CLASSIFICATION OF SIMPLE CLOSED GEODESICS ON CONVEX DELTAHEDRA

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ABSTRACT. We obtain a complete classification of all simple closed geodesics on the eight convex deltahedra. In particular, up to symmetry and parallel translation, there is exactly one simple closed geodesic on the triangular dipyramid, one on the pentagonal dipyramid, five on the snub disphenoid, two on the triaugmented triangular prism, and four on the gyroelongated square dipyramid. It has been shown previously [FF07] that the octahedron contains two simple closed goedesics and the icosahedron contains three; it is well known that the tetrahedron contains infinitely many simple closed geodesics.

1. INTRODUCTION

A *geodesic* γ on a polyhedron $\mathscr{P}$ is a locally shortest curve $\gamma : \mathbb{R} \to \mathscr{P}$. A *geodesic segment* is similar except that the domain is a closed, bounded interval. In each face of the polyhedron, a geodesic is a line segment. In two adjacent faces, a geodesic will form a straight line after one of the faces is rotated about their common edge by the dihedral angle. If one develops the polyhedron into the plane, the image of the geodesic in a development will be a line. A *closed geodesic* is a geodesic segment $\gamma : [a,b] \to \mathscr{P}$ so that $\gamma(a) = \gamma(b)$ and so that the tangent vectors at a and b have the same direction (if $\gamma(a)$ and $\gamma(b)$ lie on an edge, then the tangent vectors have the same direction after rotation by the dihedral angle). A *simple closed geodesic* is a closed geodesic whose only self-intersection occurs at the endpoints.

The classification of simple closed geodesics on polyhedra has recently been studied by a number of authors. Galperin [Gal03] has shown that a generic polyhedron will have *no* simple closed geodesics; a necessary condition for a polyhedron to have a simple closed geodesic is that the vertices may be partitioned into two sets, each having total curvature 2π. It is known that there are arbitrarily long closed geodesics on a regular tetrahedron; furthermore, all geodesics on a regular tetrahedron are simple. (These remarkable properties hold because the flat torus is a double cover of the tetrahedron.) Fuchs and Fuchs [FF07]

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classified the geodesics on three other Platonic solids: the cube, the octahedron (both also considered by [CFG+05]), and the icosahedron. Fuchs [Fuc09] and Lawson [Law10] have recently independently completed the classification of simple closed geodesics on the dodecahedron.

A *deltahedron* is a polyhedron with equilateral triangle faces. It is known [FvdW47] that there are exactly 8 convex deltahedra: the tetrahedron, triangular dipyramd, octahedron, pentagonal dipyramid, snub disphenoid, triaugmented triangular prism, gyroelongated square dipyramid, and the icosahedron. (We examine only properly convex deltahedra and so exclude the possibility of 6 triangles meeting at a vertex. The techniques here apply equally well to non-properly convex deltahedra.) In this paper, we complete the classification of simple closed geodesics on the 8 convex deltahedra. In a forthcoming paper [LPTW10], we detail an algorithm that finds all simple closed geodesics up to a given length on an arbitrary deltahedron.

2. Terminology and results required for the main theorem

Our proof relies on the ability to label the edges of a deltahedron based on the angle at which they are crossed by a simple closed geodesic. For this, we require the following theorem.

Theorem 2.1. *All segments of a simple closed geodesic on a deltahedron that lie in a fixed face of the deltahedron are parallel.*

Proof. Let γ be a simple closed geodesic, and consider a face containing at least two geodesic segments γ_1 and γ_2 of γ; call the edge crossed by both segments $\overline{AB}$ and the face $\triangle ABC$. Define α_i to be the angle between $\overline{AB}$ and γ_i measured counterclockwise using the inward pointing normal.

We claim that $\alpha_1 = \alpha_2 + \frac{k\pi}{3} \pmod{\pi}$ for some $k \in \{0,1,2\}$. To see this, develop forwards and backwards from the segment γ_1 until the entire geodesic γ is developed. Since the faces crossed by the geodesic are equilateral triangles, the development will coincide with the tiling of the plane by equilateral triangles. Images of $\triangle ABC$ encountered in the developing may differ by a rotation by a multiple of $\frac{\pi}{3}$, and the segment γ_2 will appear in some (possibly rotated) image of $\triangle ABC$ in this development. We conclude that $\alpha_1 = \alpha_2 + \frac{k\pi}{3} \pmod{\pi}$.

Define Corridor 1 to be the region between line $\overleftrightarrow{AC}$ and the line through B parallel to $\overleftrightarrow{AC}$. Define Corridor 2 to be the region between line $\overleftrightarrow{BC}$ and the line through A parallel to $\overleftrightarrow{BC}$. These two corridors are shown in Figure 2.1. We now consider two simultaneous developments of γ into the plane (and into Corridors 1 and 2) by fixing an image of $\triangle ABC$ containing images of γ_1 and γ_2 and developing γ forwards and backwards from both segments. Let $\overleftrightarrow{\gamma_i}$ denote the development of geodesic γ_i into the plane. Since γ_i is a geodesic segment, the development of γ_i will be a straight line contained in corridor 1 (resp. corridor 2) until possibly the line $\overleftrightarrow{\gamma_i}$ leaves corridor 1 (resp. corridor 2).

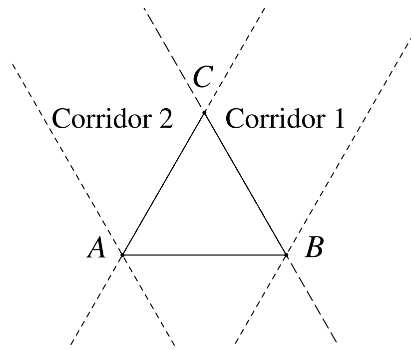


FIGURE 2.1. Two corridors in the plane; the horizontal edge $\overline{AB}$ is crossed by two geodesic segments.

Partition the set of possible values $(0,\pi)$ for angles α_i into the following intervals and singleton sets: $(0,\pi/3)$, $\{\pi/3\}$, $(\pi/3,2\pi/3)$, $\{2\pi/3\}$, and $(2\pi/3,\pi)$. Assume towards a contradiction that γ_1 and γ_2 are not parallel in $\triangle ABC$. Thus $\alpha_1 = \alpha_2 + \frac{k\pi}{3} \pmod{\pi}$ for some $k \in \{1,2\}$, and so α_1 and α_2 are elements of *different* sets from the list above. Note that if $\overleftrightarrow{\gamma_1}$ and $\overleftrightarrow{\gamma_2}$ intersect in some corridor, they intersect on the polyhedron. We consider two cases. If $\alpha_1 = \frac{l\pi}{3}$ for some $l \in \{1,2\}$, then $\alpha_2 = \frac{(3-l)\pi}{3}$, and the lines $\overleftrightarrow{\gamma_1}$ and $\overleftrightarrow{\gamma_2}$ will cross in the intersection of Corridors 1 and 2. Otherwise, we orient the corridors by aligning $\overline{AB}$ horizontally so that the corridors inherit the orientation shown in Figure 2.1. If the intersection of $\overleftrightarrow{\gamma_i}$ with the left (resp. right) boundary of the corridor is below segment $\overline{AB}$, say that $\overleftrightarrow{\gamma_i}$ crosses the corridor from left to right (resp. right to left). If $\overleftrightarrow{\gamma_1}$ and $\overleftrightarrow{\gamma_2}$ cross the same corridor in different directions (one left to right and one right to left), then they *must* intersect in that corridor. Table 1 shows that for any choice of α_1, and for any choice of corresponding value of α_2, there exists a corridor in which the two developments of $\overleftrightarrow{\gamma_1}$ and $\overleftrightarrow{\gamma_2}$ have opposite orientation, and therefore intersect. Since such an intersection contradicts our initial assumption that γ was a simple closed geodesic, we conclude that γ_1 and γ_2 are parallel. $\square$

