Multiplicative functionals and entire functions, II

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September 1, 1996

Abstract. Let $\mathcal{A}$ be a complex uniform algebra with a unit e, let F be a nonsurjective and nonconstant entire function, and let T be a linear functional with $T(e) = 1$ and such that $T \circ F$ is nonsurjective. Then T is multiplicative.

1. Introduction

Let T be a nonzero multiplicative functional on a complex Banach algebra $\mathcal{A}$ with a unit e, and let $\mathcal{A}^{-1}$denote the set of all invertible elements of $\mathcal{A}$. Then $T(e) = 1$, and for any $x \in \mathcal{A}^{-1}$ we have $T(x) \neq 0$. A. M. Gleason [5] and, independently, J. P. Kahane & W. Żelazko [8], [9] proved that the converse implication also holds.

Theorem 1 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$, such that $T(e) = 1$ and*

$$T(x) \neq 0, \quad \text{for } x \in \mathcal{A}^{-1},$$

then T is multiplicative.

In fact, they proved even a stronger result.

Theorem 2 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$, such that $T(e) = 1$ and*

$$T(x) \neq 0, \quad \text{for } x \in \exp \mathcal{A}, \tag{1}$$

then T is multiplicative.

[0]Research was supported in part by a grant from the International Research & Exchanges Board, with funds provided by the National Endowment for the Humanities and the U.S. Department of State.

Mathematics Subject Classification (1990) Primary 46J05, Secondary 46H05, 46H30, 46J15, 46J20.

Where $\exp\mathcal{A}=\{\exp y : y\in\mathcal{A}\}$. In 1987 R. Arens asked if the exponential function above can be replaced by other entire function F. If F is a nonsurjective function then the Arens hypothesis has to involve some natural additional continuity assumptions and has been solved in [7] (see also [1] and [2] for earlier partial solutions). In this note we prove that the Gleason-Kahane-Żelazko theorem remains valid, for uniform algebras, if we replace the exponential function in (1) by any nonconstant and nonsurjective entire function. A uniform algebra is a closed subalgebra of an algebra of all continuous functions on a compact set K. Equivalently a Banach algebra $\mathcal{A}$ is (isometrically isomorphic to) a uniform algebra if the norm and the spectral norm coincide.

The Gleason-Kahane-Żelazko theorem has also been extended in several other directions; a number of problems remains open [6].

2. THE RESULT

Theorem 3. *Let $\mathcal{A}$ be a complex uniform algebra with a unit e, let F be a nonsurjective and nonconstant entire function, and let T be a linear functional on $\mathcal{A}$. Then the function $T\circ F:\mathcal{A}\to\mathbb{C}$ is nonsurjective if and only if $T=0$, or $T/T(e)$ is multiplicative.*

Proof of the theorem. We first prove the easy "if" part of the theorem. Assume that T is a multiplicative functional and F is a nonsurjective entire function. By the Weierstrass Factorization Theorem [3] any nonsurjective entire function F is of the form $F(z)=a+\exp g(z)$ for some entire function g and a constant a. For any x in $\mathcal{A}$ we have

$$T\circ F(x)=T\left(ae+\exp g(x)\right)=a+\exp\left(g\left(Tx\right)\right)\neq a.$$

To prove the "only if" part we consider two cases:

$$Te\neq 0 \quad \text{or} \quad Te=0,$$

we show that in the first case $T/T(e)$ is multiplicative and in the second case $T=0$.

CASE Te $\neq$ 0. Dividing by Te we may then assume, without loss of generality, that $Te=1$.

For any x,y in a commutative algebra we have $xy=\left((x+y)^2-(x-y)^2\right)/4$ so to prove that T is multiplicative it is sufficient to show that it preserves the operation of taking the square. Thus it is sufficient to prove that T is multiplicative on subalgebras with one generator. Consequently, we may assume that $\mathcal{A}$ has a single generator.

STEP 1. We now show that without loss of generality we may also make several other assumptions, namely that $F(z)=\exp g(z)$, for some entire function g, that $F'(0)\neq 0$, that $F(0)=1$, and that the range of $T\circ F:\mathcal{A}\to\mathbb{C}$ is $\mathbb{C}\setminus\{0\}$.

