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Local isometries of function spaces

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Abstract. We show that for a large class of function spaces any isometry that coincides locally with a surjective isometry must be automatically surjective. This class includes finite-codimensional subspaces of $C(X)$ and spaces $C(X, E)$ of E-valued continuous functions for finite-dimensional or uniformly convex and algebraically reflexive E.

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1. Introduction

A linear map T from a Banach space A *into* itself is called a locally surjective isometry if for each $a \in A$ there exists a surjective isometry $T_a : A \to A$ such that $T(a) = T_a(a)$. A Banach space A is called algebraically reflexive if any locally surjective isometry on A is automatically surjective. Clearly not all Banach spaces are algebraically reflexive, for example an infinite-dimensional Hilbert space fails that property. On the other hand any separable Banach space can be renormed so that the group of isometries is trivial [12]. Reflexivity problems for subalgebras of the algebra of all bounded linear maps on a Hilbert space have been among the most active research areas in operator theory. Similar study concerning sets of linear transformations on a Banach space was initiated more recently by Kadison [14] and Larson [15], and continued in a series of papers by Molnàr and others [5, 7, 17–20, 23]. Since we have a precise description of surjective isometries for most of the classical function spaces (see for example [2, 3, 6, 8, 11, 12, 16, 21]), one

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has a good idea of how T locally looks like; getting global conclusions from local hypothesis is a very basic problem.

In this paper we concentrate on locally surjective isometries of spaces of continuous scalar and vector-valued functions. In the first section we prove that any finite-codimensional closed subspace of $C(X)$ is algebraically reflexive provided X is first countable. This extends a result of Molnàr and Zalar [20] on locally surjective isometries of the $C(X)$ spaces for first countable X. If the space X is not first countable, then even $C(X)$ need not be algebraically reflexive [20].

In the following sections we study locally surjective isometries of spaces of vector-valued functions. We show that $C(X,E)$ is algebraically reflexive for any compact first countable set X and any finite-dimensional Banach space E as well as for any uniformly convex and algebraically reflexive space E. We next investigate algebraically reflexive spaces E with trivial centralizer. Under the assumption that $e^*(e) \neq 0$ for all extreme points e of the unit ball of E and for all extreme points e^* of the dual unit ball , we show that any locally surjective isometry on $C(X,E)$ whose adjoint preserves the extreme points, is surjective. We derive a similar result for $\oplus_{\ell^\infty} E$.

In the concluding part of the paper we consider the space $A(K,E)$ of affine E-valued continuous functions defined on a compact convex set K. If K is a Choquet simplex such that the set $\partial_e K$ of the extreme points is a closed set then $A(K,E)$ is isometric to $C(\partial_e K, E)$ via the restriction map. Thus these spaces are a natural generalization of the ones considered before. For a metrizable Choquet simplex K and for a Banach space E satisfying the conditions imposed in the previous paragraph and such that E^* is strictly convex, we show that any locally surjective isometry whose adjoint preserves extreme points is surjective. This gives the vector-valued version of the main theorem in [23].

Unless otherwise stated our results are valid only over the complex scalar field.

2. Notation and introductory results

Our notation and terminology is standard. For a convex set K by $\partial_e K$ we denote the set of extreme points of K. For a Banach space E, E_1denotes the closed unit ball and $\mathcal{I}(E)$ the group of all surjective isometries of E. By $\mathbb{D}$ we denote the closed unit disc. For a function space $A \subset C(X)$ and $x \in X$ we define $\delta_x \in A^*$ by $\delta_x(f) = f(x)$, we put $eX = \partial_e A_1^* \cap \{\delta_x : x \in X\}$.

For a family of Banach spaces $\{W_j\}_{j \in J}$ we denote by $\bigoplus_{j \in J} W_j$ the l^∞-direct sum of the spaces W_j; that means, $\bigoplus_{j \in J} W_j$ is the Banach space of all bounded functions $w : J \to \bigcup_{j \in J} E_j$ such that $w(j) \in E_j$, where

$$\|w\| = \sup\{\|w(j)\| : j \in J\}.$$

For a compact Hausdorff space X and a Banach space E by $C(X,E)$ we mean the space of all continuous E-valued functions on X with the sup-norm: $\|f\| = \sup\{\|f(x)\| : x \in X\}$. In this setting we denote by δ_x the operator from $C(X,E)$ into E defined by $\delta_x(f) = f(x)$.

We will frequently refer in our proofs to the following result from [10] concerning finite codimensional subspaces of $C(X)$.

Theorem 1. *Let X be a compact Hausdorff space and let A be a finite-codi mensional subspace of the complex $C(X)$ space. Assume that any function in A has at least $n \geq 1$ distinct zeros. Then there exists a point $x_1 \in X$ such that $A \subset \{f \in C(X) : f(x_1) = 0\}$. Moreover if X is first countable then there are distinct points $x_1, x_2, ..., x_n$ in X such that $A \subset \{f \in C(X) : f(x_1) = ... = f(x_n) = 0\}$.*

We may notice that a compact space X is first countable if and only if for each point $x \in X$ the set $\{x\}$ is G_δ. We will also refer to the following theorem (see [6], page 147) which describes surjective isometries of $C(X,E)$ spaces.

