When is a linear functional multiplicative?

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Abstract. We discuss the problem of how to recognize a multiplicative functional from its behavior on a small subset of the algebra, by its local behavior, or by other properties.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{M}(\mathcal{A})$ be the set of all (nonzero) linear and multiplicative functionals on $\mathcal{A}$, that is, the set of functionals that preserve both the linear and the multiplicative structures of the algebra. Let us consider few examples:

- for the algebra $C[0,1]$ of all continuous functions defined on the unit segment $[0,1]$ any linear multiplicative functional is equal to δ_x - the evaluation at a point $x \in [0,1]$ [**11**],
- for the algebra $H^\infty(\mathbb{D})$ of all bounded analytic functions defined on the open unit disc $\mathbb{D}$, the functionals δ_z, $z \in \mathbb{D}$ are obviously linear and multiplicative, but there is also a large set of other linear multiplicative functionals related to the boundary of $\mathbb{D}$ [**12**],
- for the group algebra $L^1(\mathbb{T})$ of the functions integrable on the unit circle $\mathbb{T}$, with the convolution multiplication, any linear multiplicative functional is given by an element of the dual group $\mathbb{Z}$ of all integers [**32**],
- for the operator algebra $B(X)$ of all bounded linear maps on a given Banach space X the set $\mathfrak{M}(B(X))$ is empty for many classical Banach spaces including the Hilbert spaces, though for some Banach spaces $\mathfrak{M}(B(X))$ may be surprisingly large [**25**].

In this note we discuss the problem of how to recognize a multiplicative functional from its behavior on a small subset of the algebra, by its local behavior, or by other properties.

2. Gleason-Kahane-Żelazko Theorem

Let us first consider the algebra $C_{\mathbb{R}}[0,1]$ of all continuous real-valued functions defined on $[0,1]$, and put

$$F(f) = \int_0^1 f dm,\ f \in C_{\mathbb{R}}[0,1],$$

where m is the Lebesgue measure. Any function f in the kernel of F must be nonnegative on a part of the segment and nonpositive on the other part, consequently

$f(x)=0$ for at least one point x in the segment; of course the point x depends on the function f. We can extend this observation: if $F(f)=a$, then f must be in part above a and in part below a consequently $f(x)=a$ for some x in $[0,1]$. So for any fixed $f \in C_{\mathbb{R}}[0,1]$ the functional F coincides with one of the multiplicative functionals δ_x; however F is obviously not a multiplicative functional itself. This is a fairly trivial and non-surprising observation, still one may ask for a similar example in the complex case. The previous formula no longer works - the function $f(t)=\exp(2\pi i t)$ does not vanish but has a zero integral. The surprise is that in the complex case the opposite is true:

Proposition 2.1. For any linear functional $F: C_{\mathbb{C}}[0,1] \to \mathbb{C}$ we have

$$(\forall f \in \ker F \; \exists x \in [0,1] \quad f(x)=0) \quad \Rightarrow \quad (\exists x \in [0,1] \; \forall f \in \ker F \quad f(x)=0).$$

Equivalently,

Proposition 2.2. Let $F: C_{\mathbb{C}}[0,1] \to \mathbb{C}$ be a linear functional. Assume that for any $f \in C_{\mathbb{C}}[0,1]$ there is an $x \in [0,1]$ such that $F(f)=f(x)$. Then $F=\delta_x$, for some $x \in [0,1]$.

The above propositions represent a special case of the Gleason-Kahane-Żelazko theorem [**13**], [**22**], [**31**].

Theorem 2.3 (Gleason-Kahane-Żelazko). *Let $\mathcal{A}$ be a complex Banach algebra with a unit e and let F be a linear functional on $\mathcal{A}$. Assume that*

$$F(f) \neq 0, \text{ for } f \in \mathcal{A}^{-1}.$$

Then $F/F(e)$ is multiplicative.

Here we denote by $\mathcal{A}^{-1}$ the set of invertible elements of the algebra $\mathcal{A}$.

