

## PEAK SET WITHOUT PEAK POINTS

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**ABSTRACT.** We give an example of a natural Banach function algebra on the unit disc such that a smaller disc is a peak set for the algebra, but it does not contain any peak point.

A Banach function algebra on a compact Hausdorff space  $X$  is a Banach algebra  $A$  consisting of continuous functions on  $X$ , such that  $A$  separates points of  $X$  and contains the constant functions. If the norm of the algebra  $A$  coincides with the sup norm on  $X$ , it is called a uniform algebra. If any linear and multiplicative functional on  $A$  is of the form  $f \mapsto f(x)$  for some  $x \in X$ , the algebra is called natural. A subset  $K$  of  $X$  is a peak set for  $A$  if there is an  $f \in A$  such that  $f \equiv 1$  on  $K$  and  $|f(x)| < 1$  for  $x \notin K$ ; if  $K = \{x_0\}$ , then  $x_0$  is a peak point. If no proper subset of  $K$  is a peak set we call it a minimal peak set for  $A$ . It is well known [2] that if  $A$  is a uniform algebra on a metrizable set  $X$  then any peak set contains a peak point. In [1] H. G. Dales constructed a natural Banach function algebra on a compact subset of  $\mathbb{C}^2$  having a peak set without any peak point. T. G. Honary [3] provided an example in  $\mathbb{R}^3$ , however his algebra is not natural. In this note we give a very simple example of a natural Banach function algebra on the unit disc with peak sets not containing any peak point.

Let  $K$  be an open nonempty subset of the complex plane  $\mathbb{C}$ . By  $C^1(K)$  we denote the algebra of all bounded complex valued functions on  $\overline{K}$  with continuous and bounded first order partial derivatives on  $K$ .  $C^1(K)$  is a Banach function algebra on  $\overline{K}$  if equipped with the norm

$$\|f\| = \|f\|_\infty + \|f_x\|_\infty + \|f_y\|_\infty,$$

where  $\|\cdot\|_\infty$  is the sup norm. By  $A(K)$  we denote the uniform algebra of all continuous functions on  $\overline{K}$  which are analytic on  $K$ . We put  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$  and  $\mathbb{C}_- = \{z : \operatorname{Re} z < 0\}$ .

**Theorem 1.** *Let  $A$  be a subalgebra of  $C^1(\mathbb{D}_1)$  consisting of all functions which are analytic on  $\mathbb{D}_{\frac{1}{2}}$ . Then  $\overline{\mathbb{D}_{\frac{1}{2}}}$  is a minimal peak set for  $A$ ; in particular  $\overline{\mathbb{D}_{\frac{1}{2}}}$  does not contain any peak point.*

**Lemma 2.** *There is no  $h \in A(\mathbb{D}_1)$  such that  $h(1) = 0$  and  $|1 + (z - 1)h(z)| < 1$  for  $z \neq 1$ .*

*Proof of the lemma.* Assume that such a function does exist. If we compose  $1 + (z - 1)h(z)$  with a suitable fractional linear transformation we get an  $f \in A(\mathbb{C}_-)$

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such that  $f(z) = 1 + zg(z)$ ,  $g(0) = 0$ , and  $|f(z)| < 1$  for  $z \in \overline{\mathbb{C}_-} \setminus \{0\}$ . Let  $\alpha$  be a positively oriented curve consisting of the segment from  $-i$  to  $+i$  and the half of the unit circle contained in  $\mathbb{C}_-$ . We shall show that  $\gamma = g \circ \alpha$  has negative orientation, which will contradict the fact that  $g$  is analytic.

For any  $z \in \overline{\mathbb{C}_-} \setminus \{0\}$  we have  $\operatorname{Re} zg(z) < 0$ , so

- as  $z$  moves up from  $0$  to  $+i$  along the curve  $\alpha$ ,  $\operatorname{Im} g(z) > 0$ ,
- as  $z$  moves along the half circle of  $\alpha$ ,  $g(z)$  is in  $\mathbb{C} \setminus \{z : \operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$ ,
- as  $z$  moves from  $-i$  to  $0$  on the vertical line,  $\operatorname{Im} g(z) < 0$ .

Consequently, if  $z_0$  is a positive real number such that the segment  $(0, z_0]$  does not intersect  $\gamma$ , then the orientation of  $\gamma$  around  $z_0$ , as well as around any point from  $(0, z_0]$ , is negative. □

*Proof of the theorem.* It is well known that there is a  $C^1$ -function, and even a  $C^\infty$ -function on  $\mathbb{D}_1$ , peaking exactly on  $\overline{\mathbb{D}_{\frac{1}{2}}}$ ; thus  $\overline{\mathbb{D}_{\frac{1}{2}}}$  is a peak set.