By the Weierstrass Factorization Theorem F is of the form $F(z) = a + \exp g(z)$. Since T is linear, $T \circ F$ is nonsurjective if and only if $T \circ \exp g$ is nonsurjective. Hence we may assume that $a = 0$, so that

$$F(z) = \exp g(z).$$

It follows that the value missing from the range of F considered as an entire function on the complex plane is the number zero. Notice that the number missing from the range of $T \circ F : \mathcal{A} \to \mathbb{C}$ is also the number zero. Indeed if

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

is the power series expansion of the function F then, since $Te = 1$, we have

$$T(F(\lambda e)) = T\left(\sum_{n=0}^{\infty} a_n (e\lambda)^n\right) = Te\left(\sum_{n=0}^{\infty} a_n \lambda^n\right) = F(\lambda),$$

hence the range of $T \circ F : \mathcal{A} \to \mathbb{C}$ contains the range of $F : \mathbb{C} \to \mathbb{C}$ and so it must be equal to $\mathbb{C}\setminus\{0\}$.

Assume

$$F(z_0 + z) = \sum_{n=0}^{\infty} b_n z^n$$

is a power series expansion of the same function around a point z_0, so that

$$\sum_{n=0}^{\infty} a_n (z_0 + z)^n = \sum_{n=0}^{\infty} b_n z^n \quad \text{for } z \in \mathbb{C}.$$

It is easy to check that these two expansions define the same function on $\mathcal{A}$, that is,

$$\sum_{n=0}^{\infty} a_n (z_0 e + x)^n = \sum_{n=0}^{\infty} b_n x^n \quad \text{for } x \in \mathcal{A}.$$

Hence by shifting the origin we may assume that $F'(0) \neq 0$. Finally, we may also assume, replacing F by $F/F(0)$, that $F(0) = 1$. $\square$

STEP 2. We now show that, as a consequence of our assumptions, T is continuous.

Since $F'(0) \neq 0$ there is a neighborhood U of 0 such that $F_{|U}$ is a homeomorphism onto a neighborhood of 1 in $\mathbb{C}$. Let

$$\left(F_{|U}\right)^{-1}(w) = \sum_{n=0}^{\infty} \beta_n (1 + w)^n$$

for w in a neighborhood of zero. Elementary computations show that the same power series defines also a local inverse of $F : \mathcal{A} \to \mathcal{A}$. Hence the range of $F : \mathcal{A} \to \mathcal{A}$ contains a neighborhood of e, the unit element of $\mathcal{A}$. Since according to our assumption T is not zero on the range of F it follows that $\ker T$ is not dense, and consequently T is a continuous functional. $\square$

STEP 3. We now prove the theorem for the disc algebra $A(\mathbb{D})$ and later we show how the general result follows from this special case.

So suppose that $\mathcal{A} = A(\mathbb{D})$. Denote by $\mathbf{Z}$ the identity function on $\mathbb{C}$, that is, $\mathbf{Z}(z) = z$ and fix a nonzero complex number $\lambda = re^{i\theta}$. Since g is a nonconstant entire function there is a region Ω in $\mathbb{C}$ such that g is one-to-one on the closure $\overline{\Omega}$ of Ω and $g(\overline{\Omega})$ is a closed disc of radius r and center at some point w_0. The existence of such a region Ω is obvious if g is a linear function; if g is a nonlinear entire function then the derivative of g is unbounded and the existence of Ω follows from the Bloch's Theorem [3]. By the Riemann Mapping Theorem there is a conformal homeomorphism $\varkappa$ from the unit disc onto Ω, and since the boundary of Ω is homeomorphic with the unit circle, $\varkappa$ can be extended to a homeomorphism between the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\Omega}$ [4]. The function

$$f(z) = \left(g \circ \varkappa(z) - w_0\right)/r$$

is an analytic homeomorphism of $\overline{\mathbb{D}}$ onto itself. Put

$$\psi(z) = \varkappa\left(f^{-1}(e^{i\theta}z)\right).$$

We have

$$\psi \in A(\mathbb{D}), \text{ and } g \circ \psi = \lambda \mathbf{Z} + w_0.$$

Hence, by our assumption,

$$T(e^{\lambda \mathbf{Z}}) = T(e^{g\circ\psi - w_0}) = e^{-w_0} T(F(\psi)) \neq 0. \tag{2}$$

The rest of Step 3 runs now exactly as in the original papers by Gleason [5] and Kahane & Żelazko [8].