We present a proof of the Proposition, the same idea may also be applied in the general case though the presentation would have to be much more technical. The method is different than the original proofs by Gleason and by Kahane and Żelazko and can be applied to subspaces of higher codimension.

Proof. We first need to move $[0,1]$ away from zero; that is, in place of the algebra $C_{\mathbb{C}}[0,1]$ we shall consider the (isometrically isomorphic) algebra $C_{\mathbb{C}}[1,2]$ of functions on the segment $[1,2]$. Assume that $F: C_{\mathbb{C}}[1,2] \to \mathbb{C}$ is a linear functional whose kernel is contained in the set of noninvertible elements of $C_{\mathbb{C}}[1,2]$, in particular $F(\mathbf{1}) \neq 0$, so we may normalize F and assume that $F(\mathbf{1})=1$. Notice that the set of noninvertible elements is closed in $C_{\mathbb{C}}[1,2]$ so $\ker F$ is not dense and consequently F is continuous [**27**].

For any complex number λ, the function $f(t)=t^{\lambda}$ is an invertible element of $C_{\mathbb{C}}[1,2]$, so $F(t^{\lambda}) \neq 0$ and we may define an entire function $\varphi: \mathbb{C} \to \mathbb{C}$ by

$$\varphi(\lambda)=F(t^{\lambda+1})/F(t^{\lambda}).$$

We have $t^{\lambda+1}-\varphi(\lambda)t^{\lambda} \in \ker F$ so $t^{\lambda+1}-\varphi(\lambda)t^{\lambda}$ must vanish at some point of the segment $[1,2]$, however the only zero of $t^{\lambda+1}-\varphi(\lambda)t^{\lambda}=t^{\lambda}(t-\varphi(\lambda))$, other than at the origin, is at the point $\varphi(\lambda)$, hence $\varphi(\lambda) \in [1,2]$. Consequently φ is a bounded entire function and by the Liouville Theorem it is constant, $\varphi(\lambda) \equiv c \in [1,2]$. We get

$$\{t^n(t-c): n \in \mathbb{N}\} \subset \ker F,$$

and by the Stone-Weierstrass Theorem, the closure of $\{t^n (t-c) : n \in \mathbb{N}\}$ is equal to $\ker \delta_c = \{f \in C_{\mathbb{C}}[1,2] : f(c) = 0\}$, so $F = \delta_c$. □

3. Generalizations of the Gleason-Kahane-Żelazko Theorem

There are several obvious questions concerning the possible extensions or generalizations and the depth of the G-K-Ż Theorem; for example:

1. Is the result purely algebraic? That is: if $\mathcal{A}$ is a complex algebra with no topology imposed and F a linear functional on $\mathcal{A}$ that does not vanish on the set $\mathcal{A}^{-1}$ of invertible elements of $\mathcal{A}$, and $F(e) = 1$, is F a linear-multiplicative functional?
2. Is the result local? That is: if any function f from a linear subspace M of $C_{\mathbb{C}}[0,1]$ (or of a general Banach algebra $\mathcal{A}$) is zero at some point of $[0,1]$ (of $\mathfrak{M}(\mathcal{A})$ in general) does it follow that all the functions from M have a common zero?
3. Can we replace the assumption
$$F(a) \neq 0 \qquad \text{for } a \in \mathcal{A}^{-1}$$
in the G-K-Ż theorem, by an assumption that F does not vanish on a smaller set?
4. Can the result be extended to maps between two Banach (topological) algebras? That is: if $T : \mathcal{A} \to \mathcal{B}$ is a linear map from an algebra $\mathcal{A}$ into (onto) an algebra $\mathcal{B}$ that preserves the set of invertible elements, does it follow that T is multiplicative, that is, preserves the multiplicative structure?

The answer to some of the questions is an easy *NO*, in some cases the answer is *YES*, or *YES if ...* which usually leads to deep and interesting results, some problems are still open. We start again with simple examples.

Example 3.1. Let $\mathcal{A} = P[z]$ be the algebra of all complex polynomials and let F be any linear functional on $\mathcal{A}$ with $F(\mathbf{1}) = 1$. Any polynomial p in the kernel of F is nonconstant so is equal to zero at some point of the plane, but the point may depend on p and F need not be multiplicative.

