PERTURBATIONS OF UNIFORM ALGEBRAS

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1. *Introduction.* Let A be a Banach algebra. By an ε-perturbation of A we mean a commutative associative multiplication $\times$ on the Banach space A such that

$$\|f \times g - fg\| \leqslant \varepsilon \|f\| \|g\|$$

for all f, g in A.

A Banach algebra is said to be rigid if any sufficiently small perturbation produces a new algebra which is algebraically isomorphic to the original one. Conditions which are sufficient to ensure that a Banach algebra is rigid have been given by B. E. Johnson [**1**] and independently by I. Raeburn and J. L. Taylor [**3**]. R. Rochberg [**4**] considered a wider class of algebras and has proved that if A is a uniform algebra such that its Shilov boundary is equal to its Choquet boundary and each point of the Shilov boundary is G_δ then the Shilov boundaries of A and $A_\times$ are homeomorphic for any sufficiently small perturbation. In this paper we prove that the assumption "each point of the Shilov boundary is G_δ" can be omitted. As a consequence we obtain a proof that the algebra $C(S)$ is rigid for any compact Hausdorff space S.

Before passing to the theorem let us formulate some general facts about ε-perturbations. Notice that if $\times$ is an ε-perturbation of a Banach algebra $(A, \cdot, \|\cdot\|)$ then $(A, \times, \|\cdot\|)$ need not be a Banach algebra (because the original norm on A need not be $\times$-submultiplicative) but the new multiplication is certainly continuous. It is well known that in this situation there exists an equivalent norm N on A which is $\times$-submultiplicative. If A is a uniform algebra this new norm can possess some special additional properties. By the definition of an ε-perturbation we have

$$(1-\varepsilon)\|f\|^2 \leqslant \|f \times f\| \leqslant (1+\varepsilon)\|f\|^2 .$$

Hence, by induction, the norms $\|\cdot\|$ and N being equivalent, we can estimate the spectral norm of f in the algebra $(A, \times, N)$

$$(1-\varepsilon)\|f\| \leqslant \|f\|_{sp} \leqslant (1+\varepsilon)\|f\| . \tag{1}$$

We obtain the following proposition (see also [**4**, §2]).

PROPOSITION. *Let $(A, \cdot, \| \ \|)$ be a uniform algebra and let $\times$ be an ε-perturbation of A. Then there exists an equivalent norm $|||\cdot|||$ on the Banach space A such that*

$$(1-\varepsilon)\|f\| \leqslant |||f||| \leqslant (1+\varepsilon)\|f\|$$

for all f in A, and such that $(A, \times, ||| \ |||)$ is a uniform algebra.

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Following Rochberg we call the algebra $(A, \times, |||\ |||)$ an ε-deformation of A.

2. Let A be a Banach algebra and let ∂A, $\operatorname{Ch} A$, $M(A)$ denote the Shilov boundary, the Choquet boundary and the maximal ideal space of A respectively.

Our main result is the following:

THEOREM 1. *Suppose that A is a uniform algebra with $\partial A = \operatorname{Ch} A$. There are positive constants ε, k, which are independent of A, such that for all $\varepsilon_0 < \varepsilon$ if $A_\times$ is an ε_0-deformation of A then there is a homeomorphism ϕ of $\partial A_\times$ onto ∂A such that, for all s in $\partial A_\times$ and all f in A*

$$(*) \qquad |f(\phi(s)) - \hat{f}(s)| \leqslant k\varepsilon_0 \|f\|,$$

$\hat{f}$ being the Gelfand transformation of $f \in A_\times$ (that is, $\hat{}: A \to C(M(A_\times))$).

Proof. Let A and $\times$ be as in the theorem. The first part of the proof consists of proving the following lemma.

