A NOTE ON SOME UPPER BOUNDS FOR PERMANENTS OF $(0,1)$-MATRICES

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ABSTRACT. In [1], Al-Kurdi proves two new upper bounds on the permanents of $(0,1)$-matrices. In this note, we provide a very short combinatorial proof of these bounds by interpreting the permanent of a $(0,1)$-matrix as the number of ways to choose a system of distinct representatives. We also sharpen these bounds and show that in certain situations they improve on the well-known Minc-Brégman bound. We provide numerical evidence that the bound here also improves on the Minc-Brégman and a bound of Liang and Bai a fraction of the time for the generic case of a very sparse $(0,1)$ matrix.

1. INTRODUCTION

The permanent of a $n \times n$ matrix A with entries a_{ij} is defined by

$$\operatorname{Per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n is the symmetric group on n-letters. Permanents have myriad applications in combinatorics and graph theory; one particular combinatorial application involves the number of ways to choose a system of distinct representatives from a collection of sets. Let $A_1, A_2, \ldots A_n \subseteq \{1, 2, \ldots n\}$. A *system of distinct representatives* is an n-tuple $(a_1, a_2, \ldots, a_n)$ where $a_j \in A_j$ and $a_i = a_j$ if and only if $i = j$. The *Hall matrix* corresponding to sets $A_1, A_2, \ldots A_n$ is the matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $j \in A_i$ and $a_{ij} = 0$ otherwise. The permanent of the Hall matrix, $\operatorname{Per}(A)$, gives the number of different systems of distinct representatives.

To see this combinatorial interpretation of the permanent, note that the position of ones in the jth column of the Hall matrix indicates the sets of which j is an element, i.e., $a_{ij} = 1$ iff $j \in A_i$. Since a system of distinct representatives consists of a choice of n distinct elements from the set $\{1, 2, \ldots n\}$ each element is chosen as the representative of some subset. Thus, a system of distinct representatives corresponds to a choice of n locations a_{ij} so that each $a_{ij} = 1$ and one element is chosen from each row and column. In other words, each system of distinct representatives corresponds to an element $\sigma \in S_n$ such that $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = 1$. Therefore, $\operatorname{Per}(A)$ counts the number of possible systems of distinct representatives.

We use this combinatorial technique to provide a short intuitive proof of the following bounds that were the subject of [1]:

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Theorem 1.1 (from [1]). *Let A be a $(0,1)$ $n \times n$ matrix. Let $\{r_i\}$ be the set of row sums of A, ordered so that $r_1 \leq r_2 \leq \cdots \leq r_n$. Then*

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} \min(r_i, n-i+1).$$

Theorem 1.2 (from [1]). *Let A be a $(0,1)$ $n \times n$ matrix. Let $H_i = \{j \mid a_{ij} = 1\}$ (the set of non-zero column indicies of row i). Assume that $H_1 \leq H_2 \leq \cdots \leq H_n$. Then*

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n-1} \left| \bigcup_{j>i} (H_i \cap H_j) \right|.$$

A good upper bound for the permanent of a $n \times n$ nonnegative matrix is the Minc-Brégman [2] bound,

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} (r_i)!^{1/r_i},$$

where r_i is the i-th row sum. For a random $(0,1)$-matrix, this is a quite good bound. In certain special cases, however, other upper bounds are better than the Minc-Brégman bound. For example, in [4], Liang and Bai prove that if A is a $n \times n$ $(0,1)$-matrix with rows sums r_i, then

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} \sqrt{a_i(r_i - a_i + 1)}$$

where $a_i = \min(\lceil \frac{r_i+1}{2} \rceil, \lceil \frac{i}{2} \rceil)$. They proceed to show that if A is a matrix consisting of $n-k$ rows containing all ones, followed by k rows containing only one or two ones, then their bound is smaller than the Minc-Brégman bound.

In this note we prove the following refinement of Theorem 1.2 and show that it sometimes improves on the bounds of Liang-Bai and Minc-Brégman.

Theorem 1.3. *Let A be a $(0,1)$ $n \times n$ matrix. Let $H_i = \{j \mid a_{ij} = 1\}$ (the set of non-zero column indicies of row i). Assume that $H_1 \leq H_2 \leq \cdots \leq H_n$. Then*

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n-1} \min \left\{ \left| \bigcup_{j>i} (H_i \cap H_j) \right|, n-i+1 \right\}.$$

2. Proof of the theorems

We provide here a combinatorial proof of Theorem 1.3. It is clear that Theorem 1.3 implies Theorem 1.2, and Theorem 1.1 also follows since $\left| \bigcup_{j>i} (H_i \cap H_j) \right| \leq r_i$.

Proof of Theorem 1.3. The permanent of a matrix is invariant under row and column permutations, so we lose no generality by assuming that the row sums are non-decreasing. We consider choosing a representative for each A_k beginning with $k = 1$, and we make these choices in order of increasing k. Note that $\left| \bigcup_{j>i} (H_i \cap H_j) \right|$ is the number of choices of representative for A_i that are *also* possible choices of representative for other rows that have not already been chosen.