Since every segment of a simple closed geodesic in a face is parallel, the angle α between the geodesic and any edge, measured counterclockwise using the inward pointing normal, depends only on the edge. We therefore can label each edge of $\mathscr{P}$ as U, R, H, or L depending on whether the edge is uncrossed (U) or whether α lies in the interval $[2\pi/3,\pi)$, $[\pi/3,2\pi/3)$, or $(0,\pi/3)$ respectively. Angle considerations immediately yield the following:

Corollary 2.2. *Either every edge is uncrossed in a face, or at least one edge of the face is labeled H and another is labeled L or R. If one edge of a face is labeled L, then the edge*

Angle α_i	Corridor 1	Corridor 2
$(0, \pi/3)$	left to right	left to right
$(\pi/3, 2\pi/3)$	right to left	left to right
$(2\pi/3, \pi)$	right to left	right to left

TABLE 1. The angle α_i of geodesic γ_i determines the way in which it crosses the corridors.

of that face located immediately counterclockwise along the boundary of the face must be labeled H; similarly, if an edge is labeled R then the edge immediately clockwise along the boundary of the face must be labeled H.

Every geodesic on a deltahedron can be developed into the plane so that the image of the deltahedron's vertices lie in the lattice generated by 1 and $\zeta = e^{i\pi/3}$. Let γ be a closed geodesic on a deltahedron $\mathscr{P}$ passing through an edge $\overline{AB}$. Since γ is a geodesic, some development of γ into the plane will be a line segment bounded at each end by an image of $\overline{AB}$. Now, since the geodesic γ is closed, the difference λ between the endpoints of this line segment will lie in the lattice. λ is called a *generator of γ with respect to edge $\overline{AB}$*. It is clear that each closed geodesic determines a unique generator.

We now describe how to determine the generator of a geodesic by only knowing how the geodesic alternates among edges. Fix an edge of $\mathscr{P}$ labeled H (such an edge must exist since every face crossed by γ must contain an edge labeled H). We consider the sequence of edge labels corresponding to the list of crossings made sequentially by γ. Angular considerations imply that the next two edge crossings by γ are labeled either R,H or L,H, as illustrated in Figure 2.2.

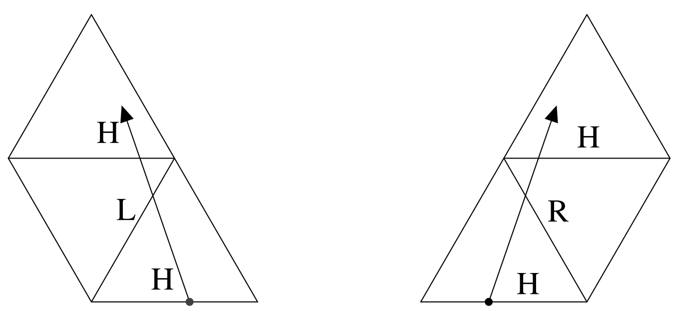


FIGURE 2.2. An edge crossing labeled 'H' implies that the next two crossings are either 'LH' or 'RH'.

We indicate the 'RH' sequence as $+$ and the 'LH' sequence as $-$. Since this two-face process ended on an edge labeled H, we can continue labeling in this fashion to obtain a *geodesic sequence* of $+$ and $-$ associated to γ and to the initial edge. Changing the initial edge results in a cyclic permutation of the sequence of $+/-$.

The next definition and proposition pertain to geodesic *segments*; the result is an important tool in the proof of Theorem 2.6.

Definition 2.3. *Let γ be a geodesic segment. Define m to be the sum of the number of $+$ and $-$ in the geodesic sequence, and define n to be the number of $-$. The* signature *of γ is the pair (m,n). The* increment *of γ is $\Delta x/\Delta y$ in the coordinate system given by -1 and ζ.*

Proposition 2.4. *If a geodesic segment has signature (m,n), then the increment of γ is strictly between $\frac{n-1}{m}$ and $\frac{n+1}{m}$.*

Proof. Since the signature is (m,n), the geodesic intersects m horizontal lines in addition to the initial edge, so $\Delta y = m$. Similarly, the geodesic crosses exactly n lines parallel to the vector ζ, so $n-1 < \Delta x < n+1$. $\square$

If γ is a simple closed geodesic, the increment of any segment of γ is the same as the increment of γ. However, two segments may have different signatures that provide different bounds on the increment.

Lemma 2.5. *If the generator of a closed geodesic γ is $m\zeta - n$, where m and n are integers, then m is the sum of the number of $+$ and $-$ in the geodesic sequence, while n is the number of $-$. The signature of γ is the pair (m,n) and the increment of γ is $\frac{n}{m}$.*

Proof. Since each 'LH' or 'RH' sequence takes the geodesic from a horizontal line to a horizontal line, m is the sum of the number of $+$ and $-$ in the geodesic sequence. Also, each 'LH' pair will necessarily cross exactly one of the lines parallel to the vector ζ, so n is the number of $-$. The statements about signature and increment are immediate consequences. $\square$

Theorem 2.6. *Let γ be a simple closed geodesic on a deltahedron $\mathscr{P}$. If $\mathscr{P}$ has no vertices of degree 3, then γ crosses each edge at most once.*

Proof. Assume by way of contradiction that $\mathscr{P}$ has no vertices of degree 3 and that γ crosses some edge more than once. Since γ is simple closed, γ divides $\mathscr{P}$ into two regions each with curvature 2π. There is some face so that γ crosses all three edges; if not, then one of the regions would contain no vertices which violates the curvature condition. In this face, the vertex X between the segments must either have degree 4 or degree 5.

For the remainder of this proof we refer to Figure 2.3. Suppose that X has degree 4. $\triangle XCD$ must be labeled so that $\overline{XC}$ is labeled L, $\overline{CD}$ is labeled H, and $\overline{XD}$ is labeled R. Since $\overline{XC}$ is labeled L, $\overline{XB}$ must be H; similarly, $\overline{XA}$ must be labeled H by Corollary 2.2. However, this causes $\triangle XAB$ to have two edges labeled H; a contradiction.

If instead X has degree 5, $\triangle XCD$ is labeled L, H, and R. Then $\overline{XB}$ is labeled H, and $\overline{XE}$ is labeled H. $\overline{XA}$ can be labeled neither L nor R since either $\triangle XAB$ or $\triangle XEA$ would have

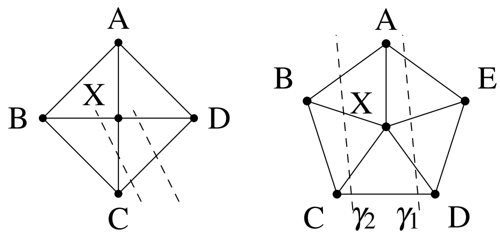


FIGURE 2.3. If a simple closed geodesic crosses an edge more than once, there must be degree 3 vertices.

two sides labeled H. Therefore, $\overline{XA}$ is uncrossed and so $\overline{AB}$ is L and $\overline{AE}$ is R. Call these two geodesic segments γ_1 and γ_2; γ_1 has signature $(2,0)$ and γ_2 has signature $(2,2)$. Now, the increment of γ_1 is equal to the increment of γ_2 because they are segments of the same closed geodesic. However, γ_1 bounds the increment strictly between $-1/2$ and $1/2$ while γ_2 bounds the increment strictly between $1/2$ and $3/2$; these bounds are incompatible. $\square$

Finally, the main ingredient of our proof is that there is only a small number of possible configurations around a degree four or degree five vertex:

Theorem 2.7. *Let γ be a simple closed geodesic on a convex deltahedron. The labeling induced by γ around degree 4 and degree 5 vertices is one of the labelings indicated in Figures 2.4 and 2.5.*

Proof. We consider first the cases for both degrees four and five where only 0, 1, or 2 edges incident to the central vertex are crossed; in what follows we use the notation in Figure 2.3. If no edge incident to X is crossed, then the only possibility is **4e** or a similar diagram for the degree 5 case (which only occurs on the icosahedron). If only one edge incident to X is crossed, then the remaining edges incident to X are uncrossed. The crossed edge can be labeled neither L nor R since either of these would force another of the edges adjacent to X to be labeled H. Therefore, the crossed edge must be labeled H; without loss of generality we let the crossed edge be $\overline{XB}$. The remaining edges are forced as shown in **4a** and **5a** by Corollary 2.2.