LEMMA. *There is a positive constant c such that:*

(a) *For all s_0 in ∂A there is a measure $\Delta\mu_{s_0}$ on ∂A, of variation not greater than $c\varepsilon_0$, such that the functional on A which is represented by the measure $\mu = \delta_{s_0} + \Delta\mu_{s_0}$ is in the Shilov boundary of $A_\times$.*

(b) *Each functional from $\partial A_\times$ can be represented by a measure μ of the above form, and the point s_0 is uniquely determined.*

Proof. Fix $s_0 \in \partial A$ and let $(U_\alpha)_{\alpha \in \Gamma}$ be the net of the open neighbourhoods of s_0. By the assumption $s_0 \in \operatorname{Ch} A$, so there is a net $(f_\alpha)_{\alpha \in \Gamma} \subset A$ such that

$$(2) \qquad \|f_\alpha\| = 1 = f_\alpha(s) \text{ and } |f_\alpha(s)| < \varepsilon \qquad \text{for } s \in \partial A \setminus U \text{ and all } \alpha \in \Gamma.$$

By (1) for each $\alpha \in \Gamma$ there exists a measure μ_α on ∂A, which represents a functional from $\partial A_\times$, such that

$$(3) \qquad |\mu_\alpha(f_\alpha)| \geqslant 1 - \varepsilon_0 \quad \text{and} \quad \|\mu_\alpha\|_{A^*} = \operatorname{var}(\mu_\alpha) \leqslant 1 + \varepsilon_0.$$

The inequality $\|\mathbf{1} \times f - f\| \leqslant \varepsilon_0 \|f\|$ provides

$$|\mu_\alpha(\mathbf{1}) - 1|\,|\mu_\alpha(f)| = |\mu_\alpha(\mathbf{1})\mu_\alpha(f) - \mu_\alpha(f)|$$

$$= |\mu_\alpha(\mathbf{1} \times f) - \mu_\alpha(f)| = |\mu_\alpha(\mathbf{1} \times f - f)| \leqslant \varepsilon_0 \|\mu_\alpha\| \|f\|.$$

Hence

$$(4) \qquad |\mu_\alpha(\mathbf{1}) - 1| \leqslant \varepsilon_0.$$

From (2), (3), and (4) we get

$$(5) \qquad \operatorname{Re}(\mu_\alpha(U_\alpha)) \geqslant 1 - c'\varepsilon_0.$$

Taking a net finer then $(\mu_\alpha)_{\alpha \in \Gamma}$, and using weak $*$ compactness of $\partial A_\times$, we can assume, without loss of generality, that $(\mu_\alpha)_{\alpha\in\Gamma}$ converges to $\mu_0 \in \partial A_\times$. From (3) and (5) it follows that μ_0 is of the form

$$\mu_0 = \delta_{s_0} + \Delta\mu,$$

where $\|\Delta\mu\| \leqslant (2c'+1)\varepsilon_0 = c\varepsilon_0$, and this proves assertion (a).

The subset S of $\partial A_\times$ consisting of all functionals which can be represented by a measure of the form $\delta_s + \Delta\mu$ with $\|\Delta\mu\| \leqslant c\varepsilon_0 < 1$ is a closed subset of $\partial A_\times$ and from assertion (a) we derive

$$\begin{aligned}\|f_{|S}\|_\infty &= \sup\{|(\delta_s+\Delta\mu)(f)| : \delta_s+\Delta\mu \in S\}\\ &\geqslant \sup\{|f(s)| - |\Delta\mu(f)| : \delta_s+\Delta\mu \in S\}\\ &\geqslant \|f\| - c\varepsilon_0\|f\| \geqslant (1-\varepsilon_0)(1-c\varepsilon_0)\|\hat{f}\|_\infty .\end{aligned}$$

This means that the closed subset S of $\partial A_\times$ provides an equivalent norm on A, hence $S = \partial A_\times$ which completes the proof of the lemma.

Now, using our lemma, we can define the function $\phi : \partial A_\times \to \partial A$ as follows: $\phi(l) = s$ if and only if the functional l can be represented by a measure on ∂A, of the form $\delta_s + \Delta\mu$ with $\|\Delta\mu\| \leqslant c\varepsilon_0$. By the lemma the function ϕ is continuous and onto ∂A and (*) is satisfied. It remains to be proved that ϕ is one to one. To this end let $s_0 \in \partial A$ and let K denote the set $\phi^{-1}(s_0)$. We need to demonstrate that K consists of only a single point. This is a consequence of the following two propositions:

(i) If $\phi(x_1) = \phi(x_2) = s_0$ then $|\hat{f}(x_1) - \hat{f}(x_2)| \leqslant 2c\varepsilon_0\|f\|$ for any $f \in A$.

