

Topological algebras in which all maximal two-sided ideals are closed

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ABSTRACT. We characterize unital topological algebras in which all maximal two-sided ideals are closed.

1. Introduction

In a unital Banach algebra all maximal two-sided ideals are obviously closed. The problem of characterizing other classes of topological algebras with the same property has been investigated by a number of authors. For example for a complex unital commutative Fréchet algebra that property is equivalent to being a Q -algebra ([5], Proposition 17); for a complex unital commutative complete locally m -convex algebra for which the set of all non-zero continuous linear multiplicative functionals is compact in the Gelfand topology our property is equivalent to any proper finitely generated ideal being non-dense ([16], Proposition 2). However there are also non- Q -algebras with that property ([8], p. 81). In this note we provide a characterization of that property in terms of compactness of the structure space and properties of finitely generated ideals. We also show that every proper finitely generated two-sided ideal in a complex unital locally m -pseudoconvex Fréchet algebra is contained in a closed maximal two-sided ideal of A if A is topologically almost commutative or $\text{comm}A \neq A$.

Unless otherwise stated the result is valid in both the real and the complex case.

2. Definitions and Notation

By a topological algebra we mean an algebra A that is also a topological vector space such that the multiplication operation is separately continuous.

We call a topological algebra A *locally pseudoconvex* if it has a base of neighborhoods of zero consisting of *balanced* and *pseudoconvex* sets, that is of sets U for which $\mu U \subset U$, whenever $|\mu| \leq 1$, and $U + U \subset \lambda U$ for some $\lambda \geq 2$. In such an algebra the topology can be defined by a family $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ -homogeneous seminorms with $k_\lambda \in (0, 1]$, where a seminorm p is called k -homogeneous if $p(\mu a) = |\mu|^k p(a)$ for any $a \in A$ and any scalar μ ([12], p. 4). If A is also metrizable then one can select a countable family $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ -homogeneous seminorms. A locally pseudoconvex algebra A is *locally m -pseudoconvex* if every seminorm p_λ in the family $\{p_\lambda : \lambda \in \Lambda\}$ is submultiplicative, is *locally convex*, if $k_\lambda = 1$ for each $\lambda \in \Lambda$, and *k -normed* if the topology of A can be define by a single k -homogeneous norm. A metrizable and complete algebra is called a *Fréchet algebra*.

For a unital ring A , $M(A)$ denotes the set of all maximal two-sided ideals in A . The space $M(A)$ is equipped with the hk -topology: $S \subset M(A)$ is closed if $S = H(K(S))$, where $K(S)$ is the intersection of all ideals in S and $H(I) = \{M \in M(A) : I \subset M\}$ for any two-sided ideal I of A . If A is also equipped with a compatible topology we consider a subset $m(A)$ of $M(A)$ consisting of closed ideals, we use small letters h and k to indicate operations H and K restricted to $m(A)$.

A unital topological algebra A is called *simplicial* (with respect to two-sided ideals) or *normal* in the sense of Michael ([10], p. 68) if every closed two-sided ideal of A is contained in a closed maximal two-sided ideal of A . For an algebra A let $\text{rad}A$ denote the topological radical ([3], p. 27) of A , $\langle a_1, \dots, a_n \rangle$ the two-sided ideal of A generated by $a_1, \dots, a_n \in A$, and $\text{comm}A$ the *commutator ideal* of A , that is the closure of the two-sided ideal of A generated by the set $\{ab - ba : a, b \in A\}$. A is an *topologically almost commutative algebra* if $A/\text{rad}A$ is commutative.

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3. Topological algebras, in which all maximal two-sided ideals are closed

The next theorem characterizes topological algebras with closed maximal ideals.

THEOREM 1. *Let A be a unital topological algebra. Then $M(A) = m(A)$ if and only if*

- (i) *every proper finitely generated two-sided ideal of A is contained in a closed maximal two-sided ideal of A , and*
(ii) *$m(A)$ is compact in the hk -topology.*

To prove the theorem we will need the compactness of $M(A)$, that fact is well known but typically it is presented only for certain classes of algebras or topological algebras (e.g. [7], p. 301, [11], p. 84, or [15]). Below we show that the standard arguments work for an arbitrary unital ring.