The algebra $P[z]$ is countably dimensional so it can not be made into a *complete* topological algebra, however with more effort we could construct a complete topological algebra with the same property, e.g.,
$$\mathcal{A} = \left\{ f \in Hol(\mathbb{C}) : \|f\|_k = \sup_{z\in\mathbb{C}} \left| \frac{f(z)}{e^{|z|/k}} \right| < \infty \text{ for } k \in \mathbb{N} \right\}.$$
The crucial property of $\mathcal{A}$ that facilitates the construction is the abundance of elements with unbounded spectrum. It turns out that the G-K-Ż Theorem remains valid in complete topological algebras if the spectrum off all the elements is bounded [**18**], [**26**].

The next example shows that the G-K-Ż theorem is not local.

Example 3.2. Let $\mathcal{A}$ be equal to the algebra $C_{\mathbb{C}}\left(\overline{\mathbb{D}}\right)$ of all continuous complex valued functions on the closed unit disc $\overline{\mathbb{D}}$, or to the disc algebra $A(\mathbb{D}) \stackrel{df}{=} C_{\mathbb{C}}\left(\overline{\mathbb{D}}\right) \cap H^\infty(\mathbb{D})$. Fix $z_1 \neq z_2$ in the open unit disc, and let
$$B_j(w) = \frac{w - z_j}{1 - \overline{z_j} w}, \qquad \text{for } j = 1, 2$$

be the corresponding Blaschke products. The maps B_j are automorphisms of the disc, they map the boundary of $\mathbb{D}$ onto itself, $B_j(z_j) = 0$ and $B_j(w) \neq 0$ for all other points in the disc. Let M be the two dimensional subspace of $\mathcal{A}$ spanned by B_1 and B_2. Consider a typical element $\alpha B_1 + \beta B_2$ of M. If $|\alpha| < |\beta|$ then $|\alpha B_1| < |\beta B_2|$ on the boundary of the disc so by the Rouché Theorem [**9**] $\alpha B_1 + \beta B_2$ has the same number of zeros in $\mathbb{D}$ as βB_2, that is, one zero. Similarly, if $|\beta| < |\alpha|$, the function $\alpha B_1 + \beta B_2$ has a zero in $\mathbb{D}$. Since a uniform limit of a sequence of functions with zeros in $\overline{\mathbb{D}}$ must have a zero in $\overline{\mathbb{D}}$, $\alpha B_1 + \beta B_2$ must also have a zero if $|\beta| = |\alpha|$. Clearly, there is no common zero for the subspace M.

A similar construction can be done in almost any uniform algebra. For example, any uniform algebra that contains a function with an uncountable spectrum contains an infinite dimensional subspace contained in the set of noninvertible elements but not contained in the kernel of any multiplicative functional. On the other hand the following are open problems.

Open Problem 1: Let M be a finite codimensional subspace of a commutative unital complex Banach algebra $\mathcal{A}$. Assume that M contains only noninvertible elements of $\mathcal{A}$. Is M contained in a kernel of a multiplicative functional?

Open Problem 2: Let $\mathcal{A} = A(\mathbb{D})$ be the disc algebra and let M be a codimension two subspace of $\mathcal{A}$. Assume that for any $f \in M$ there is a point z in the disc such that $f(z) = 0$. Does it follow that all the functions from M have a common zero in the disc?

The second problem is a special case of the first one. However, based on the general theory of Banach algebras, the positive answer to the second problem would imply the same answer for a large class of Banach algebras. It is not difficult to prove [**17**] that the second problem is equivalent to the following one that does not refer to any topology on $\mathcal{A}$:

Open Problem 3: Let $\mathcal{A} = P[z]$ be the algebra of all complex polynomials and let M be a codimension two subspace of $\mathcal{A}$. Assume that for any $f \in M$ there is a point z in the *unit* disc such that $f(z) = 0$. Does it follow that all the functions from M have a common zero in the disc?