Put

$$\varphi(\lambda) = T(e^{\lambda \mathbf{Z}}) = T\left(\sum_{n=0}^{\infty} \frac{(\lambda \mathbf{Z})^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{T(\mathbf{Z}^n)}{n!}\lambda^n. \tag{3}$$

For any λ we have

$$|\varphi(\lambda)| \leq \|T\| \left\|e^{\lambda \mathbf{Z}}\right\| = \|T\| \, e^{|\lambda|}$$

and by (2),

$$\varphi(\lambda) \neq 0,$$

so by the Weierstrass Factorization Theorem and by the Hadamard's Factorization Theorem [3]

$$\varphi(\lambda) = e^{\alpha\lambda+\beta}.$$

Since $\varphi(0) = T(e^0) = T(e) = 1$, we have $\beta = 0$, hence

$$\varphi(\lambda) = e^{\alpha\lambda} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\lambda^n \ . \tag{4}$$

Comparing (3) and (4) we get

$$T(\mathbf{Z}^n) = (T(\mathbf{Z}))^n\,, \quad \text{for } n \in \mathbb{N}.$$

Since polynomials are dense in $A(\mathbb{D})$ it follows that T is multiplicative of the form $T(f) = f(\alpha)$ for $f \in A(\mathbb{D})$, where $\alpha = T(\mathbf{Z})$. $\square$

STEP 4. Let now $\mathcal{A}$ be arbitrary uniform algebra with a single generator x. The spectrum K of x is simply connected and without loss of generality we may assume that K is a subset of the unit disc. We denote by $\|\cdot\|_K$ the original norm of $\mathcal{A}$ and by $\|\cdot\|_{\mathbb{D}}$ the supnorm on the unit disc $\mathbb{D}$. A restriction of any element of the disc algebra $A(\mathbb{D})$ to K is an element of $\mathcal{A}$ and

$$\|\cdot\|_K \leq \|\cdot\|_{\mathbb{D}}\,,$$

so $T_{A(\mathbb{D})} : A(\mathbb{D}) \to \mathbb{C}$ defined by

$$T_{A(\mathbb{D})}(a) = T\left(a_{|K}\right)$$

is a linear continuous functional which maps $\left\{e^{g(a)} : a \in A(\mathbb{D})\right\}$ into $\mathbb{C}\setminus\{0\}$ and such that $T_{A(\mathbb{D})}(e) = 1$. By the previous step there is a number α such that

$$T(a) = a(\alpha) \quad \text{ for } a \in A(\mathbb{D}).$$

Since T is continuous and the restrictions of the functions from $A(\mathbb{D})$ are dense in $\mathcal{A}$ it follows that $\alpha \in K$ and

$$T(a) = a(\alpha) \quad \text{ for } a \in \mathcal{A}. \quad \square$$

CASE Te = 0. Assume that $T \circ F : \mathcal{A} \to \mathbb{C}$ is not surjective; without loss of generality we may assume that $T \circ F$ does not assume value 1. As before, we first consider the case when $\mathcal{A}$ is the disc algebra $A(\mathbb{D})$.

Fix a nonzero complex number $\lambda = re^{i\theta}$ and a positive integer n. Let Ω be a region in $\mathbb{C}$ such that F is injective on the closure $\bar{\Omega}$ of Ω and $F(\bar{\Omega})$ is a closed disc of radius r and center at some point w_0. Notice that unlike in Step 3, the region Ω is selected for the function F, not for the function g. Put

$$f(z) = (F \circ \varkappa(z) - w_0)/r$$

and

$$\psi(z) = \varkappa\left(f^{-1}(e^{i\theta}z^n)\right).$$

We have

$$\psi \in A(\mathbb{D}), \text{ and } F \circ \psi = \lambda \mathbf{Z}^n + w_0.$$

Hence, by our assumptions, we have for arbitrary λ

$$\lambda T(\mathbf{Z}^n) = T(\lambda \mathbf{Z}^n) = T(F \circ \psi - w_0 e) = T(F \circ \psi) \neq 1.$$

Hence

$$T(\mathbf{Z}^n) = 0 \text{ for } n \in N \cup \{0\},$$

so $T = 0$.

Now, if $\mathcal{A}$ is an arbitrary uniform algebra and $\mathcal{A}_0$ its subalgebra with a single generator, it follows from the disc algebra case, exactly as in Step 4, that T is zero on $\mathcal{A}_0$, consequently T is zero on the algebra $\mathcal{A}$. □

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