PROPOSITION 1. *Let A be a ring with a unit e , then the space $M(A)$ is compact in the hk -topology.*

PROOF. Assume $\mathcal{F} = \{F_j : j \in J\}$ is a non empty family of hk -closed subsets of $M(A)$ with $\bigcap_{j \in J} F_j = \emptyset$. We have

$$\begin{aligned} \emptyset &= \bigcap_{j \in J} F_j = \bigcap_{j \in J} HK(F_j) = \bigcap_{j \in J} \{M \in M(A) : K(F_j) \subset M\} \\ &= \{M \in M(A) : K(F_j) \subset M \text{ for all } j\} = \left\{ M \in M(A) : \bigcup_{j \in J} K(F_j) \subset M \right\} \end{aligned}$$

so the two-sided ideal generated by $\bigcup_{j \in J} K(F_j)$ is equal to the entire ring A . Hence there are $a_1, \dots, a_n \in \bigcup_{j \in J} K(F_j)$ and $b_1, \dots, b_n, b'_1, \dots, b'_n \in A$ with $\sum_{k=1}^n b'_k a_k b_k = e$; let $F_1, \dots, F_n \in \mathcal{F}$ be such that $a_j \in K(F_k)$ for $k = 1, 2, \dots, n$. It follows that the two-sided ideal generated by $\bigcup_{k=1}^n K(F_k)$ is equal to the entire ring A and consequently we can reverse the last sequence of equations:

$$\begin{aligned} \emptyset &= \left\{ M \in M(A) : \bigcup_{k=1}^n K(F_k) \subset M \right\} = \{M \in M(A) : K(F_k) \subset M \text{ for } k = 1, \dots, n\} \\ &= \bigcap_{k=1}^n \{M \in M(A) : K(F_k) \subset M\} = \bigcap_{k=1}^n HK(F_k) = \bigcap_{k=1}^n F_k, \end{aligned}$$

which shows that $M(A)$ is compact. \square

PROOF OF THEOREM 1. Assume $M(A) = m(A)$. In a unital algebra any proper two-sided ideal is contained in a maximal ideal. Hence, if $M(A) = m(A)$ any proper two-sided ideal is contained in a closed maximal two-sided ideal. Since $M(A)$ is compact in the hk -topology so is $m(A)$.

Assume now A is a unital topological algebra which satisfies the conditions of the Theorem and let $M_0 \in M(A)$. Put

$$Z(a) \stackrel{\text{df}}{=} \{M \in m(A) : a \in M\}, \quad \text{for } a \in M_0.$$

Since for any $a_1, \dots, a_n \in M_0$ we have $I \stackrel{\text{df}}{=} \langle a_1, \dots, a_n \rangle \subset M_0 \neq A$, by the first condition there is an ideal $M \in m(A)$ with $\{a_1, \dots, a_n\} \subset I \subset M$. Hence $M \in \bigcap_{k=1}^n Z(a_k)$, so $\{Z(a) : a \in M_0\}$ is a collection of hk -closed subsets of $m(A)$ having the finite intersection property. By (ii) there is a closed maximal two-sided ideal $M_1 \in \bigcap_{a \in M_0} Z(a)$. The ideal M_1 contains all elements of M_0 so $M_0 \subset M_1$, however M_0 is maximal so $M_0 = M_1 \in m(A)$. \square

COROLLARY 1. *Let A be a unital simplicial (with respect to two-sided ideals) algebra. If*

- (a) *every proper finitely generated two-sided ideal of A is not dense in A*
and
(b) *$m(A)$ is compact in the hk -topology,*
then $M(A) = m(A)$.

PROOF. Let I be a proper finitely generated two-sided ideal in A . Since $\text{cl}_A(I)$ is a closed two-sided ideal in a simplicial algebra A , by (a) there is $M \in m(A)$ with $\text{cl}_A(I) \subset M$. Hence A has the property (i) of Theorem 1 and consequently $M(A) = m(A)$. \square

EXAMPLE 1. *We construct a commutative unital norm algebra A such that both $M(A)$ and $m(A)$ are compact but different. Let A be the space of all C^∞ (real or complex valued) functions f defined on the real line \mathbb{R} such that both restrictions $f|_{(-\infty, 0]}$ and $f|_{[1, \infty)}$ are polynomials. We equip A with a submultiplicative norm: $\|f\| = \sup\{|f(t)| : 0 \leq t \leq 1\}$. Notice that $\|\cdot\|$ is indeed a norm rather than just a seminorm since if $f = 0$ on*

the unit segment, then all the derivatives of f at the points 0 and 1 are equal to zero and consequently $f = 0$ on the entire line.