(ii) If K consisted of more than one point then $\sup\{\|\hat{f}\| : f \in \mathscr{B}\} = 2$, where $\mathscr{B} = \{f \in A : \text{there are } x_1, x_2 \in K \text{ such that } \hat{f}(x_1) = 0 \text{ and } \hat{f}(x_2) = 1 = \sup_K |\hat{f}|\}$.

The first proposition is an immediate consequence of the lemma. To prove the second one it is sufficient to show that, for all $f \in \mathscr{B}$, there is $h \in \mathscr{B}$ such that $\|h\| \leqslant \max\{2, \|f\|/2\}$. For this purpose let $f \in \mathscr{B}$, $\hat{f}(x_1) = 0$, $\hat{f}(x_2) = 1 = \sup_K |\hat{f}|$, let U be a neighbourhood of K such that $|\hat{f}(x)| < 1+\varepsilon_0$ for $x \in U$ and let $g \in A$ with $g(s_0) = 1 = \|g\|$, $|g(s)| < \varepsilon_0$ for $s \in \partial A \setminus \phi(U)$. From the lemma

$$\widehat{fg}(x_1) = 0,$$

$$|\widehat{fg}(x_2) - 1| = |\hat{g}(x_2) - 1| \leqslant c\varepsilon_0,$$

$$|\widehat{fg}(x)| \leqslant (1+\varepsilon_0)\|\hat{g}\| \leqslant (1+\varepsilon_0)(1+c\varepsilon_0) \text{ for } x \in U,$$

and

$$|\widehat{fg}(x)| \leqslant \|\hat{f}\| \sup_{x\notin U}\{|g(\phi(x)) + \Delta\mu_x(g)|\} \leqslant \|\hat{f}\|(\varepsilon_0 + c\varepsilon_0).$$

The desired result is now obtained by substituting $h = fg$ and taking ε_0

sufficiently small. This completes the proof of the second proposition and hence that of the theorem.

COROLLARY 1. *The algebra $C(S)$ is rigid for any compact Hausdorff space S.*

Proof. Let $A = C(S)$. For any sufficiently small ε_0 the theorem allows us to introduce an operator $\Phi : C(S) \to C(S)$ defined as follows

$$\Phi(f)(s) = \hat{f}(\phi^{-1}(s)).$$

The distance between Φ and the identity operator is less than $k\varepsilon_0$, hence, for $k\varepsilon_0$ less than 1, Φ is an isomorphism from $C(S)$ onto $C(S)$. Consequently the Gelfand transformation from the algebra $C(S)_\times$ onto $C(M(C(S)_\times)) = C(S)$ is also an isomorphism which means that $C(S)$ is rigid.

From the Bishop '1/4–3/4 criterion' (see e.g. [**2**, p. 263]) and from our theorem we also get:

COROLLARY 2. *If the Shilov and the Choquet boundaries of a uniform algebra A coincide then, for any sufficiently small perturbation $\times$ of A, the Shilov and the Choquet boundaries of $A_\times$ also coincide.*

3. Investigating deformations of Banach algebras R. Rochberg [**4**] has defined a natural measure of closeness of two algebras A and B:

$$D(A, B) = \inf\{\varepsilon : B \text{ is algebraically isomorphic to an } \varepsilon\text{-deformation of } A\}.$$

R. Rochberg has shown several interesting properties of $D(\cdot, \cdot)$ and also formulated the following problem:

Problem. Assume that the Shilov boundaries of uniform algebras A and B consist entirely of peak points. Does $D(A, B) = 0$ imply that A is isomorphic to B?

The following theorem gives a negative answer:

THEOREM 2. *There are two compact subsets D_1 and D_2, of the complex plane $\mathbb{C}$, such that the following hold:*

(a) *Algebras $A(D_1)$ and $A(D_2)$ are not algebraically isomorphic.*

(b) *For each $\varepsilon > 0$ there is an ε-perturbation $\times$ of $A(D_1)$, such that the algebras $(A(D_1), \times)$ and $A(D_2)$ are algebraically isomorphic.*

(c) *The Shilov boundaries of $A(D_1)$ and $A(D_2)$ consist entirely of peak points.*

Here $A(D)$ denotes the algebra of all continuous functions on D which are analytic on int D.