If two edges are crossed, those edges are either adjacent or not. If the two crossed edges are not adjacent, these must both be labeled H, since otherwise a third adjacent edge would also be labeled. The remaining edges are forced by Corollary 2.2 and the resulting diagrams are **4b** and **5b**. If the crossed edges are adjacent, one edge must be H and one must be either L or R. If X has degree four, then Corollary 2.2forces all but one edge label and results in diagrams **4cl** and **4cr**. If X has degree five then Corollary 2.2 forces all of the remaining edges and yields diagrams **5cl** and **5cr**.

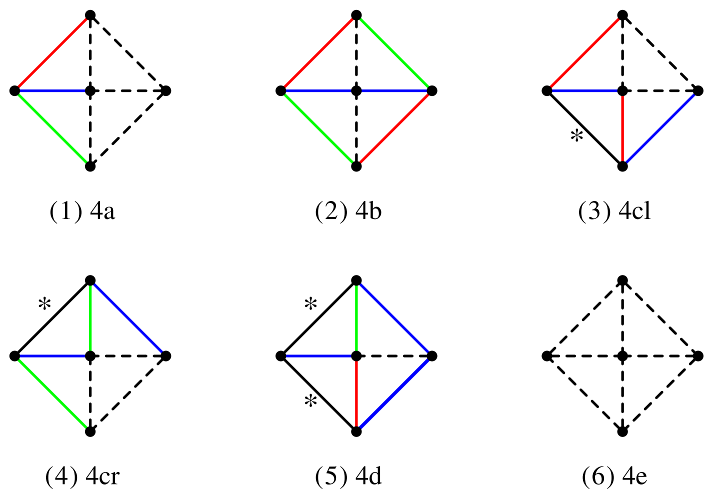


FIGURE 2.4. The six possible configurations around a degree 4 vertex. Red edges are labeled ‘L’, blue edges ‘H’, green edges ‘R’. Edges marked with an asterisk are inconclusive on a triangular dipyramid; on all other deltahedra those edges are uncrossed.

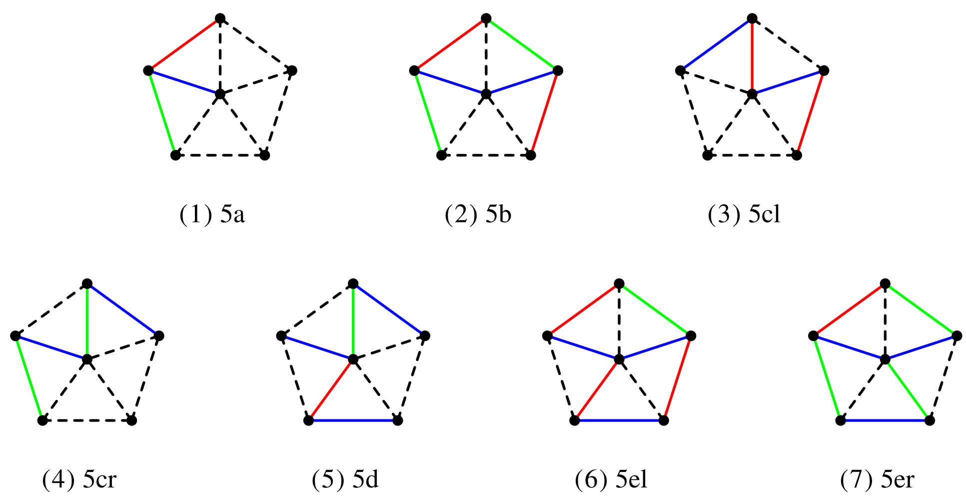


FIGURE 2.5. The seven possible configurations around a degree 5 vertex (an eighth diagram, with all edges uncrossed, is possible if all neighbors have degree 5).

Now we consider possibilities when X has degree four and more than 2 edges are crossed. One edge must be horizontal; again assume $\overline{XB}$ is labeled H. This implies that $\overline{XD}$ is uncrossed, because if $\overline{XD}$ were H then both $\overline{XA}$ and $\overline{XC}$ would be uncrossed because of the competing demands placed by the two edges labeled H, and $\overline{XD}$ cannot be either L or R since that would force a triangle to have two edges labeled H. Therefore, it is not possible to have all four edges crossed. The only possibility is for $\overline{XA}$ to be R and $\overline{XC}$ to be L. While this forces $\overline{AD}$ and $\overline{CD}$ to be H, the remaining two edges are undetermined; this is the case **4d**.

If X has degree five and more than 2 edges are crossed, then at most two of them can be labeled H (no two edges labeled H can be adjacent). If exactly one edge is labeled H, we assume it is $\overline{XB}$. Then $\overline{XA}$ is labeled R and $\overline{XC}$ is labeled L; the remaining edges must be uncrossed, and this is case **5d**. If two edges are labeled H, by symmetry they are $\overline{XB}$ and $\overline{XE}$. $\overline{XA}$ must be uncrossed because the H labels put conflicting demands on $\overline{XA}$. We can then have either $\overline{XC}$ labeled L or $\overline{XD}$ labeled R (these are cases **5el** and **5er**). Since at least one edge in every face is uncrossed, we can not have both $\overline{XC}$ and $\overline{XD}$ labeled. □

Finally, we observe that some generators are equivalent under symmetry.

Lemma 2.8. *Consider a simple closed geodesic with generator $m\zeta - n$. After a reflection in a symmetry plane perpendicular to a horizontal edge, the geodesic is generated by $m\zeta - (m-n)$.*

Note that after a reflection in such a symmetry plane, the sequence of edges crossed by a geodesic is the same except that each "L" becomes an "R" and vice versa.

3. Proof and statement of the main theorems

In the previous section we showed that every simple closed geodesic induces a labeling of the edges of $\mathscr{P}$ with the labels U, L, H, and R. Sufficiently small parallel translations of a simple closed geodesic will preserve both the sequence of edge crossings and the lack of self-intersections. Call a parallel translation of a geodesic γ *vertex-avoiding* if it does not move γ past a vertex in the planar development of γ. (If a parallel translation sweeps the geodesic past a vertex, the sequence of edge crossings will change and so will the $+/-$ sequence.) The goal of this paper is to classify, up to symmetry and vertex-avoiding parallel translations, the simple closed geodesics on deltahedra.

The Table 2 lists the number of simple closed geodesics on each deltahedron, the lengths of each geodesic, and the generator for each geodesic. In this section we verify this table.