Theorem 2. *Assume that X is a compact Hausdorff space and E is a Banach space with a trivial centralizer. Then any surjective isometry $\Psi : C(X,E) \to C(X,E)$ is of the form*

$$\Psi f(x) = \tau(x)(f(\psi(x))), \quad \text{for } x \in X, f \in C(X,E),$$

where ψ is a homeomorphism from X onto itself and $\tau : X \to \mathcal{I}(E)$.

3. Finite-codimensional subspaces of $C(X)$

We first need to get a description of *surjective* isometries of finite-codimensional closed subspaces A of $C(X)$ that can be used to investigate locally surjective isometries. Notice that the set X is not uniquely determined by A. Indeed, if $A \subset C(X_1)$ then we can take any functional F on A with $\|F\| \leq 1$, put $X_2 = X_1 \cup \{x_0\}$ and define an isometric embedding J of A into $C(X_2)$ by

$$J(f)(x) = \begin{cases} f(x) & \text{for} \quad f \in A, x \in X_1 \\ F(f) & \text{for} \quad f \in A, x = x_0. \end{cases}$$

Hence to get a reasonable description of a surjective isometry $T : A \to A$ in terms of maps on X we need to assume that the set X is not artificially large. We say that X is the smallest possible for $A \subset C(X)$ if

$$\dim(C(X)/A) = \inf\left\{\dim\left(C\left(X'\right)/J(A)\right) : J : A \to C\left(X'\right) \text{ is an isometric embedding}\right\}.$$

One can show that such a smallest set is unique up to a homeomorphism, however we will not need the uniqueness property here.

Notice that if X is the smallest possible then A separates the points of X since otherwise we could replace X with an appropriate quotient space and lower the dimension of $C(X)/A$. For an x in X we denote by $[x]$ the set of points x' in X such that

$$f(x) = \eta f(x'), \quad \text{for all } f \in A$$

for some complex number $\eta = \eta(x, x')$. Since A is of finite codimension only finitely many of the sets $[x]$ are nonsingletons and all of the sets $[x]$ are finite. Also none of the numbers $\eta = \eta(x, x')$ for $x \neq x'$ has the absolute value one. To show this assume x_1, x_2 are two different points of X such that there is a real number θ with

$$f(x_1) = e^{i\theta} f(x_2), \quad \text{for all } f \in A\ .$$

Let h be a unimodular continuous function on X such that $h(x_1) = 1 = e^{-i\theta} h(x_2)$. Define $S : C(X) \to C(X)$ by $Sf = hf$. For all g in the range of S we have $g(x_1) = g(x_2)$ hence $S(A)$ and consequently A can be isometrically embedded in $C(X')$, where X' is obtained from X by identifying the points x_1 and x_2. This contradicts our assumptions that X is the smallest possible.

Theorem 3. *Assume A is a finite-codimensional closed subspace of $C(X)$, where X is the smallest possible compact set. If $\Phi : A \to A$ is a surjective isometry, then there is a surjective homeomorphism $\varphi : X \to X$ and a continuous unimodular function $\chi \in C(X)$ such that*

(3.1) $$\Phi(f) = \chi f \circ \varphi, \quad \textit{for } f \in A.$$

Proof. For any subspace A of $C(X)$ we have

$$\partial_e A_1^* = \{\lambda \delta_x : x \in eX, |\lambda| = 1\},$$

where $eX = \partial_e A_1^* \cap \{\delta_x : x \in X\}$. Since X is the smallest possible, eX is dense in X, as otherwise we could replace X with the closure of eX. The conjugate map Φ^* is a homeomorphism of $\partial_e A_1^*$ onto itself so

(3.2) $$\Phi^*(\delta_x) = \chi(x)\, \delta_{\varphi(x)}, \quad \text{for } x \in eX,$$

where $\chi(x)$ is a complex-valued unimodular function on X.

To show that $(\chi, \varphi) : eX \to \partial\mathbb{D} \times X$ is continuous and can be extended to a continuous function on X let $\left(x_\gamma^i\right)_{\gamma \in \Gamma}$, $i = 1, 2$ be nets in eX convergent to a point $x_0 \in X$ and such that $\chi\left(x_\gamma^i\right) \to \chi_0^i$, and $\varphi\left(x_\gamma^i\right) \to y_0^i$. We have

$$\Phi^*\left(\delta_{x_\gamma^i}\right) \to \Phi^*(\delta_{x_0}), \quad \text{and } \chi\left(x_\gamma^i\right) \delta_{\varphi\left(x_\gamma^i\right)} \to \chi_0^i \delta_{y_0^i},$$

so

$$\chi_0^1 \delta_{y_0^1} = \chi_0^2 \delta_{y_0^2}.$$

Hence the functionals $\delta_{y_0^1}$ and $\delta_{y_0^2}$ are not only proportional on A but have the same norm which as we noticed earlier is possible only if $y_0^1 = y_0^2$; consequently $\chi_0^1 = \chi_0^2$ as well. We will use the same symbols χ, φ for the extended functions.