Proof. Let us denote

$$P(z_0, R, r) = \{z \in \mathbb{C} : r \leqslant |z_0 - z| \leqslant R\},$$

and define D_1 and D_2 as follows:

$$D_1 = \bigcup_{n,m=1}^{\infty} P\left(\frac{1}{n} + \frac{i}{m+n}, \frac{1}{4^{n+m}}, \frac{1}{4^{n+m}\left(2+\frac{1}{m}\right)}\right) \cup \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\},$$

$$D_2 = D_1 \cup P(2, 1/2, 1/4).$$

Assume that $A(D_1)$ and $A(D_2)$ are isomorphic. Then there is a one to one correspondence between factors of $A(D_1)$ and isomorphic factors of $A(D_2)$. It is well known that algebras $A(P(z_0, R_0, r_0))$ and $A(P(z_1, R_1, r_1))$ are isomorphic if and only if $R_0/r_0 = R_1/r_1$, but we have, for all $n, m \in \mathbb{N}$,

$$\frac{1/4^{n+m}}{1/4^{n+m}\left(2+\frac{1}{m}\right)} \neq \frac{1/2}{1/4}.$$

This contradicts our assumption and hence proves (a).

To get the second part of the thesis it is sufficient to prove ([**4**, Theorem 4.1]) that the Banach–Mazur distance between Banach spaces $A(D_1)$ and $A(D_2)$ is 1. For this purpose let us define the following sequence of operators T_k from $A(D_1)$ onto $A(D_2)$:

$$T_k f(z) = \begin{cases} f(z), \quad \text{for } z \in D_2 \setminus \bigcup_{n=1}^{\infty} P\left(\frac{1}{k} + \frac{i}{k+n}, \frac{1}{4^{n+k}}, \frac{1}{4^{n+k}\left(2+\frac{1}{k}\right)}\right), \\ f\left(\frac{1}{4}\left(z - i\left(\frac{1}{k+n} - \frac{1}{k+n+1}\right)\right)\right), \\ \qquad\qquad \text{for } z \in \bigcup_{n=1}^{\infty} P\left(\frac{1}{k} + \frac{i}{k+n}, \frac{1}{4^{n+k}}, \frac{1}{4^{n+k}\left(2+\frac{1}{k}\right)}\right), \\ \sum_{p=-\infty}^{0} \frac{a_p}{4^{kp}\left(2+\frac{1}{k}\right)^p}\left(z - 2 + \frac{1}{k} + \frac{i}{k+1}\right)^p \\ \qquad + \sum_{p=1}^{\infty} \frac{a_p}{4^{kp}2^p}\left(z - 2 + \frac{1}{k} + \frac{i}{k+1}\right), \quad \text{for } z \in P(2, 1/2, 1/4), \end{cases}$$

where

$$f(z) = \sum_{-\infty}^{+\infty} a_p \left(z - \frac{1}{k} - \frac{i}{k+1}\right)^p$$

for $z \in P\left(\frac{1}{k} + \frac{i}{k+1}, \frac{1}{4^{k+1}}, \frac{1}{4^{k+1}\left(2+\frac{1}{k}\right)}\right)$.

Now the equality $\lim \|T_k\| \|T_k^{-1}\| = 1$ is an immediate consequence of the following well known proposition:

PROPOSITION. *Let an operator $T: A(P(0, R, r)) \to A(P(0, R', r'))$ be defined by*

$$T\left(\sum_{-\infty}^{+\infty} a_p z^p\right) = \sum_{-\infty}^{0} \left(\frac{r}{r'}\right)^p a_p z^p + \sum_{1}^{\infty} \left(\frac{R}{R'}\right)^p a_p z^p .$$

Then $\|T\| \|T^{-1}\|$ tends to 0 if R'/r' tends to R/r.

Part (c) of the theorem is obvious so this completes the proof.

References

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