We do not know any commutative complex Banach algebra that would violate the property described in Problem 1; we know that some algebras have this and related properties [**16**], [**19**]:

THEOREM 3.3. *Let X be a compact Hausdorff space, let $C(X)$ be the algebra of all continuous complex valued functions on X, and let M be a finite codimensional subspace of $C(X)$ consisting of noninvertible elements. Then there is an $x \in X$ such that $M \subset \ker \delta_x = \{f \in C(X) : f(x) = 0\}$.*

As we already mentioned before the above Theorem is not valid for subspaces of $C(X)$ of infinite codimension.

THEOREM 3.4. *Let $\mathcal{A} = A(\mathbb{D})$ be the disc algebra, or any Banach algebra with one generator, and let M be an $n < \infty$ codimensional subspace of A. Assume that for any $f \in M$ there are n distinct points $z_1, ..., z_n$ in the disc such that $f(z_1) = ... = f(z_n) = 0$. Then the functions from M have n common zeros in the disc, so $M = \bigcap_{j=1}^{n} \ker \delta_{z_j}$.*

The proof of the last result is based on the same general idea as the proof we presented earlier for Proposition 1 but, as the next example shows, the result does not apply to general Banach algebras.

EXAMPLE 3.5. Put $B = \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leq 1\right\}$ and let $A(B)$ be the algebra of functions continuous on B and analytic on $intB$. Let M be a subspace of $\{f \in B : f(0,0) = 0\}$. Since zeros of analytic functions are not isolated, any function f in M has infinitely many zeros in the ball B; however the origin is, in general, the only common zero for M.

The assumption in the G-K-Ż Theorem is that the functional does not vanish on the set $\mathcal{A}^{-1}$ of noninvertible elements, or if we prefer to think in terms of the subspaces, that the subspace is contained in $\mathcal{A} - \mathcal{A}^{-1}$. An examination of the original proof of the G-K-Ż Theorem actually shows that it is enough to assume that the functional does not vanish on $\exp \mathcal{A} = \{\exp f : f \in \mathcal{A}\}$, a subset of $\mathcal{A}^{-1}$. In 1987 Arens asked [**1**] if the exponential function could be replaced by other entire functions. Partial answers were obtained by Badea [**4**] and by Berntzen and Sołtysiak [**5**], a complete answer was given very recently in [**15**] (see also [**20**]):

THEOREM 3.6. *Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let φ be a nonconstant entire function, and let F be a linear functional on $\mathcal{A}$. If the function $F \circ \varphi : \mathcal{A} \to \mathbb{C}$ is nonsurjective then $F = 0$, or $F/F(e)$ is multiplicative.*

It is not known whether there is a common generalization of Theorems 3.6 and 3.3 or of Theorems 3.6 and 3.4.

4. Invertibility preserving maps between Banach algebras

The G-K-Ż Theorem states that linear maps from a complex Banach algebra into the complex field that map invertible elements of the algebra into invertible elements of the field, and the unit of the algebra onto the unit of the field, are automatically multiplicative. Is it true in general that a linear map between two Banach algebras that preserves the set of invertible elements, and the unit, must be multiplicative? As before we begin with easy counterexamples.

EXAMPLE 4.1. Let E be a Banach space and $\times, *$ be two different multiplications on E such that both $(E, \times)$ and $(E, *)$ are commutative radical Banach algebras. Such multiplications can be constructed for any nontrivial Banach space E. To have a specific example we may put $E = L^1[0,1]$, and

$$f \times g \equiv 0, \text{ and } f * g(t) = \int_0^1 f(s)\, g(t-s)\, ds, \quad \text{for } f, g \in L^1[0,1].$$

Let $\hat{E}$ be a direct sum $E \bigoplus \mathbb{C}\mathbf{e}$ of E and a one dimensional subspace spanned by a new element $\mathbf{e}$. We can extend the multiplications $\times, *$ from E to $\hat{E}$ by

$$\begin{aligned} (f + \alpha\mathbf{e}) \times (g + \beta\mathbf{e}) &= \alpha g + \beta f + \alpha\beta\mathbf{e}, \\ (f + \alpha\mathbf{e}) * (g + \beta\mathbf{e}) &= f * g + \alpha g + \beta f + \alpha\beta\mathbf{e}, \quad \text{for } f, g \in L^1[0,1]. \end{aligned}$$