Let I be a maximal proper ideal of A . Assume there is $f \in I$ which does not vanish on $[0, 1]$. If p_n is a sequence of polynomials convergent uniformly to $1/f$ on $[0, 1]$ then the sequence fp_n converges in A to the unit of that algebra, so I is not closed. On the other hand if all of the functions from I vanish somewhere on $[0, 1]$ then the sup norm closure of I in the Banach algebra $C[0, 1]$ is a proper ideal so $I \subset \{f : f(x_I) = 0\}$ for some x_I in $[0, 1]$. Hence $m(A)$ can be identified with the unit segment.

We show that the hk -topology on $[0, 1]$, denoted by τ_{hk} , coincides with the usual topology τ_{std} of that segment. Clearly any hk -closed set is closed in the standard topology so $\tau_{hk} \subset \tau_{std}$. On the other hand, since for any $0 \leq a < b \leq 1$ there is a function $f_0 \in A$ such that $f_0(t) \neq 0$ exactly when $a < t < b$, we have $hk([0, 1] \setminus (a, b)) = [0, 1] \setminus (a, b)$ so all of the segments (a, b) are hk -open; as such segments form a basis of τ_{std} we get $\tau_{std} \subset \tau_{hk}$.

Hence $m(A) = [0, 1]$ while $M(A)$ is much bigger containing as a proper subset all of \mathbb{R} .

4. Properties of locally m -pseudoconvex Fréchet algebras

The next theorem shows that the first condition considered in the previous section is valid for a large class of topological algebras.

THEOREM 2. *Let A be a complex unital locally m -pseudoconvex Fréchet algebra. If $\text{comm}A \neq A$ or A is topologically almost commutative then every proper finitely generated two-sided ideal in A is contained in a closed maximal two-sided ideal of A .*

To prove the theorem we will need the following Lemma ([7], p. 233).

LEMMA 1. *Let A and B be topological algebras with unit elements e_A and e_B , respectively and let h be a homomorphism from A onto a dense subset of B with $h(e_A) = e_B$. Assume $a_1, \dots, a_n, c_1, \dots, c_n \in A$ and $d_1, \dots, d_n \in B$ are such that*

$$\sum_{v=1}^n c_v a_v = e_A \quad \text{and} \quad \sum_{v=1}^n d_v h(a_v) = e_B.$$

Then for any neighborhood O of zero in B there exist $b_1, \dots, b_n \in A$ such that

$$\sum_{v=1}^n b_v a_v = e_A \quad \text{and} \quad h(b_v) \in d_v + O, \quad \text{for } v = 1, \dots, n.$$

PROOF OF THEOREM 2. Assume A is a complex locally m -pseudoconvex Fréchet algebra with a unit element e_A and the topology given by a family $\{p_n : n \in \mathbb{N}\}$ of k_n -homogeneous submultiplicative seminorms, with $k_n \in (0, 1]$, for $n \in \mathbb{N}$. We may also assume ([6], Proposition 4.6.1), that

$$(4.1) \quad p_n(a)^{\frac{1}{k_n}} \leq p_{n+1}(a)^{\frac{1}{k_{n+1}}}, \quad \text{for } n \in \mathbb{N}, a \in A, k_{n+1} \leq k_n.$$

Put

$$B \stackrel{\text{df}}{=} \begin{cases} A/\text{comm}A & \text{if } \text{comm}A \neq A \\ A/\text{rad}A & \text{if } A \text{ is topologically almost commutative,} \end{cases}$$

let $\kappa : A \rightarrow B$ be the canonical homomorphism, and let

$$q_n(b) \stackrel{\text{df}}{=} \inf \{p_n(a) : \kappa(a) = b, a \in A\}, \quad n \in \mathbb{N}, b \in B.$$

Since $\text{comm}A$ and $\text{rad}A$ are closed two-sided ideals, B is a commutative complex locally m -pseudoconvex Fréchet algebra with a unit element $e_B = \kappa(e_A)$ and the topology given by the family $\{q_n : n \in \mathbb{N}\}$ of k_n -homogeneous submultiplicative seminorms. Furthermore by (4.1)

$$(4.2) \quad q_n(b)^{\frac{1}{k_n}} \leq q_{n+1}(b)^{\frac{1}{k_{n+1}}}, \quad \text{for } n \in \mathbb{N}, b \in B, k_{n+1} \leq k_n.$$

Next for $n \in \mathbb{N}$ let π_n be the canonical homomorphism of B onto $B_n \stackrel{\text{df}}{=} B/\ker q_n$ and let r_n be the quotient k_n -homogeneous norm on B_n . Let \tilde{B}_n be the completion of B_n , let \tilde{r}_n be the extension of r_n to a k_n -homogeneous norm on \tilde{B}_n , and let denote by $\mu_n : B \rightarrow \tilde{B}_n$ the composition of π_n with the embedding into \tilde{B}_n .