Proving that Table 2 is correct requires the following three steps:

(1) Analyze which edge labelings can possibly be induced by a simple closed geodesic (the existence of an edge labeling is a necessary condition for a simple closed geodesic to exist)
(2) Determine the generator that corresponds to that labeling

Name (Number of SCG)	Lengths	Generators
tetrahedron (∞)		
triangular dipyramid (1)	$2\sqrt{3}$	$4\zeta-2$
octahedron (2)[FF07]	$3, 2\sqrt{3}$	$3\zeta, 4\zeta-2$
pentagonal dipyramid (1)	$2\sqrt{3}$	$4\zeta-2$
snub disphenoid (5)	$4, 2\sqrt{3}, 2\sqrt{7}, \sqrt{13}, \sqrt{19}$	$4\zeta, 4\zeta-2, 3\zeta-1, 4\zeta-1, 5\zeta-2$
triagumented triangular prism (2)	$4, \sqrt{19}$	$4\zeta, 5\zeta-2$
gyroelongated square dipyramid (4)	$4, 2\sqrt{7}, \sqrt{19}, \sqrt{21}$	$4\zeta, 6\zeta-2, 5\zeta-2, 5\zeta-1$
icosahedron (3) [FF07]	$5, 3\sqrt{3}, 2\sqrt{7}$	$5\zeta, 6\zeta-3, 6\zeta-2$

TABLE 2. Simple closed geodesics on the 8 convex deltahedra.

(3) Check that the generator corresponds to a geodesic by ensuring that the development given by the labeling contains a line segment given by the generator.

In the figures that follow, we provide the edge labeling analysis (1) for the dipyramids, the snub disphenoid, the triaugmented triangular prism, and the gyroelongated square dipyramid. Although it is in principal possible to provide a prosaic proof, we find it much more illuminating to organize our work as a sequence of diagrams. We always begin at the "red" degree four vertex. In a labeling, the red vertex must correspond to one of the possibilities **4a** - **4e** shown in Figure 2.4. We begin by making a choice for this vertex (a choice in the diagrams is indicated by <u>an underlined</u> symbol). We then move to the vertex labeled "1". This vertex is either forced to be a particular configuration (indicated by, for example, a **5d** with no underline) or we must make a choice (indicated by an underline). Note that each time we make a choice, either the reader will find a later case with another choice indicated, or the other choice(s) were already covered by a previous symmetric case. If the labeling can be completed, then the generator corresponding to the labeling (2) is a consequence of Lemma 2.5; these generators are indicated in the figures as well. If the labeling can not be completed, then we indicate what contradiction arises. These contradictions take two forms. Sometimes, a vertex (indicated with a numeral) has a partial labeling that can not be completed to any of the degree 4 or 5 labelings. Sometimes, however, the partial labeling that exists would force the vertices to be partitioned into two sets that violate the the criterion that each set must contain vertices of total curvature exactly 2π.

We finally must check that the generator corresponds to a geodesic by ensuring that the development given by the labeling contains a line segment given by the generator. We call a labeling *compatible* with its generator if the development defined by the labeling contains a line segment given by the generator. In all of the cases we consider, the generator and the labeling are compatible; we do not know whether this is the case more generally or only in the case of the convex deltahedra.

We now describe how to determine which labelings are compatible with a given generator. Assume that a generator $m\zeta - n$ is *primitive*, that is, assume that that $\gcd(m,n) = 1$. A geodesic must connect corresponding points on a given horizontal line segment and its translation by $m\zeta - n$. This determines a parallelogram (shaded in gray in Figure 3.1) that contains exactly $m-1$ vertices. There are m different $+/-$ sequences possible. (By sweeping a line segment parallel to the left edge of the quadrilateral from left to right, the $+/-$ sequence will change each time a vertex is crossed.) Some adjacent $+-$ pair (namely, if the vertex being passed over is on the kth horizontal, then the symbols in positions k and $k+1$ will be transposed) will change to $-+$ or vice versa. For the generators below, this results in a cyclic permutation of the previous sequence. If $\gcd(m,n) = 1$, then the generator $a(m\zeta - n)$ is not primitive and there are only m possible sequences. We are working with a relatively small set of generators, and so it is easy to write down all of the possible sequences; they are shown in Table 3.

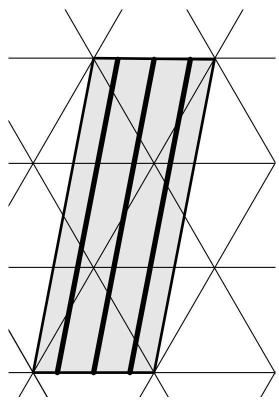


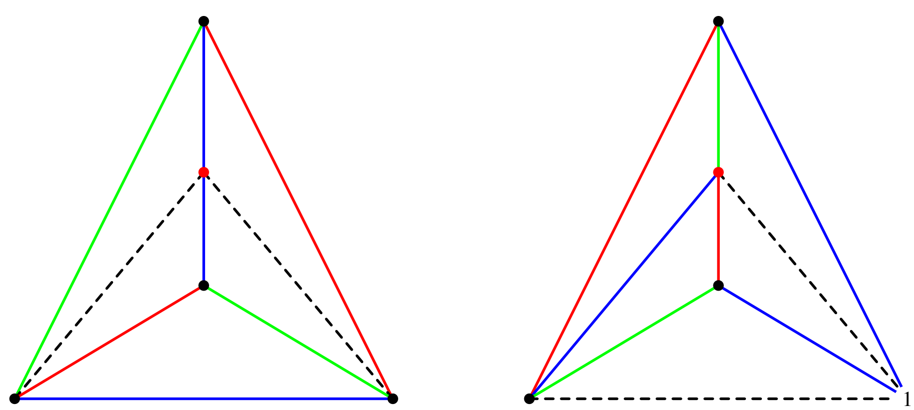
FIGURE 3.1. The three possible geodesics which determine different $+/-$ sequences corresponding to $3\zeta - 1$ are shown by the 3 dark line segments; the sequences are $-++$, $+-+$, and $++-$. Each is a cyclic permutation of the others. A geodesic for a given generator exists only if the $+/-$ sequence for the generator appears in this list.

3.1. **Triangular Dipyramid.** To begin the case analysis, we choose the initial vertex to be one of the degree four vertices, colored red in Figure 3.2. Because there are two vertices of curvature π and three vertices of curvature $2\pi/3$, a simple closed geodesic must divide the surface into two regions, one of which contains both of the degree 3 vertices. If an edge is uncrossed then the vertices on that edge will be in the same component; these conditions immediately rule out all possible degree four configurations except **4b** and **4d**, both situated so that uncrossed edges are equatorial.

Generator	Sequence	Generator	Sequence
ζ	$+$	$4\zeta-1$	$-+++$
$2\zeta-1$	$-+$	$5\zeta-1$	$-++++$
$3\zeta-1$	$-++$	$5\zeta-2$	$-+-++$

TABLE 3. Each generator is compatible only with cyclic permutations of the sequence shown.

If the initial vertex is **4b**, then the only edge not colored by this configuration is forced to be labeled H, since every triangle must contain an H. If the initial choice is **4d**, then there remain three unlabeled edges. However, the vertex labeled 1 is incident to two edges labeled H, hence the vertex 1 is **4b**, which forces the remaining labels. Note that this configuration is obtained from the previous configuration by a rotation of the dipyramid about the axis containing the apical vertices. Therefore, up to symmetry there is only one possible labeling of the triangular dipyramid, and so there is at most one geodesic. To see that there is exactly one, we need to ensure that the development defined by the labeling contains the line segment defined by the generator. The $+/-$ sequence determined by the labeling is $+--+-$; this is compatible with the related primitive generator $2\zeta-1$.