$\mathcal{A} \stackrel{df}{=} \left(\hat{E}, \times\right)$ and $\mathcal{B} \stackrel{df}{=} \left(\hat{E}, *\right)$ are both unital Banach algebras; an element $f + \alpha\mathbf{e}$ is invertible in $\mathcal{A}$ if and only if it is invertible in $\mathcal{B}$ and if and only if $\alpha \neq 0$. Hence, the identity map preserves invertibility but does not preserve the multiplication.

Example 4.2. Let $\mathcal{A}$ be the algebra of upper triangular 2×2 matrices, and $\mathcal{B}=M_2(\mathbb{C})$ the algebra of all 2×2 matrices. Define $T:\mathcal{A}\to\mathcal{B}$ by

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \overset{T}{\longmapsto} \begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}.$$

The map T preserves the determinant so it preserves invertibility but

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2\right) = T\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \text{ while}$$

$$\left(T\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\right)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

so it is not multiplicative.

The algebras and the maps from the previous two examples are very different - in the first example the algebras are commutative and the map is surjective, in the second example the algebras are noncommutative and the map is nonsurjective. Still there is a crucial similarity - in both examples the set of invertible elements forms an unusually large part of the algebra. Let us explain the idea. Consider the algebra $C[0,1]$ and its element f. In order to check if f is invertible we must check if $f(x)\neq 0$ for all $x\in[0,1]$; that is, we need to check infinitely many independent linear conditions. On the other hand if $f+\alpha\mathbf{e}$ is an element of one of the algebras from Example 4.1 we need to check only one condition: $\alpha\neq 0$, to conclude that $f+\alpha\mathbf{e}$ is invertible. Similarly, if $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is an element of one of the algebras from the last example, we only need to check if the determinant of $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is nonzero to conclude invertibility - again just one condition (though this time a nonlinear condition). This is a basic idea - if there are so many invertible elements then there should be a lot of linear maps that preserve invertibility but the number of maps that preserve multiplication may be much smaller. There is yet another important aspect - the algebra $C[0,1]$ has a large number of multiplicative functionals - the zero function is the only one that is contained in the kernel of all such functionals; on the other hand the algebras from Example 4.1 have only one multiplicative functional and the algebra $M_2(\mathbb{C})$ has none.

The above are indeed exactly the right ideas in the commutative case: we can replace the complex field by an algebra $\mathcal{B}$, in the G-K-Ż Theorem, if and only if $\mathcal{B}$ is semisimple. For the noncommutative algebras the situation is much more complicated and the main problem remains open.

We discuss the commutative case first.

Definition 4.3. A commutative Banach algebra $\mathcal{B}$ is semisimple if for any nonzero element b of $\mathcal{B}$ there is a linear and multiplicative functional G such that $G(b)\neq 0$.

Equivalently, a commutative Banach algebra $\mathcal{B}$ is semisimple if and only if 0 is the only element whose spectrum is equal to $\{0\}$. Any commutative semisimple Banach algebra can be identified with an algebra of continuous functions with the usual pointwise multiplication. The map

$$\mathcal{B}\ni b\longmapsto \hat{b}\in C(\mathfrak{M}(\mathcal{B})) : \hat{b}(G)\overset{df}{=} G(b)$$

is an isomorphism from B onto a subalgebra of the algebra of continuous functions on $\mathfrak{M}(\mathcal{B})$.

THEOREM 4.4. *If $T : \mathcal{A} \to \mathcal{B}$ is a linear map from a complex Banach algebra into a commutative, semisimple Banach algebra such that $T(e_{\mathcal{A}}) = e_{\mathcal{B}}$ and*

$$T(a) \in \mathcal{B}^{-1} \qquad \text{for } a \in \mathcal{A}^{-1},$$

then T is multiplicative.