To show that φ is injective assume that $\varphi(x_1) = \varphi(x_2)$. We have

$$\delta_{x_1} = (\Phi^*)^{-1} \left(\chi(x_1) \delta_{\varphi(x_1)}\right) = (\Phi^*)^{-1} \left(\frac{\chi(x_1)}{\chi(x_2)} \chi(x_2) \delta_{\varphi(x_2)}\right)$$
$$= \frac{\chi(x_1)}{\chi(x_2)} \delta_{x_2},$$

hence as before the functionals δ_{x_1} and δ_{x_2} are not only proportional on A but have the same norm, so $x_1 = x_2$.

The map φ is clearly surjective since it is continuous on a compact set and its range contains eX which is dense in X. □

Theorem 4. *Assume X is a first countable compact Hausdorff space and A a finite-codimensional subspace of the complex space $C(X)$. Then A is algebraically reflexive.*

Proof. Without loss of generality we may assume that X is the smallest possible for $A \subset C(X)$. Put $n_0 \overset{df}{=} \dim(C(X)/A)$. We show that $\operatorname{card}(X \backslash eX) \leq n_0$. Assume to the contrary that $x_1, x_2, ..., x_{n_0}, x_{n_0+1}$ is a sequence of distinct points in $X \backslash eX$ and let μ_n be a sequence of Borel measures on eX such that

$$\int_X f d\mu_n = f(x_n) \ , \text{ for } f \in A.$$

Notice that

$$A \subset \bigcap_{j=1}^{n_0+1} \ker\left(\delta_{x_j} - \mu_j\right),$$

and that the functionals $\delta_{x_j} - \mu_j \in C(X)^*$ are linearly independent; indeed, if

$$\mu \overset{df}{=} \sum_{j=1}^{n_0+1} \lambda_j \left(\delta_{x_j} - \mu_j\right) = 0 \ ,$$

then $0 = \mu(\{x_j\}) = \lambda_j$, for all j. This is a contradiction since $n_0 = \dim(C(X)/A)$.

Hence eX is dense in X. Otherwise the finite set $X \backslash eX$ would contain an isolated point $\widetilde{x}$ and the restriction map $f \longmapsto f_{|X \backslash \{\widetilde{x}\}}$ would be an isometric embedding of A into $C(X \backslash \{\widetilde{x}\})$ with codimension $n_0 - 1$.

To finish the proof of our theorem assume $T : A \to A$ is a locally surjective isometry. Since A separates the points of X there is at most one point in X where all the functions from A vanish; we define k_0 to be 0 if such a point does not exist, and to be 1 if there is such a point in X, which we would call x_0.

For $x \in X \setminus \{x_0\}$ let

$$A_x = \ker \delta_x \circ T = \{f \in A : Tf(x) = 0\},$$

let k_x be the minimum number of zeros in X for the functions from A_x, and let g_x be any function in A_x such that $\operatorname{card}\left(g_x^{-1}(\{0\})\right) = k_x$. Put $\Psi(x) = g_x^{-1}(\{0\}) \setminus \{x_0\}$; we have $\operatorname{card}(\Psi(x)) = k_x - k_0$. Since T coincides locally with a surjective isometry by Theorem 3 $Tg_x \in \ker \delta_x$ has the same number of zeros in X as g_x so $k_x \geq 1 + k_0$.

Since A_x is finite-codimensional, by Theorem 1 there are exactly k_x points in X where *all* of the functions from A_x are equal to zero. Notice that for any $y \in X$

$$\delta_y \text{ is proportional to } T^*(\delta_x) \quad \text{iff } y \in \Psi(x),$$

hence for any x, $\Psi(x)$ is an equivalence class of the relation $[\cdot]$ so

$$\text{(3.3)} \quad \bigcup \{\Psi(x) : \operatorname{card}(\Psi(x)) > 1\} \subset \bigcup \{[x] : \operatorname{card}[x] > 1\} \overset{df}{=} \Omega.$$

Notice that Ω is equal to the set of points x' in X such that $\delta_{x'}$ is proportional to $\delta_{x''}$ for some $x'' \neq x'$, so Ω is finite. By (3.1) Tg_x has exactly k_x zeros in X. Since Tg_x is equal to zero at each point of $[x]$ it follows that

$$\operatorname{card}[x] \leq \operatorname{card}(\Psi(x)),$$

hence

$$\operatorname{card}\left(\bigcup \{[x] : \operatorname{card}[x] > 1\}\right) \leq \operatorname{card}\left(\bigcup \{\Psi(x) : \operatorname{card}(\Psi(x)) > 1\}\right). \tag{3.4}$$

Combining (3.4) and (3.3) we get

$$\operatorname{card}[x] = \operatorname{card}(\Psi(x)), \quad \text{for } x \in X.$$

Since the elements of $[x]$ have all different norms as functionals on A we can define a function $\psi : X \to X$ such that $\psi(x) \in \Psi(x)$ and the largest element of $[x]$ is mapped onto the largest element of $\Psi(x)$, the second largest onto the second largest, etc. By definition ψ is injective.