(1) <u>4b apical</u>. $4\zeta-2$ (2) <u>4d</u>, 4b. $4\zeta-2$

FIGURE 3.2. Cases 1 - 2 for the triangular dipyramid.

3.2. **Pentagonal Dipyramid.** To obtain curvature of 2π in each region, the geodesic must split the vertices so that both degree 5 vertices are in the same region; the curvature then splits as (2,2) and (0,3) (here we indicate (n_1,n_2) to mean that there are n_1 degree 5 vertices and n_2 degree 4 vertices). All degree 4 vertices are equatorial, and since each is incident to two apical and two equatorial edges, we must distinguish the orientation of some configurations. We indicate this in the figures by **4a** (apical) or **4a** (equatorial) depending on whether the apical or equatorial edges are labeled H. Choices for the initial vertex of **4a** (apical), **4c** (both), **4d** (apical), and **4e** immediately yield curvature contradictions. The four remaining cases are shown in Figure 3.3. One can easily check the the three completed labelings are symmetric by a rotation of the dipyramid about the axis connecting the apices. Therefore, these three geodesics are equivalent under symmetry.

3.3. **Snub Disphenoid.** We use the notation $(n_1,n_2);(m_1,m_2)$ to mean that there are n_1 degree 5 vertices and n_2 degree 4 vertices in one region determined by a simple closed geodesic and m_1 and m_2 such vertices in the other region. In the case of the snub disphenoid, the vertices must be partitioned as $(0,3);(4,1)$ or $(2,2);(2,2)$. Moreover, there are three different types of edges incident to degree four vertices, which requires us to consider multiple orientations of the initial vertex configuration. The fourteen possible resulting cases are illustrated in Figure 3.3. After a reflection symmetry (see Lemma 2.8), the generators obtained are $4\zeta-1$ (case 1), $4\zeta-2$ (case 2), $5\zeta-2$ (cases 6 and 12), 4ζ (cases 7 and 8), and $3\zeta-1$ (case 13).

The two geodesics obtained by the generator 4ζ are equivalent after a reflection of the snub disphenoid through a symmetry plane. The two $5\zeta-3$ geodesics are equivalent after a reflection and a rotation taking one pair of adjacent degree four vertices to the other pair.

3.4. **Triaugmented Triangular Prism.** Using the same notation as above, the only possible vertex partitions are $(6,0);(0,3)$ or $(4,1);(2,2)$. Although all individual edges incident to a degree four vertex are symmetric to one another, there are two types of faces incident to the degree 4 vertices. Thus, for some vertex configurations, we will again need to consider different orientations. The ten possible cases, shown in Figure 3.4, give generators $5\zeta-2$ (cases 3 and 9) and 4ζ (case 8). To see that cases 3 and 9 are equivalent under symmetry, first reflect so that both generators are $5\zeta-2$. Then, rotate the triaugmented triangular prism 180° about the line through the vertex labeled **4cr** that is perpendicular to a square face of the original prism.

3.5. **Gyroelongated Square Dipyramid.** The possible vertex partitions are $(6,0);(2,2)$ or $(4,1);(4,1)$. The nine possible cases for the gyroelongated square dipyramid are illustrated in Figure 3.5. After a symmetry, we obtain the possible generators $5\zeta-2$ (cases 1 and 2), $5\zeta-1$ (case 5), $6\zeta-2$ (case 8), and 4ζ (case 9). One can easily check that the sequences corresponding to these labeling are compatible with the generator; it remains to see that the geodesics identified in cases 1 and 2 are symmetric. After a reflection, one obtains the other geodesic by rotating 180° about the axis connecting the two degree four vertices.

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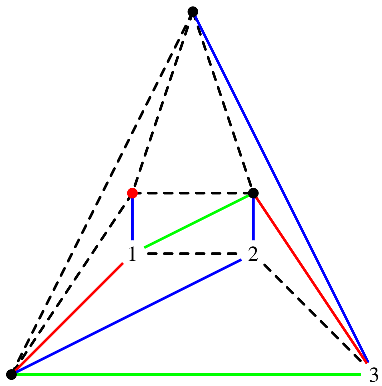
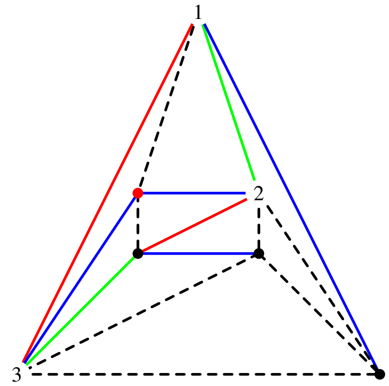


(1) 4a (equatorial), 4d, 4b, 4d. $4\zeta - 2$



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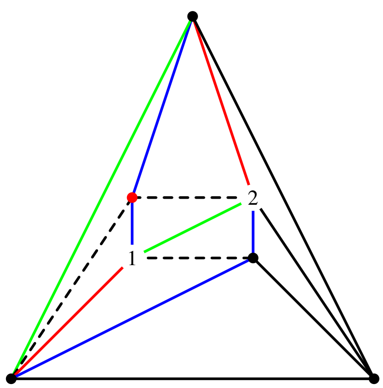
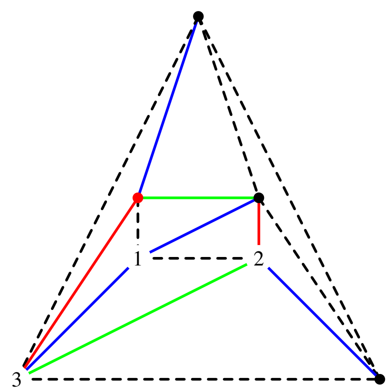
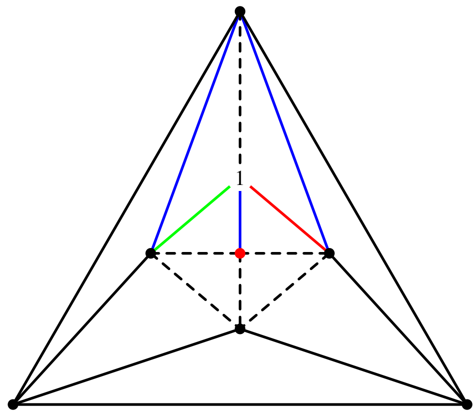


(3) 4b (equatorial), 4d, contradiction (not an allowable degree 5 vertex)

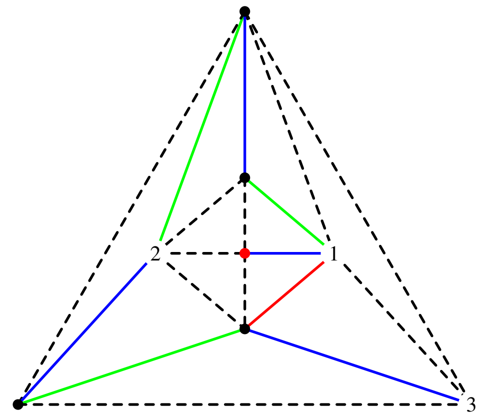


(4) 4d (equatorial), 4b, 4d, 5d. $4\zeta - 2$

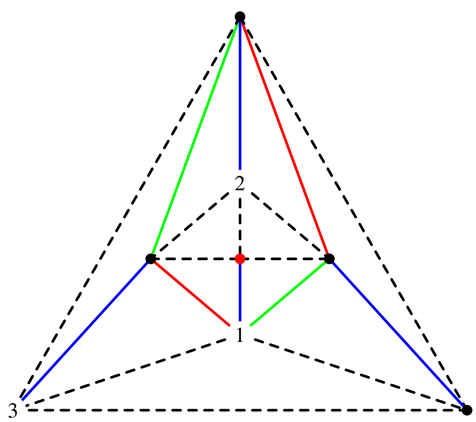
FIGURE 3.3. Cases 1 - 4 of the pentagonal dipyramid.