For $m \leq n$ we have $B_n \subset B_m$ and by (4.2)

$$(4.3) \quad r_m(\pi_m(b))^{\frac{1}{k_m}} = q_m(b)^{\frac{1}{k_m}} \leq q_n(b)^{\frac{1}{k_n}} = r_n(\pi_n(b))^{\frac{1}{k_n}}, \quad b \in B, m \leq n.$$

For $n, m \in \mathbb{N}$ with $m \leq n$ let

$$f_{m,n} : B_n \rightarrow B_m, \quad \text{be defined by } f_{m,n}(\pi_n(b)) = \pi_m(b).$$

By (4.3) the homomorphism f_{mn} is a uniformly continuous map from B_n onto B_m ([6], Theorem 4.3.11) so it can be continuously extended to homomorphism $\tilde{f}_{m,n}$ the commutative k_n -Banach algebra \tilde{B}_n onto a dense subalgebra of k_m -Banach algebra \tilde{B}_m ([9], Proposition 5, p. 129). We have

$$(4.4) \quad \tilde{f}_{l,n} = \tilde{f}_{l,m} \circ \tilde{f}_{m,n}, \text{ and } \tilde{r}_m(\tilde{f}_{m,n}(\tilde{b}_n))^{\frac{1}{k_m}} \leq \tilde{r}_n(\tilde{b}_n)^{\frac{1}{k_n}}, \quad \text{for } l \leq m \leq n, \tilde{b}_n \in \tilde{B}_n.$$

To finish the proof assume I is a finitely generated ideal in A and let $a_1, \dots, a_s \in A$ be such that $I = \langle a_1, \dots, a_s \rangle$. Suppose that

$$\langle \mu_n(\kappa(a_1)), \dots, \mu_n(\kappa(a_s)) \rangle = \tilde{B}_n, \quad \text{for all } n \in \mathbb{N},$$

and let $\tilde{b}_1^n, \dots, \tilde{b}_s^n \in \tilde{B}_n$ be such that

$$\sum_{v=1}^s \tilde{b}_v^n \mu_n(\kappa(a_v)) = \mu_n(\kappa(e_A)) = e_{\tilde{B}_n}.$$

Put $\tilde{d}_v^{n-1} = \tilde{f}_{n-1,n}(\tilde{b}_v^n)$. We have

$$\sum_{v=1}^s \tilde{d}_v^{n-1} \mu_{n-1}(\kappa(a_v)) = \mu_{n-1}(\kappa(e_A)) = e_{\tilde{B}_{n-1}}.$$

Hence by Lemma 1, with $A = \tilde{B}_n$, $B = \tilde{B}_{n-1}$, $O = \{x \in \tilde{B}_{n-1} : \tilde{r}_{n-1}(x) < 2^{-n}\}$, and $h = \tilde{f}_{n-1,n}$ there are $\tilde{d}_1^n, \dots, \tilde{d}_s^n \in \tilde{B}_n$ such that

$$(4.5) \quad \sum_{v=1}^s \tilde{d}_v^n \mu_n(\kappa(a_v)) = e_{\tilde{B}_n}$$

and

$$\tilde{r}_{n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1}) < 2^{-n}, \quad n \in \mathbb{N}, v = 1, \dots, s.$$

By (4.4) for $m \leq n-1$ and $v = 1, \dots, s$

$$\begin{aligned} \tilde{r}_m(\tilde{f}_{m,n}(\tilde{d}_v^n) - \tilde{f}_{m,n-1}(\tilde{d}_v^{n-1})) &= \tilde{r}_m(\tilde{f}_{m,n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n)) - \tilde{f}_{m,n-1}(\tilde{d}_v^{n-1})) \\ &= \tilde{r}_m(\tilde{f}_{m,n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1})) \\ &\leq (\tilde{r}_{n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1}))^{\frac{k_m}{k_{n-1}}} \leq \left(\frac{1}{2^n}\right)^{\frac{k_m}{k_{n-1}}} \end{aligned}$$

therefore, since $k_l \leq k_n$ for $l \geq n$, for any $n \leq p < q$ we get

$$\tilde{r}_n(\tilde{f}_{n,q}(\tilde{d}_v^q) - \tilde{f}_{n,p}(\tilde{d}_v^p)) \leq \sum_{t=p+1}^q \tilde{r}_n(\tilde{f}_{n,t}(\tilde{d}_v^t) - \tilde{f}_{n,t-1}(\tilde{d}_v^{t-1})) < \sum_{t=p+1}^q \left(\frac{1}{2^t}\right)^{\frac{k_n}{k_{t-1}}} \leq \sum_{t=p+1}^q \left(\frac{1}{2}\right)^t.$$