From the definition of A_x we have for all $f \in A$

$$Tf(x) = 0 \Longrightarrow f(\psi(x)) = 0, \text{ for any } x \in X,$$

so

$$\ker \delta_{T^*(\delta_x)} \subset \ker \delta_{\psi(x)} \text{ , for any } x \in X.$$

Therefore the above functionals are proportional and we get

$$Tf(x) = \alpha(x) f(\psi(x)), \quad \text{for } x \in X, f \in A, \tag{3.5}$$

where α is a scalar-valued function on X.

Assume $x_\gamma \to x$, $\alpha(x_\gamma) \to \alpha_0 \in \mathbb{C}$, and $\psi(x_\gamma) \to y_0 \in X$. For any $f \in A$ we have

$$\begin{aligned} Tf(x_\gamma) &\to Tf(x) = \alpha(x) f(\psi(x)) \text{ , and} \\ Tf(x_\gamma) &= \alpha(x_\gamma) f(\psi(x_\gamma)) \to \alpha_0 f(y_0). \end{aligned}$$

Hence

$$\alpha(x) f(\psi(x)) = \alpha_0 f(y_0),$$

so $[y_0] = [\psi(x)]$, thus ψ and α are continuous at least on the set

$$\{x \in X : \operatorname{card}[x] = 1\} \cup \{x \in X : x \text{ is an isolated point of } X\}.$$

Put

$$\widetilde{A} = \left\{ f \in A : f_{|\Omega} = 0 \right\}.$$

$\widetilde{A}$ is a finite codimensional subspace of A. Let $\widetilde{X}$ be the set obtained from X by identifying all the points of $\Omega \cup \{x_0\}$. $\widetilde{A}$ can be considered as a subspace of $C\left(\widetilde{X}\right)$. Furthermore since $\psi(\Omega) = \Omega$, ψ induces the quotient function $\widetilde{\psi} : \widetilde{X} \to \widetilde{X}$ such that by (3.5) we have

$$Tf(x) = \alpha(x) f\left(\widetilde{\psi}(x)\right), \quad \text{for } x \in \widetilde{X}, f \in \widetilde{A}. \tag{3.6}$$

On $\widetilde{A} \subset C\left(\widetilde{X}\right)$ the functionals δ_x are not proportional for different points x, and $\widetilde{\psi}$ is continuous. By (3.6) $\widetilde{\psi}$ is a surjective homeomorphism, and α is continuous and unimodular. Hence the formula (3.6) extends the map $T_{|\widetilde{A}}$ to a surjective isometry $\widetilde{T}$ of $C\left(\widetilde{X}\right)$ onto itself. $\widetilde{T}$ maps finite-codimensional subspaces of $C\left(\widetilde{X}\right)$ onto subspaces of the same codimension hence $\widetilde{T}$ maps $\widetilde{A}$ *onto* itself. Since $\dim\left(A/\widetilde{A}\right)$ is equal to $\dim\left(TA/T\widetilde{A}\right) = \dim\left(TA/\widetilde{A}\right)$ it follows that T maps A onto itself. $\square$

4. Local isometries of $C(X, E)$, the finite-dimensional case

In this section we consider local isometries of spaces of E-valued continuous functions for a finite-dimensional space E. Such a space E is obviously algebraically reflexive. We first need a description of surjective isometries of $C(X, E)$ type spaces.

Theorem 5. *Assume that $X_1, ..., X_n$ are compact Hausdorff spaces, $E_1, ..., E_n$ are pairwise nonisometric Banach spaces all with trivial centralizers, and $\Phi : \bigoplus_{j=1}^n C(X_j, E_j) \to \bigoplus_{j=1}^n C(X_j, E_j)$ is a surjective isometry. Then*

$$\Phi f(x) = \tau_j(x)(f(\psi_j(x))),$$
$$\text{for } x \in X_j, j = 1, ..., n, f \in \bigoplus_{j=1}^{n} C(X_j, E_j),$$

where ψ_j is a homeomorphism from X_j onto itself and $\tau_j : X_j \to \mathcal{I}(E_j)$.

Proof. The proof is an easy application of the techniques developed in [6]. First we notice that the centralizer of $\bigoplus_{j=1}^n C(X_j, E_j)$ is equal to $C(X)$, where X is the disjoint union of the sets $X_j, j = 1, ..., n$. Since the centralizer is preserved by a surjective isometry by the Banach-Stone Theorem we get a homeomorphism ψ of X onto itself such that for any $x \in X$, $\Phi f(x)$ depends only on the value of f at the point $\psi(x)$. Hence for each x we get a linear map τ such that $\Phi f(x) = \tau(x)(f(\psi(x)))$. Since Φ is surjective and an isometry, it follows that all $\tau(x)$ are surjective isometries. □

The main results of this section are corollaries of the following more technical theorem which describes locally surjective isometries of $C(X, E)$ spaces. For a topological space X we denote by X_{G_δ} the subset of X consisting of all points $x \in X$ such that $\{x\}$ is a G_δ set.