Suppose that $k \in H_i$ but that $k \notin \bigcup_{j>i} (H_i \cap H_j)$. Since each element of $\{1, 2, \ldots, n\}$ must be chosen as a representative for some row in order to have a system of distinct representatives, we know that either k has already been chosen as a representative for some row or k must be chosen as the representative for A_i (or no system of distinct representatives exists, in which case the permanent is 0 and the bound trivially holds). Therefore, the number of choices for a representative of A_i is either 1 or $\left|\bigcup_{j>i}(H_i \cap H_j)\right|$.

On the other hand, since we make the choices of representatives of A_i in increasing order of i, we know that when choosing a representative of A_i we have already chosen $i-1$ representatives, so at most $n-(i-1) = n-i+1$ possible choices remain. The number of choices of representative for A_i is therefore no larger than

$$\min\left\{\left|\bigcup_{j>i}(H_i \cap H_j)\right|, n-k+1\right\},$$

and we multiply these to obtain a bound for the permanent.

□

3. Numerical comparison of Minc-Brégman, Liang-Bai, and Theorem 1.3

In this section we provide some numerical evidence that the bound of Theorem 1.3 is an improvement over the bounds of Liang-Bai and Minc-Brégman.

In [4], the following matrix is given as an example where the bound of Liang-Bai beats the Minc-Brégman bound:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this particular example, the Minc-Brégman bound for $\operatorname{Per}(A)$ is approximately 1189.74, the Liang-Bai bound is approximately 509.12, the bound of Theorem 1.3 yields 192, while the exact permanent is 48. However, it is far from the case that the bound of Theorem 1.3 always improves on the Liang-Bai bound.

In numerical computations, we see that when the fraction of ones in a $(0,1)$-matrix is small, then the bound of Theorem 1.3 beats the Minc-Brégman and Liang-Bai bound a portion of the time for randomly generated matrices. Using the computer algebra system *Mathematica*, we created $10,000$ $n \times n$, $(0,1)$-matrices where the probability of any single entry being 1 was p. Table 1 show the percentage of time that a matrix of size $n \times n$ summarizes the results of the numerical experiment. They show that the bound of Theorem 1.3 does not work only in finely crafted cases; it is occasionally the best known bound for random $(0,1)$-matrices.

Since calculating the permanent of a matrix is computationally intensive, while the bounds mentioned in this paper are not, one can compute several different bounds for the permanent of a matrix in a fraction of the time required for a precise computation of the permanent. The practical impact of the above results is that when the matrix is sparse, the bound of Theorem 1.3 is probably worth adding to the list of bounds to be computed.

$p \backslash n$	10	15	20	25	30	35	40	45	50	55	60	65
.05	0.01	0	0.01	0.04	0.05	0.15	0.41	0.51	0.62	0.19	0.01	0
.06	0.05	0.04	0.04	0.20	0.53	1.00	1.12	0.48	0.07	0	0	-
.07	0.11	0.27	0.61	1.38	2.05	1.79	0.52	0.01	0.01	-	-	-
.08	0.36	0.69	1.62	2.97	3.29	0.81	0.05	0	0	-	-	-
.09	0.87	1.54	3.45	4.84	2.01	0.10	0	0	0	-	-	-
.10	1.32	3.1	6.02	4.46	0.73	0.01	0	-	-	-	-	-
.11	2.31	5.07	8.82	3.4	0.09	0	0	-	-	-	-	-
.12	3.56	8.74	9.12	1.17	0	0	0	-	-	-	-	-
.13	5.74	11.52	7.34	0.39	0	0	0	-	-	-	-	-
.14	7.98	13.97	5.02	0.11	0	0	0	-	-	-	-	-
.15	10.59	15.08	2.58	0	0	0	0	-	-	-	-	-
.16	13.73	15.26	1.0	0	0	0	0	-	-	-	-	-
.17	17.27	13.44	0.39	0	0	0	0	-	-	-	-	-
.18	20.5	10.7	0.1	0	0	0	0	-	-	-	-	-
.19	23.14	8.42	0.04	0	0	0	0	-	-	-	-	-
.20	25.29	5.42	0	0	0	0	0	-	-	-	-	-
.25	25.25	0.13	0	0	-	-	-	-	-	-	-	-
.30	11.60	0	0	0	-	-	-	-	-	-	-	-
.35	2.26	0	0	0	-	-	-	-	-	-	-	-
.40	0.37	0	0	0	-	-	-	-	-	-	-	-

TABLE 1. An $n \times n$ $(0,1)$-matrix is randomly generated with probability p of any single entry being 1. For each n, 10,000 trials are conducted; the entries in the table are the percentage of the time that the bound of Theorem 1.3 improves on the bound of Minc-Brégman and Liang-Bai. A "-" indicates that the results were not computed.

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