Hence, as $\sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^t$ is convergent, $(\tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}))_{l \in \mathbb{N}}$ is a Cauchy sequence in \tilde{B}_n , for any $n \in \mathbb{N}$ and $v = 1, \dots, s$; as \tilde{B}_n are complete we may put

$$\lim_l \tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}) \stackrel{df}{=} \tilde{e}_v^n \in \tilde{B}_n.$$

By (4.4) $(\tilde{e}_v^n)_{n \in \mathbb{N}} \in \varprojlim \left\{ \tilde{B}_n, \tilde{f}_{m,n}, \mathbb{N} \right\}$ for each $v = 1, \dots, s$. Since $B \ni b \mapsto (\mu_n(b)) \in \varprojlim \left\{ \tilde{B}_n, \tilde{f}_{m,n}, \mathbb{N} \right\}$ is a surjective topological isomorphism ([2], p. 18–22, or [6], Theorem 4.5.3) there are elements $e_1, \dots, e_s \in A$ such that

$$(4.6) \quad \mu_n(\kappa(e_v)) = \tilde{e}_v^n = \lim_{l \rightarrow \infty} \tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}), \quad \text{for all } v \text{ and } n.$$

Therefore by (4.5) and (4.6) for each $n \in \mathbb{N}$ we get

$$\begin{aligned}
 p_n \left(\sum_{v=1}^s e_v a_v - e_A \right) &= \tilde{r}_n \left(\mu_n \left(\kappa \left(\sum_{v=1}^s e_v a_v - e_A \right) \right) \right) \\
 &= \tilde{r}_n \left(\sum_{v=1}^s \mu_n(\kappa(e_v)) \mu_n(\kappa(a_v)) - \mu_n(\kappa(e_A)) \right) \\
 &= \tilde{r}_n \left(\sum_{v=1}^s \lim_{l \rightarrow \infty} \tilde{f}_{n, n+l}(\tilde{d}_v^{n+l}) \mu_n(\kappa(a_v)) - \mu_n(\kappa(e_A)) \right) \\
 &= \lim_{l \rightarrow \infty} \tilde{r}_n \left(f_{n, n+l} \left(\sum_{v=1}^s d_v^{n+l} \mu_{n+l}(\kappa(a_v)) - \mu_{n+l}(\kappa(e_A)) \right) \right) = 0,
 \end{aligned}$$

hence $I = A$. So for every proper finitely generated ideal I of A there is $n_0 \in \mathbb{N}$ such that $\mu_{n_0}(\kappa(I)) \neq \tilde{B}_{n_0}$. Because $\mu_{n_0}(\kappa(I))$ is an ideal in a complex unital commutative k_{n_0} -Banach algebra \tilde{B}_{n_0} there is a nontrivial continuous multiplicative linear functional φ_0 on \tilde{B}_{n_0} such that $\mu_{n_0}(\kappa(I)) \subset \ker \varphi_0$ ([13], Proposition 4.3, or [14], Theorem 4.1). Let $\phi = \varphi_0 \circ \mu_{n_0} \circ \kappa$. Then ϕ is a nontrivial continuous linear multiplicative functional on A , $\ker \phi$ is a closed maximal two-sided ideal in A and $I \subset \ker \phi$. \square

COROLLARY 2. *Let A be one of the following algebras: a complex unital locally m -pseudoconvex Fréchet algebra for which $\text{comm}A \neq A$, or a unital topologically almost commutative locally m -pseudoconvex Fréchet algebra. Then $M(A) = m(A)$ if and only if $m(A)$ is compact in the hk -topology.*

5. Simplicial algebras with compact topological strong structure space

In this section we discuss the second condition listed in Theorem 1. The following proposition generalizes Corollary 3.9 of ([1]); notice that every two-sided (or commutative) Q -algebra satisfies the condition (5.1) below.