Theorem 6. *Assume X is a compact Hausdorff space and E is a finite-dimensional complex Banach space with a trivial centralizer. Then for any locally surjective isometry $T : C(X, E) \to C(X, E)$ there are continuous surjective functions $\psi : X \to X$ and $X \ni x \longmapsto S_x \in \mathcal{I}(E)$ such that*

$$Tf(x) = S_x(f(\psi(x))) \ , \text{ for } x \in X, f \in C(X, E).$$

Moreover ψ is injective on the set $\psi^{-1}(X_{G_\delta})$.

Corollary 1. *Assume that $X_1, ..., X_n$ are first countable compact Hausdorff spaces and $E_1, ..., E_n$ are finite-dimensional complex Banach spaces. Then $\bigoplus_{j=1}^n C(X_j, E_j)$ is algebraically reflexive.*

Proof. By ([6], page 110) any finite-dimensional Banach space F is isometric to a finite ℓ^{∞}-direct sum of spaces with trivial centralizer: $F = \bigoplus_{i=1}^{k} F_i$. Since $C(X, \bigoplus_{i=1}^{k} F_i) = \bigoplus_{i=1}^{k} C(X, F_i)$ we may assume without loss of generality that all the spaces E_j have trivial centralizers. If spaces E_{j_1} and E_{j_2} are isometric then $\bigoplus_{s=1}^{2} C(X_{j_s}, E_{j_s})$ is isometric with $C(X', E_{j_1})$, where X' is a disjoint union of X_{j_1} and X_{j_2}. Hence we can also assume that the spaces $E_1, ..., E_n$ are pairwise nonisometric.

Put $A = \bigoplus_{j=1}^{n} C(X_j, E_j)$ and let $T : A \to A$ be a locally surjective isometry. Fix j and let f be an element of A with the support in X_j. Since the space E_j is not isometric to any other $E_{j'}$ space and T is a local isometry, by Theorem 5, the support of Tf is contained in X_j. Hence T defines a local isometry T_j from $C(X_j, E_j)$ into itself. Since X_j-s are first countable, by Theorem 6, all the isometries T_j are surjective. □

Corollary 2. *For a complex finite-dimensional Banach space E the l^{∞}-direct sum $\bigoplus_{j\in J} E$ is algebraically reflexive if and only if J is of nonmeasurable cardinality.*

Proof. We equip J with the discrete topology. It is well known that a set J is of nonmeasurable cardinality if and only if it is realcompact ([9], page 163). On the other hand a point $x_0 \in \beta J \backslash J$ is a G_δ in the set $J \cup \{x_0\}$ if and only if there is a continuous function $f : \beta J \to [0,1]$ such that $|f| < 1$ on J but $f(x_0) = 1$. Hence J is realcompact if and only if for every point $x_0 \in \beta J \backslash J$ the set $\{x_0\}$ is a G_δ in $J \cup \{x_0\}$.

Assume J is realcompact, T is a locally surjective isometry from $\bigoplus_{j\in J} E = C(\beta J, E)$ into itself, and let ψ, S_x be continuous functions on βX given by Theorem 6. Since $J \subset (\beta J)_{G_\delta}$ is dense in βJ, ψ is continuous, and βJ is compact we conclude that ψ is surjective. Since an inverse image under ψ of an isolated point must be isolated it follows that there is a subset J_0 of J such that ψ is a bijection from J_0 onto J. Assume $y \in J \backslash J_0$. By Theorem 6 $\psi(y) \in \beta J \backslash J$. Let $g \in C(\beta J, E)$ be such that $\|g(\psi(y))\| = 1 > \|g(x)\|$ for $x \in J$. Since T is locally surjective, by Theorem 5 the norm of Tg is smaller than one on J. However by Theorem 6 $\|Tg(y)\| = \|g(\psi(y))\|$. The contradiction shows that $J_0 = J$ and consequently T is surjective.

Assume now that J is not realcompact and let $y_0 \in \beta J \backslash J$ be such that $\{y_0\}$ is not G_δ in $J \cup \{y_0\}$. Fix $x_0 \in J$, let $\varphi : \beta J \to \beta(J \backslash \{x_0\}) = \beta J \backslash \{x_0\}$ be defined by

$$\varphi(x) = \begin{cases} x & \text{for} \quad x \in \beta J \backslash \{x_0\} \\ y_0 & \text{for} \quad x = x_0 \end{cases},$$