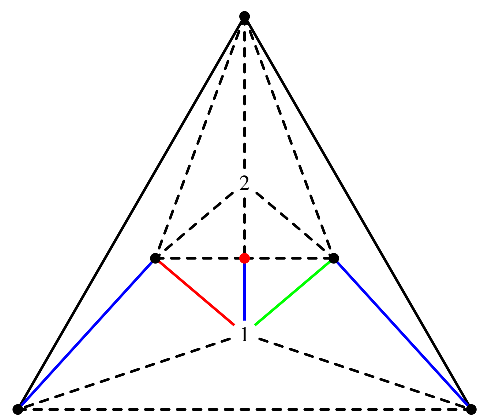
(1) 4a (orientation 1), 4d (contradiction; no such vertex partition)



(2) 4a (orientation 2), 5d, 5cr, 4a. $4\zeta - 1$

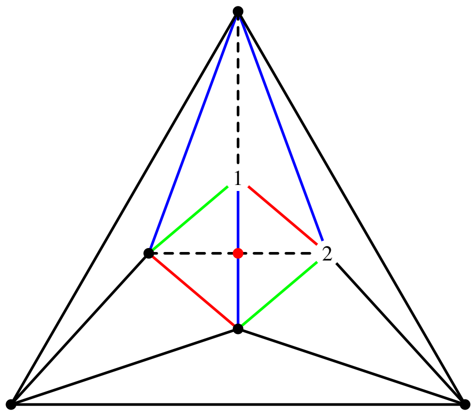


(3) 4a (orientation 3), 5d, 4a, 4a. $4\zeta - 2$

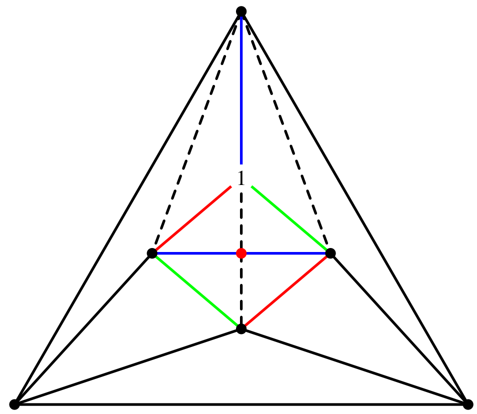


(4) 4a (orientation 3), 5d, 4e (contradiction; no such vertex partition)

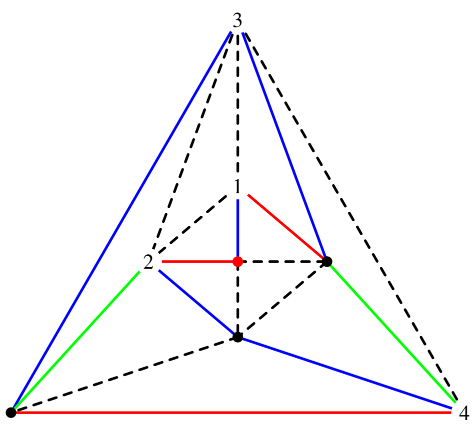
FIGURE 3.4. Cases 1-4 of the snub disphenoid



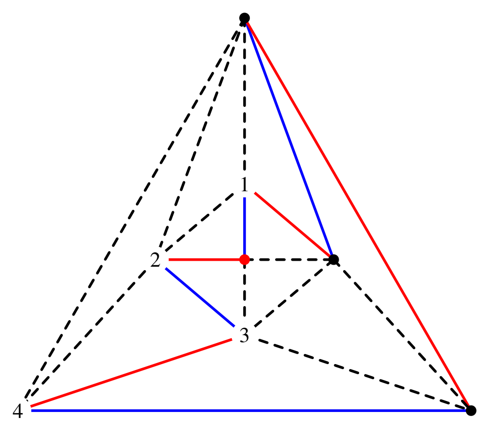
(5) 4b (orientation 1), 4d, contradiction (not an allowable degree 5 vertex)



(6) 4b (orientation 2), 4d (contradiction; no such vertex partition)

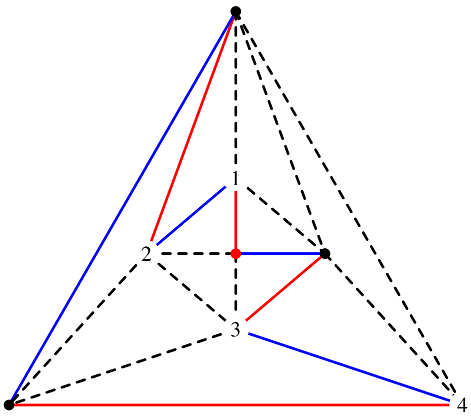


(7) 4cl (orientation 1), 4cl, 5d, 5b, 4d. $5\zeta - 3$

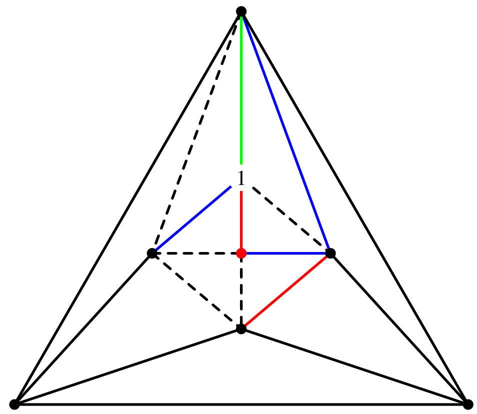


(8) 4cl (orientation 1), 4cl, 5cl, 5cl, 4cl. 4ζ

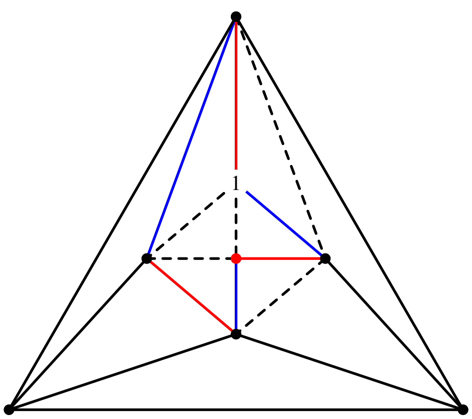
FIGURE 3.4. Cases 5-8 of the snub disphenoid



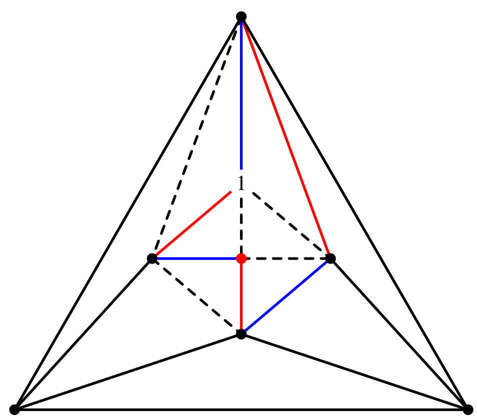
(9) 4cl (orientation 2), 4cl, 5cl, 5cl, 4cl. 4ζ



(10) 4cl (orientation 2), 4d (contradiction; no such vertex partition)