PROOF. The result immediately follows from the G-K-Ż Theorem. Let $a_1, a_2 \in \mathcal{A}$ and $G \in \mathfrak{M}(\mathcal{B})$. The functional $G \circ T$ satisfies the assumptions of the G-K-Ż Theorem, so it is multiplicative and we have

$$G(T(a_1a_2) - T(a_1)T(a_2)) = G \circ T(a_1a_2) - G \circ T(a_1) \cdot G \circ T(a_2) = 0.$$

Since $\mathcal{B}$ is semisimple it follows that $T(a_1a_2) - T(a_1)T(a_2) = 0$, so T preserves the multiplication. □

In the noncommutative case the situation is much more complicated and perhaps more interesting. First, the last definition of a semisimple Banach algebra does not apply to noncommutative algebras - noncommutative algebras usually have very few multiplicative functionals. A commutative Banach algebra is semisimple if and only if it is isomorphic with a subalgebra of an algebra of continuous functions, so if we had used the same definition in general we would have forced the algebra to be commutative.

The simplest commutative Banach algebra is the scalar field, and any semisimple commutative Banach algebra is a subalgebra of a product of the one dimensional algebras. The simplest noncommutative Banach algebra is the algebra M_n of $n \times n$ matrices, with $n > 1$, or more generally the algebra $B(X)$ of all bounded linear maps on a Banach space X. It is natural to define a semisimple noncommutative algebra as a subalgebra of a product of these building blocks.

DEFINITION 4.5. Let $\mathcal{B}$ be a Banach algebra. A representation of $\mathcal{B}$ is an isomorphism φ from $\mathcal{B}$ into an algebra $B(X)$ of all bounded linear maps on a Banach space X. A representation φ is irreducible if there is no nontrivial subspace X_0 of X such that $\varphi(b)(X_0) \subset X_0$ for all b in $\mathcal{B}$.

Notice that if $\varphi : \mathcal{B} \to B(X)$ is not irreducible then $\psi : \mathcal{B} \to B(X_0)$ defined by $\psi(b) = \varphi(b)_{|X_0}$ is a representation of $\mathcal{B}$ on a smaller space.

DEFINITION 4.6. A Banach algebra $\mathcal{B}$ is called semisimple if for any nonzero element b of $\mathcal{B}$ there is an irreducible representation φ such that $\varphi(b) \neq 0$.

The algebras M_n of $n \times n$ matrices are all semisimple.

One could expect that the G-K-Ż Theorem can be extended to cover maps into arbitrary semisimple Banach algebras. We however already know that it is not true - the algebra $M_2(\mathbb{C})$ from Example 4.2 is semisimple. On the other hand the range of the map from Example 4.2 is not equal to $M_2(\mathbb{C})$ but to a proper and *not* semisimple subalgebra. So we need to assume that the map is surjective, after all, the functional in the G-K-Ż Theorem is automatically surjective. Still, as the next example shows, there is yet another difficulty.

EXAMPLE 4.7. Fix $n > 1$ and let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices. The function $A \longmapsto A^t$ of taking the transpose of a given matrix preserves invertibility but does not preserve multiplication. One could argue that the map preserves the multiplication - it simply reverses the order:

$$(AB)^t = B^t A^t, \quad \text{for } A, B \in M_2(\mathbb{C}).$$

However, it is easy to construct a map that preserves invertibility but neither preserves nor reverses the multiplication, for example

$$M_2(\mathbb{C}) \bigoplus M_2(\mathbb{C}) \ni (A, B) \longmapsto \left(A, B^t\right) \in M_2(\mathbb{C}) \bigoplus M_2(\mathbb{C}).$$

Notice that both maps preserve the operation of taking the square and consequently multiplication on commutive subalgebras.

Definition 4.8. A linear map T between algebras $\mathcal{A}$ and $\mathcal{B}$ is called a Jordan morphism if

$$T\left(a^2\right) = (T(a))^2, \quad \text{for } a \in \mathcal{A}.$$

Conjecture: Any linear map from a complex Banach algebra *onto* a semisimple Banach algebra that preserves invertibility and the unit element is a Jordan morphism.