and define $T : C(\beta(J \backslash \{x_0\}), E) \to C(\beta J, E) \cong C(\beta(J \backslash \{x_0\}), E)$ be $Tf(x) = f(\varphi(x))$. The map T is clearly not surjective. To show it is locally surjective fix $f \in C(\beta(J\backslash\{x_0\}), E)$ and put $f(y_0) = e_0$. Since $\{y_0\}$ is not G_δ in$\{y_0\} \cup J\backslash\{x_0\}$ the set $K \overset{df}{=} f^{-1}(\{e_0\}) \cap J$ is infinite and there is a bijection $\varkappa : J\backslash\{x_0\} \to J$ which is equal to the identity map on $(J\backslash\{x_0\})\backslash K$ and maps K onto $K \cup \{x_0\}$. The map $\Psi(h) \overset{df}{=} h(\varkappa(x))$ is a surjective isometry from $\bigoplus_{j \in J\backslash\{x_0\}} E$ onto $\bigoplus_{j \in J} E$ such that $\Psi(f) = T(f)$. □

Proof of Theorem 6. Let $\{e_1, e_2, ..., e_n\}$ be a basis of vectors in E, put

$$A^j = \{g \otimes e_j \in C(X, E) : g \in C(X)\}$$

and for any $x \in X$ put

$$\begin{aligned} A_x &= \ker(\delta_x \circ T) = \{f \in C(X, E) : Tf(x) = 0\}, \\ A_x^j &= \{g \in C(X) : g \otimes e_j \in A_x\}. \end{aligned}$$

If $f \in C(X, E)$ has norm one at every point of X then by Theorem 2 Tf has the same property. Hence $E \ni e \longmapsto T(\mathbf{1} \otimes e)(x) \in E$ is an isometry which we will denote by S_x. For any nonvanishing $g \in C(X)$ and arbitrary scalars $\alpha_2, ..., \alpha_n$ by Theorem 2 the function

$$g \otimes e_1 + \sum_{j=2}^{n} \alpha_j \mathbf{1} \otimes e_j$$

does not vanish on X so $T(g \otimes e_1)(x) \notin \operatorname{span}\{e_2, ..., e_n\}$. That means that the kernel of the functional F_x^1 defined by

$$C(X) \ni g \longmapsto T(g \otimes e_1)(x) \in \mathbb{C}$$

consists exclusively of noninvertible elements of the algebra $C(X)$. By Theorem 1 there is $\psi_1(x) \in X$ such that $\{g \in C(X) : g(\psi_1(x)) = 0\} = \ker F_x^1$. By the same arguments we get for any $1 \leq j \leq n$ a point $\psi_j(x) \in X$ such that

$$\{g \in C(X) : g(\psi_j(x)) {=} 0\} {=} \{g \in C(X) : T(g \otimes e_j)(x) = 0\} \subset A_x^j.$$

All the points $\psi_j(x)$ must be identical since otherwise we could find a nonvanishing function of the form $\sum_{j=1}^n g_j \otimes e_j$ in A_x contrary to Theorem 2; we denote that common value of $\psi_j(x)$ by $\psi(x)$. We get $\ker \delta_{\psi(x)} \subset \ker(\delta_x \circ T) = A_x$. Hence

$$Tf_1(x) = Tf_2(x) \quad \text{whenever } f_1(\psi(x)) = f_2(\psi(x)),$$

so

$$Tf(x) = S_x(f(\psi(x))) \text{ , for } f \in C(X, E), x \in X. \tag{4.1}$$

Assume there are $x_1 \neq x_2$ in X such that $\psi(x_1) = \psi(x_2) \in X_{G_\delta}$. Let $g_1 \in C(X)$ be such that

$$0 = g_1(\psi(x_1)) < |g_1(x)| \text{ , for } x \neq \psi(x_1).$$

By Theorem 2 $T(g_1 \otimes e_1)$ is equal to zero at exactly one point of X however by (4.1) $T(g_1 \otimes e_1)$ is equal to zero at both x_1 and x_2.

By taking $f = e_1, f = e_2, ..., f = e_n$ we can check that the map $x \longmapsto S_x$ is continuous, consequently the map $x \longmapsto S_x^{-1}$ is also continuous, hence by (4.1) ψ is continuous.

Assume the map ψ is not surjective and let $0 \neq g_0 \in C(X)$ such that $g_0(\psi(X)) = 0$. Since T is an isometry $T(g_0 \otimes e_1) \neq 0$ whereas by (4.1) $T(g_0 \otimes e_1)(x) = S_x(g_0 \otimes e_1(\psi(x)) = 0$. The contradiction shows that ψ is a homeomorphism and T is surjective. □

5. Local isometries of $C(X,E)$, the infinite-dimensional case

We first consider local isometries of $C(X,E)$ for a uniformly convex space E; any such a space E has a trivial centralizer ([6], page 108).

Theorem 7. *Assume X is a first countable compact Hausdorff space and E is a uniformly convex and algebraically reflexive Banach space. Then $C(X,E)$ is algebraically reflexive.*

Proof. Assume $T : C(X,E) \to C(X,E)$ is a locally surjective isometry. For an $x \in X$ we define $T_x : C(X,E) \to E$ by $T_x f = \delta_x \circ T(f) = Tf(x)$. Fix an x_0 in X. We say that x is in the support of T_{x_0} if for any open neighborhood U of x there is an $f \in C(X,E)$ equal to 0 outside U and such that $T_{x_0}(f) \neq 0$.