PROPOSITION 2. *Let A be a unital simplicial algebra. If*

$$(5.1) \quad \text{cl}_A \left(\bigcup_{M \in m(A)} M \right) \subset \bigcup_{M \in M(A)} M,$$

then $m(A)$ is compact in the hk -topology.

PROOF. Suppose $m(A)$ is not compact and let $(F_\gamma)_{\gamma \in \Gamma}$ be a family of hk -closed subsets of $m(A)$ with the finite intersection property and such that $\bigcap_{\gamma \in \Gamma} F_\gamma = \emptyset$. Let J be the two-sided ideal in A generated by $\{k(F_\gamma) : \gamma \in \Gamma\}$. Since A is simplicial, if $\text{cl}_A(J) \neq A$ then $\text{cl}_A(J) \subset M$ for some $M \in m(A)$. As $k(F_\gamma) \subset \text{cl}_A(J) \subset M$ for every $\gamma \in \Gamma$, then $M \in h(k(F_\gamma)) = F_\gamma$ for each $\gamma \in \Gamma$, which contradicts our assumption. Hence $\text{cl}(J) = A$.

Fix $a \in A$ and a neighborhood of zero O in A . Let O' be a neighborhood of zero in A such that $O'a \subset O$. Since A is unital $(e_A + O') \cap J \neq \emptyset$ and we can find $o \in O'$, $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and $a_{\gamma_k} \in k(F_{\gamma_k})$ such that

$$e_A + o = \sum_{k=1}^n a_{\gamma_k} \in \sum_{k=1}^n k(F_{\gamma_k}).$$

Let $M_o \in \bigcap_{k=1}^n F_{\gamma_k}$. Since $M_o \in F_{\gamma_k} = h(k(F_{\gamma_k}))$ so $k(F_{\gamma_k}) \subset M_o$, and we get

$$e_A + o \in M_o \subset \bigcup_{M \in m(A)} M.$$

Hence

$$(e_A + O) \cap \left(\bigcup_{M \in m(A)} M \right) \neq \emptyset, \quad \text{for any neighborhood } O \text{ of zero in } A.$$

Consequently

$$e_A \in \text{cl}_A \left(\bigcup_{M \in m(A)} M \right) \subset \bigcup_{M \in M(A)} M,$$

which is impossible. \square

COROLLARY 3. *Let A be one of the following algebras: a complex unital locally m -pseudoconvex Fréchet algebra for which $\text{comm}A \neq A$, or a unital topologically almost commutative locally m -pseudoconvex Fréchet algebra. If A satisfies the condition (5.1) then $M(A) = m(A)$.*

PROPOSITION 3. *A commutative unital simplicial topological algebra A is a Q -algebra if and only if it satisfies (5.1) and $M(A) = m(A)$.*

PROOF. If A is a commutative Q -algebra then $M(A) = m(A)$ and (5.1) holds because $A \setminus \text{Inv}A$ is equal to the union of all ideals from $M(A)$. If A is a commutative unital topological algebra and $M(A) = m(A)$ then the union of all ideals from $m(A)$ and the union of all ideals from $M(A)$ coincides; by (5.1) the last union is closed so A is a Q -algebra. \square

It is clear that every Q -algebra is simplicial. The next result characterizes these simplicial algebras which are Q -algebras.

THEOREM 3. *Let A be a commutative unital simplicial algebra. Then A is a Q -algebra if and only if A satisfies (5.1) and condition (a) of Corollary 1.*

PROOF. If A is a commutative Q -algebra then $M(A) = m(A)$ and $\bigcup_{M \in m(A)} M = A \setminus \text{Inv}A$ is closed in A .

Assume A is a commutative unital simplicial algebra and I is a proper finitely generated ideal in A . If A satisfies the condition (a) of Corollary 1 then there is an ideal $M \in m(A)$ such that $I \subset M$. By Theorems 1 and Proposition 2 we have $M(A) = m(A)$. Hence, by (5.1) $\text{Inv}A = A \setminus \bigcup_{M \in m(A)} M$ is open in A and consequently A is a Q -algebra. \square

COROLLARY 4. *Let A be a commutative unital complex locally m -pseudoconvex Fréchet algebra with $\text{comm}A \neq A$ or a unital topologically almost commutative locally m -pseudoconvex Fréchet algebra. Then A is a Q -algebra if and only if A satisfies (5.1).*

PROOF. By Theorem 4.2 of [4] algebra A is simplicial and by Theorem 2 it satisfies the condition (a) of Corollary 1. Hence the result follows from Theorem 3. \square

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