1] = computeC[i + 1 + (j + 1) I];
   (*compute centers of circles in "c" points*)
   pts[i + 1, j]  = Chop[Stereographic[x + I y] /. NSolve[Apply[Plus,
            ({x, y} - toCartesian[c[i + 1, j + 1]])^2 ] == r[i + 1, j + 1]^2 &&
          Apply[Plus, ({x, y} - toCartesian[c[i, j]])^2 ] == r[i, j]^2][[1]]];
   (*the intersection of circles from the complex plane composed
    with stereographic projection at black points*)
   pts[i, j + 1] =  Chop[Stereographic[x + I y] /. NSolve[Apply[Plus,
            ({x, y} - toCartesian[c[i + 1, j + 1]])^2 ] == r[i + 1, j + 1]^2 &&
          Apply[Plus, ({x, y} - toCartesian[c[i, j]])^2 ] == r[i, j]^2][[2]]]
   , {j, -9, 9, 2}];
  ,
 {i, -9, 9, 2}]

Clear[dual]
```

```
Block[{i, j},
 For[j = -9, j ≤ 9 , j++,
  For[i = -9, i ≤ 9 , i++,
   Which[(i == -9) && (j == -9), dual[-9, -9] = {0, 0, 0},
     (i == -9) && (j ≠ -9), dual[i, j] = dual[i, j - 1] +
       (pts[i, j] - pts[i, j - 1]) / Norm[(pts[i, j] - pts[i, j - 1])]^2 ,
     i ≠ -9, dual[i, j] = dual[i - 1, j] - (pts[i, j] - pts[i - 1, j]) /
        Norm[(pts[i, j] - pts[i - 1, j])]^2 ];
  ]
 ]
](*This is the dual map which uses those
  points we get from above. That is : if i=-9 and j=9,
then the dual is the original point; if the point is in the first row,
calculate it by the dual of its lefthand side points;
otherwise, the point is calculated by the dual of
 the point located at the previous row but same column*)

Show[Table[Graphics3D[
   Polygon[{dual[i, j], dual[i + 1, j], dual[i + 1, j + 1], dual[i, j + 1]}]],
  {i, -6, 5}, {j, -6, 6 }], PlotRange → All, Boxed → False]
 (*Generate discrete Catenoid consists of "conformal squares"
 by taking the dual of Koebe Polyhedon*)
```



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