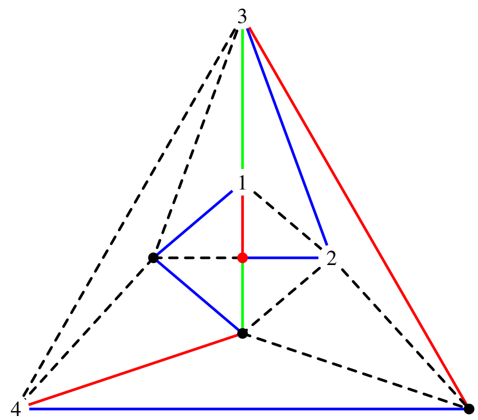


(11) 4cl (orientation 3), 4cl (contradiction; no such vertex partition)

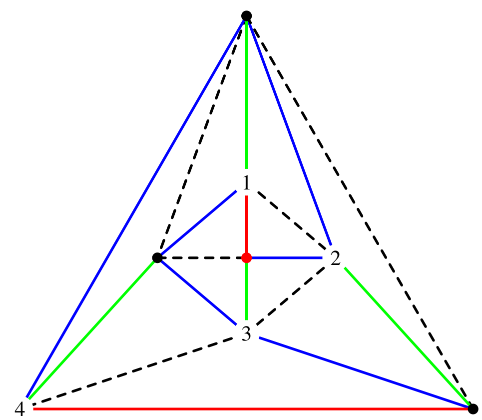


(12) 4cl (orientation 4), 4cl, (contradiction; no such vertex partition)

FIGURE 3.4. Cases 9-12 of the snub disphenoid

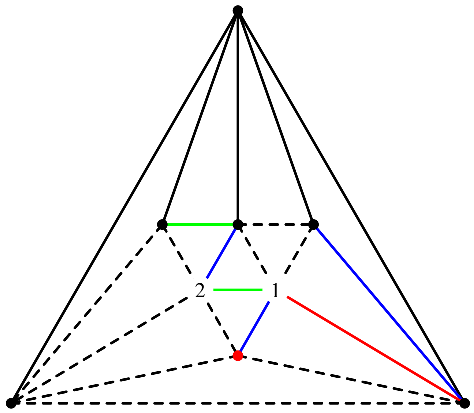


(13) <u>4d</u>, <u>4d</u>, <u>5b</u>, 5d, 4cl. $5\zeta - 3$

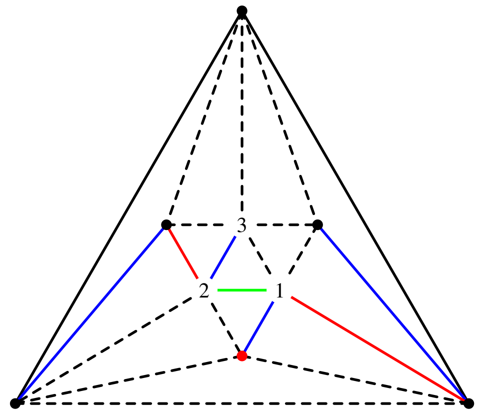


(14) <u>4d</u>, <u>4d</u>, <u>5er</u>, 5er, 4d. $6\zeta - 2$

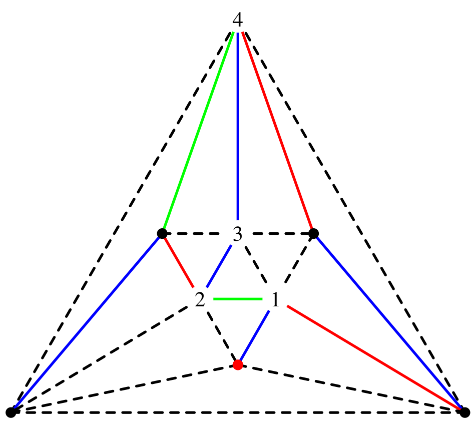
FIGURE 3.3. Cases 13-14 of the snub disphenoid



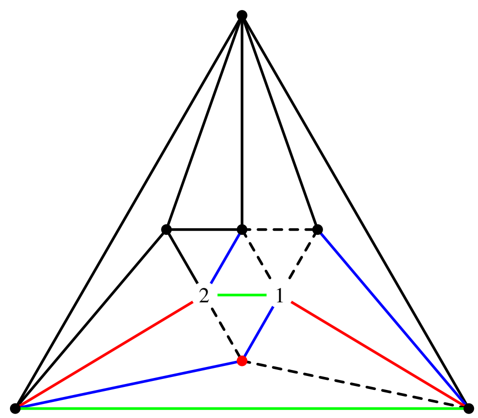
(1) 4a, 5d, 5cr (contradiction; no such vertex partition)



(2) 4a, 5d, 5d, 5a (contradiction; no such vertex partition)

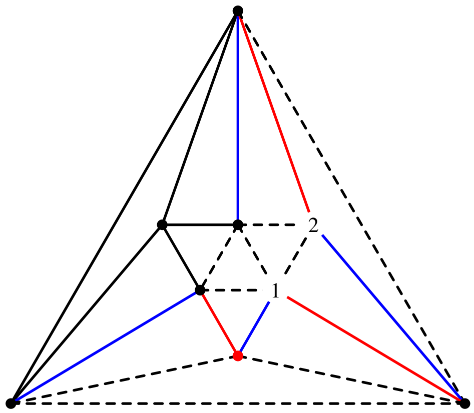


(3) 4a, 5d, 5d, 5b, 5d. $5\zeta - 3$

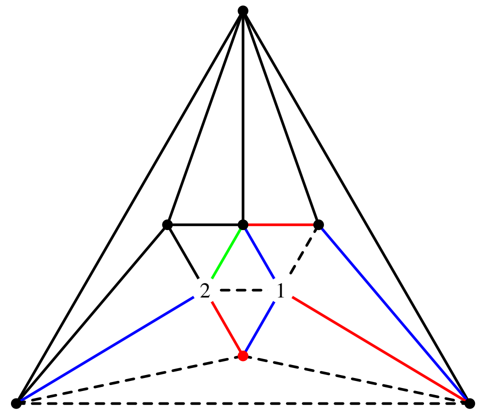


(4) 4b, 5d, contradiction (not an allowable degree 5 vertex)

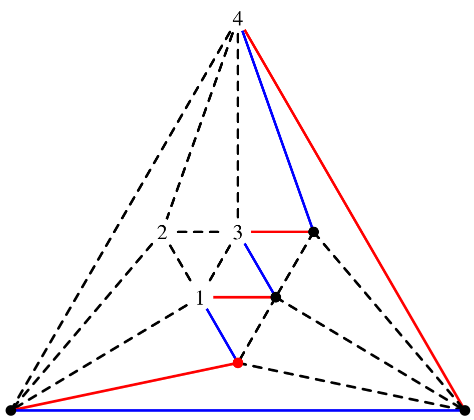
FIGURE 3.4. Cases 1-4 of the triaugmented triangular prism



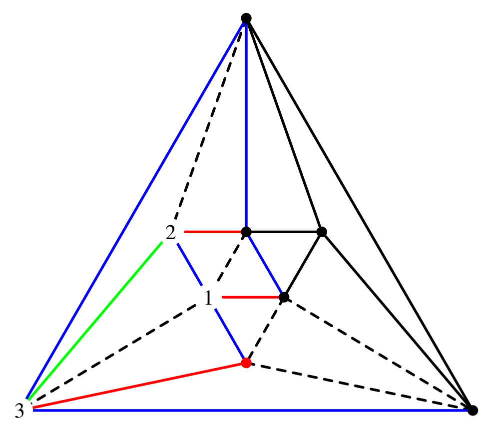
(5) 4cl (orientation 1), 5cl, 4cl (contradiction; no such vertex partition)