The Conjecture has a long history [**23**] and has inspired a large number of papers on the more general "preserver problem".

Preserver Problem: Let T be a map between two Banach algebras and assume that T preserves a given class of elements of the algebras (e.g. the invertible elements, the nilpotents, the idempotents, the finite dimensional elements, the algebraic elements, the elements with finite spectrum, normal elements, etc.). Does it follow that T is a Jordan morphism (or that it preserves some other properties)?

We refer interested readers to an extensive survey given in [**28**] (see also [**3**], [**6**], [**7**], [**8**], [**30**]). We state just three results from [**10**], [**2**] and [**30**], spanning over a century - they provide partial answers to the Conjecture.

Theorem 4.9 (Frobenius, 1897). *Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear transformation that maps the unit matrix onto itself an such that*

$$\det(T(S)) = \det(S), \quad \textit{for } S \in M_n(\mathbb{C}),$$

then T is a Jordan morphism.

Theorem 4.10 (Aupetit, 1979). *Let T be a linear map from a complex Banach algebra $\mathcal{A}$* onto *a complex Banach algebra $\mathcal{B}$. Assume that T preserves the set of invertible elements and the unit elements, assume also that $\mathcal{B}$ has a separating family of finite dimensional irreducible representations. Then T is a Jordan morphism.*

Theorem 4.11 (Sourour, 1996). *If a linear* surjective *map $T : B(X) \to B(Y)$ preserves invertibility and $T(Id_X) = T(Id_Y)$ then T is a Jordan isomorphism.*

5. Almost multiplicative functionals and small deformations

We can not discuss in this short note all problems concerning multiplicative functionals and related to the G-K-Ż Theorem. Concluding let us just mention very briefly two more.

By the G-K-Ż theorem for any linear functional F the condition

$$F(e) = 1 \text{ and } F(a) \neq 0 \qquad \text{for } a \in \mathcal{A}^{-1}$$

or equivalently

$$F(f) \in \sigma(f), \qquad \text{for } f \in \mathcal{A},$$

implies that F is multiplicative, where by $\sigma(f)$ we denote the spectrum of f. S. Kowalski and Z. Słodkowski [**24**] proved a surprising result that a condition very similar to the one above implies automatically also that F is linear.

THEOREM 5.1 (Kowalski-Slodkowski). *Let $\mathcal{A}$ be a complex Banach algebra with a unit e and let F be a functional on $\mathcal{A}$. Assume that $F(0) = 0$, and*

$$F(f) - F(g) \in \sigma(f - g), \qquad \textit{for } f, g \in \mathcal{A},$$

then F is linear and multiplicative.

Let F be a linear and multiplicative functional on a Banach algebra $\mathcal{A}$ and let Δ be a linear functional on $\mathcal{A}$ with $\|\Delta\| \le \varepsilon$. Put $F = G + \Delta$. We can easily check by direct computation that F is *δ-multiplicative,* that is,

$$|F(ab) - F(a)F(b)| \le \delta \|a\| \|b\|, \qquad \text{for } a, b \in \mathcal{A},$$

where $\delta = 3\varepsilon + \varepsilon^2$. Which means that a small deformation of a multiplicative functional is nearly multiplicative. Is the converse true? Is any δ-multiplicative functional close, in norm, to a multiplicative one? An algebra with this property is called *f-stable.* Not all algebras are f-stable. In 1986 B. E. Johnson [**21**] gave an example of a commutative Banach algebra that has δ-multiplicative functionals for any $\delta > 0$, but does not have any multiplicative functionals. More recently S. J. Sidney [**29**] gave an example of a uniform algebra that is not f-stable. Many classical algebras of analytic functions are f-stable [**14**] but for most algebras, for instance for the $H^\infty(\mathbb{D})$ algebra, the question is open. In view of the importance of the Corona Theorem for $H^\infty(\mathbb{D})$ it is particularly interesting to know whether $H^\infty(\mathbb{D})$ does not have an *almost corona,* that is, a set of almost multiplicative functionals far from the set of multiplicative functionals.

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