For $K \subset X$ put

$$\mu(K) = \inf_U \sup_f \|T_{x_0}(f)\|,$$

where the infimum is taken over all open sets U containing K and the sup over all $f \in C(X,E)$ equal to 0 outside U and such that $\|f\| \leq 1$. We show that the support of T_{x_0} contains exactly one point.

Assume first that the support of T_{x_0} is empty. Then there is a finite open cover $\mathcal{U}$ of X such that for any $U \in \mathcal{U}$ and any $f_U \in C(X,E)$ with the support in U we have $T_{x_0}(f_U) = 0$. Let $\sum g_j \equiv 1$ be a partition of unity subordinated to $\mathcal{U}$. For any $f \in C(X,E)$ we have

$$T_{x_0}(f) = T_{x_0}\left(\sum g_j f\right) = \sum T_{x_0}(g_j f) = 0,$$

that is $T_{x_0} = 0$ so all the functions in the range of T are equal to zero at x_0. However by Theorem 2 T maps nonvanishing functions onto nonvanishing functions.

Assume now that two distinct points y_1 and y_2 are both in the support of T_{x_0}. Let $\omega : X \to [0,1]$ be a continuous function such that $\omega(y_1) = 0$ and $\omega(y_2) = 1$. For $r \in [0,1]$ put

$$D_r = \{x \in X : \omega(x) < r\},$$
$$K_r = \{x \in X : \omega(x) = r\}.$$

Assume that $\mu(K_r) > 0$ for uncountably many r. Then for any $N > 0$ there is a finite sequence $0 < a^1 < a^2 < ... < a^s < 1$ such that

$$\sum_{j=1}^{s} \mu(K_{a^j}) > N.$$

Let U_j be pairwise disjoint open neighborhoods of K_{a^j} and let f_j be norm one functions from $C(X,E)$ with supports in U_j and such that

$$\sum_{j=1}^{s} \|T_{x_0}(f_j)\| > N. \tag{5.1}$$

For any sequence α_j of unimodular scalars we have $\left\|\sum_{j=1}^{s} \alpha_j f_j\right\| = 1$ so since E is uniformly convex by (5.1) we get

$$\|T\| \geq \lim_{N\to\infty} \sup_{(\alpha_j)} \left\|\sum_{j=1}^{s} \alpha_j T_{x_0}(f_j)\right\| = \infty.$$

The contradiction shows that for all but at most countably many of $r \in [0,1]$ we have $\mu(K_r) = 0$ and we can select r_0 to be such that $\mu(K_{r_0}) = 0$. Since X is compact and ω is continuous

$$\{\{x \in X : r' < \omega(x) < r''\} : 0 < r' < r_0 < r'' < 1\}$$

is a base of neighborhoods of K_{r_0} so there are r_1, r_2 with $0 < r_1 < r_0 < r_2 < 1$ and such that

$$\mu(\{x \in X : r_1 < \omega(x) < r_2\}) < \varepsilon,$$

where the value of ε will be determined later.

Fix $e_0 \in E$ with $\|e_0\| = 1$. Let g_0, h_0 be nonnegative complex-valued continuous functions on X such that

$$\|g_0\| = 1, \quad g_0 = 1 \text{ on } D_{r_1} \text{ and } g_0 = 0 \text{ outside } D_{r_0},$$
$$\|h_0\| = 1, h_0 = 1 \text{ outside } D_{r_2} \text{ and } h_0 = 0 \text{ on } D_{r_0},$$
$$\|g_0 \pm h_0\| \leq 1.$$

Put

$$f_1 = g_0 \otimes e_0, \qquad f_2 = h_0 \otimes e_0, \quad e_1 = T_{x_0}(f_1), \quad e_2 = T_{x_0}(f_2).$$

Notice that

$$\|e_1\| \leq 1 - \mu(X \backslash D_{r_0}),$$
$$\|e_2\| \leq 1 - \mu(D_{r_0}).$$

By Theorem 2 T maps functions with constant norm onto functions with constant norm so $\|T(\mathbf{1} \otimes e_0)(x_0)\| = 1$ and we have

$$\begin{aligned} 1 &\geq \|e_1 + e_2\| = \|T(f_1 + f_2)(x_0)\| \\ &= \|T(\mathbf{1} \otimes e_0)(x_0) - T(\mathbf{1} \otimes e_0 - f_1 - f_2)(x_0)\| \\ &\geq 1 - \|T_{x_0}(\mathbf{1} \otimes e_0 - f_1 - f_2)\| \geq 1 - \varepsilon, \end{aligned}$$

where the last inequality follows from the fact that the support of $\mathbf{1} \otimes e_0 - f_1 - f_2$ is contained in $\{x : r_1 < d(x, x_0) < r_2\}$.