Assume  $\frac{1}{2}$  is a peak point for  $A$  and let  $f \in A$  be a function peaking at  $\frac{1}{2}$ . Put  $u = \operatorname{Re} f$ ,  $v = \operatorname{Im} f$ . Since  $u^2 + v^2$  has the maximum at  $\frac{1}{2}$ , and  $u(\frac{1}{2}) = 1$ ,  $v(\frac{1}{2}) = 0$ , it follows that  $u_x(\frac{1}{2}) = u_y(\frac{1}{2}) = 0$ . Since the partial derivatives are continuous, and  $f$  satisfies the Cauchy-Riemann equations on  $\mathbb{D}_{\frac{1}{2}}$ , we get  $v_x(\frac{1}{2}) = v_y(\frac{1}{2}) = 0$ . Hence

$$g(z) \stackrel{df}{=} \frac{f(z) - 1}{z - \frac{1}{2}} \rightarrow 0, \quad \text{as } z \rightarrow \frac{1}{2},$$

so  $g \in A(\overline{\mathbb{D}_{\frac{1}{2}}})$ ,  $g(\frac{1}{2}) = 0$ , and  $f(z) = 1 + (z - \frac{1}{2})g(z)$  has a strict maximum at  $\frac{1}{2}$ . This contradicts the lemma. Hence  $\overline{\mathbb{D}_{\frac{1}{2}}}$  does not contain any peak point.

What remains to show is that  $\overline{\mathbb{D}_{\frac{1}{2}}}$  does not contain any proper peak set. Assume  $K \subsetneq \overline{\mathbb{D}_{\frac{1}{2}}}$  is a peak set and let  $F \in A$  be a corresponding function peaking on  $K$ . Notice that  $K$  cannot contain the entire circle  $\partial\mathbb{D}_{\frac{1}{2}}$ . Let  $z_0 \in K \cap \partial\mathbb{D}_{\frac{1}{2}}$  and  $z_1 \in \partial\mathbb{D}_{\frac{1}{2}} \setminus K$ . For any  $w \in \partial\mathbb{D}_{\frac{1}{2}} \setminus \{z_0\}$  let  $\varphi_w$  be a  $C^1$ -automorphism of  $\overline{\mathbb{D}}$  mapping  $\overline{\mathbb{D}_{\frac{1}{2}}}$  onto itself, analytic on  $\mathbb{D}_{\frac{1}{2}}$ , and such that  $\varphi_w(z_0) = z_0$  and  $\varphi_w(w) = z_1$ . Such an automorphism can be obtained by smoothly extending a suitable fractional linear automorphism of  $\mathbb{D}_{\frac{1}{2}}$ . Sets of the form  $U_w = \{z \in \overline{\mathbb{D}_{\frac{1}{2}}} \setminus \{z_0\} : |F \circ \varphi_w(z)| < 1\}$  form an open cover of  $\partial\mathbb{D}_{\frac{1}{2}} \setminus \{z_0\}$ . Let  $\{w_1, w_2, \dots\}$  be such that  $\bigcup_{j=1}^\infty U_{w_j} = \partial\mathbb{D}_{\frac{1}{2}} \setminus \{z_0\}$ . Put  $G = \sum_{j=1}^\infty \frac{F \circ \varphi_{w_j}}{2^j \|F \circ \varphi_{w_j}\|}$ . This function  $G$  belongs to  $A$  and peaks exactly at  $z_0$ , which contradicts the previous part of the proof. □

Notice that essentially the same proof can be repeated to provide a slightly more general example:

Let  $\Omega_j$ ,  $j = 1, 2, \dots$ , be a sequence of open subsets of  $\mathbb{D}_1$  such that for any  $j$ ,  $\overline{\Omega_j}$  has an open neighborhood contained in  $\mathbb{D}_1$  not intersecting  $\bigcup_{i \neq j} \Omega_i$ , and such that  $\partial\Omega_j$  is a union of finitely many smooth Jordan curves. Let  $A$  be a subalgebra of  $C^1(\mathbb{D}_1)$  consisting of all functions which are analytic on  $\bigcup \Omega_j$ . Then for any  $j$ ,  $\overline{\Omega_j}$  is a peak set not containing any peak point.

However, it is not clear whether the result can be further extended to Swiss cheese type domains to provide an example of a natural Banach function algebra where the union of all nontrivial minimal peak sets is dense in the unit disc. The

difficulty, in general, is not in proving that  $\Omega_j$  does not contain any peak point, but in proving that  $\overline{\Omega_j}$  is a peak set.

## REFERENCES

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