(6) 4cl (orientation 1), 5el, contradiction (not an allowable degree 5 vertex)

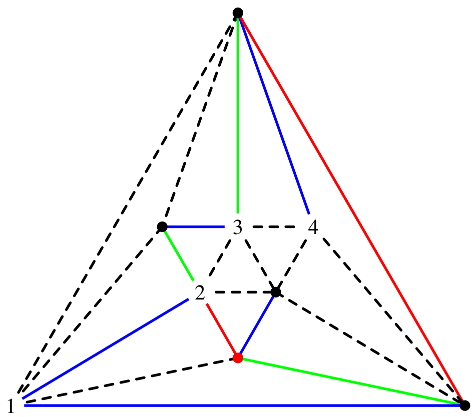


(7) 4cl (orientation 2), 5cl, 4e, 5cl, 5cl. 4ζ

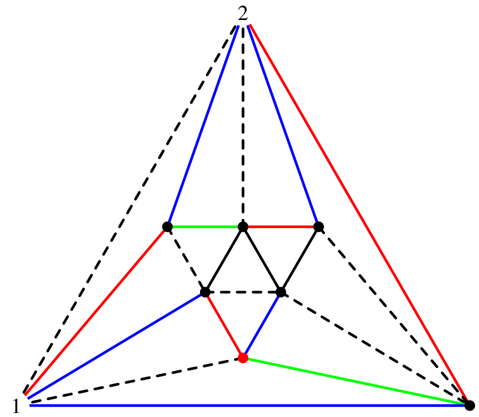


(8) 4cl (orientation 2), 5el, 4d, contradiction (not an allowable degree 5 vertex)

FIGURE 3.4. Cases 5-8 of the triaugmented triangular prism

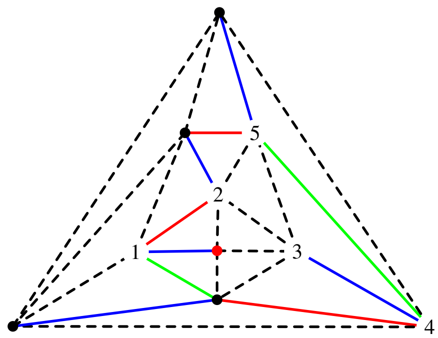


(9) 4d, 5b, 5d, 5cr, 4a. $5\zeta - 2$

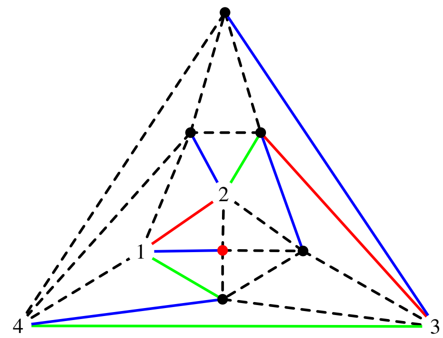


(10) 4d, 5el, 5el (contradiction; no such vertex partition)

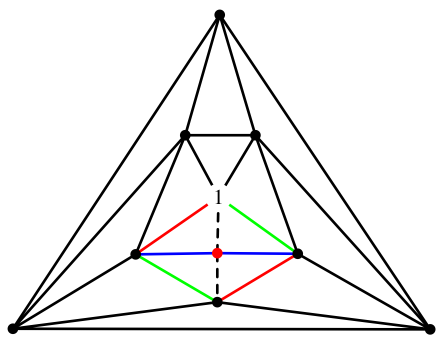
FIGURE 3.4. Cases 9-10 of the triaugmented triangular prism



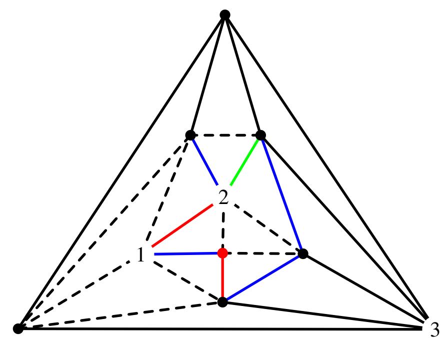
(1) 4a, 5d, 5cl, 5a, 5d, 5d. $5\zeta - 3$



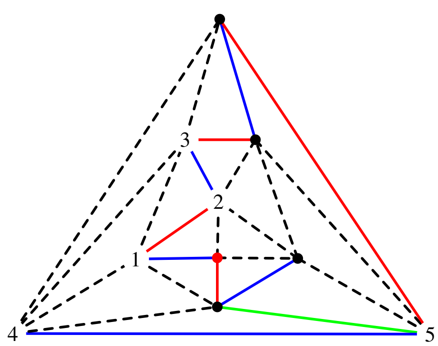
(2) 4a, 5d, 5d, 5d, 5cr. $5\zeta - 2$



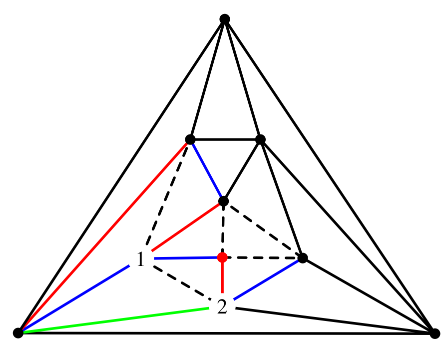
(3) 4b, contradiction (not an allowable degree 5 vertex)



(4) 4cl, 5cl, 5d, contradiction (not an allowable degree 5 vertex)

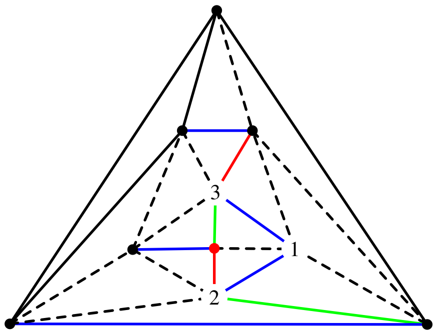


(5) 4cl, 5cl, 5cl, 5cl, 5a, 5d. $5\zeta - 4$

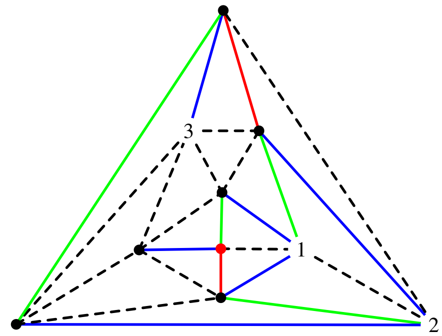


(6) 4cl, 5el, contradiction (not an allowable degree 5 vertex)

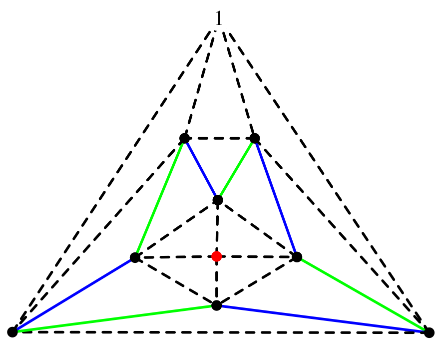
FIGURE 3.5. Cases 1-6 of the gyroelongated square dipyramid



(7) 4d, 5b, 5d, 5d (contradiction; no such vertex partition)



(8) 4d, 5er, 5er, 5a. $6\zeta - 2$



(9) 4e, 4e. 4ζ

FIGURE 3.5. Cases 7-9 of the gyroelongated square dipyramid