Let k_0 be a continuous function on X which is equal to 1 on D_{r_1} and -1 outside D_{r_2} and such that $|f_0| = 1$ on all of X. Since $\|T(k_0 \otimes e_0)(x_0)\| = 1$ we get

$$\begin{aligned} 1 &\geq \|e_1 - e_2\| = \|T(f_1 - f_2)(x_0)\| \\ &= \|T(k_0 \otimes e_0)(x_0) + T(f_1 - f_2 - k_0 \otimes e_0)(x_0)\| \\ &\geq 1 - \|T_{x_0}(f_1 - f_2 - k_0 \otimes e_0)\| \geq 1 - \varepsilon. \end{aligned}$$

To summarize we have

$$\begin{aligned} \|e_1\| &\leq 1 - \min\left\{\frac{\mu(D_{r_0})}{2}, \frac{\mu(X \backslash D_{r_0})}{2}\right\} \\ \|e_2\| &\leq 1 - \min\left\{\frac{\mu(D_{r_0})}{2}, \frac{\mu(X \backslash D_{r_0})}{2}\right\} \\ 1 &\geq \|e_1 + e_2\| \geq 1 - \varepsilon, \\ 1 &\geq \|e_1 - e_2\| \geq 1 - \varepsilon. \end{aligned}$$

Since $\min\left\{\frac{\mu(D_{r_0})}{2}, \frac{\mu(X \backslash D_{r_0})}{2}\right\}$ is a fixed positive number and $\varepsilon > 0$ is arbitrary the above contradicts our assumption that E is uniformly convex.

We proved that the support of T_{x_0} is a singleton which we will denote by $\psi(x_0)$. Hence the value of T_{x_0} depends only on the value of f at $\psi(x_0)$ and we get

$$\text{(5.2)} \qquad Tf(x) = S_x(f \circ \psi(x)), \quad \text{for } x \in X, f \in C(X, E)$$

where S_x is a linear map from E into itself.

Let $x \in X$, $e \in E$, and let Φ_e be a surjective isometry of $C(X,E)$ such that $T(\mathbf{1}\otimes e) = \Phi_e(\mathbf{1}\otimes e)$. By Theorem 2

$$S_x(e) = T(\mathbf{1}\otimes e)(x) = \Phi_e(\mathbf{1}\otimes e)(x) = \tau_x(e),$$

where τ_x is a surjective isometry. Since E is algebraically reflexive it follows that S_x is a surjective isometry. It is easy to check now, using (5.2), Theorem 2, and following the same line of arguments as the proof of Theorem 1, that ψ is a homeomorphism, that $x \longmapsto S_x$ is continuous, and consequently T is surjective. □

Up to now the results presented were valid only in the complex case. In the real case we do not even know if the space $C(X)$, for a compact metric space X, is algebraically reflexive - we do not know if the adjoint of a locally surjective isometry on $C(X)$ must preserve the set of extreme points. The following results are valid in both the real and the complex case however we need to impose an additional condition on the space E and on the local isometry under consideration. The ℓ^1 space is an example of a Banach space that satisfies the condition **a** below.

Theorem 8. *Let X be a first countable compact Hausdorff space and E an algebraically reflexive Banach space such that*

a *E has a trivial centralizer and for all $e \in \partial_e E_1 \neq \emptyset$ and all $e^* \in \partial_e E_1^*$ we have $e^*(e) \neq 0$, or*
b *E^* does not contain an isometric copy of the two-dimensional ℓ^1 space.*

Suppose $\Phi : C(X,E) \to C(X,E)$ is a locally surjective isometry such that Φ^* maps the extreme points of the dual unit ball into the same set then Φ is surjective.

Proof. By [13] any Banach space with nontrivial centralizer contains an isometric copy of the two-dimensional ℓ^∞ space, hence the assumption b implies that E has a trivial centralizer.

Since $\partial_e (C(X,E)^*) = \{\delta_x \otimes e^* : x \in X, e^* \in E_1^*\}$ by our assumption there are functions $\varphi : X \times \partial_e E_1^* \to X$, and $\Upsilon : X \times \partial_e E_1^* \to \partial_e E_1$ such that

(5.3) $\Phi^*(\delta_x \otimes e^*) = \delta_{\varphi(x,e^*)} \otimes \Upsilon(x,e^*)$, for all $x \in X, e^* \in \partial_e E_1^*$.

We show that the map φ does not depend on the second coordinate. Suppose that for some $x \in X$ and $e_1^*, e_2^* \in \partial_e E_1^*$ we have

$$\Phi^*(\delta_x \otimes e_1^*) = \delta_{y_1} \otimes e_1^{*\prime} \text{ , and}$$
$$\Phi^*(\delta_x \otimes e_2^*) = \delta_{y_2} \otimes e_2^{*\prime},$$

where $y_1 \neq y_2$. Then for all complex numbers α, β we have

$$\begin{aligned} |\alpha| + |\beta| &\geq \|\alpha e_1^* + \beta e_2^*\| = \|\alpha\delta_x \otimes e_1^* + \beta\delta_x \otimes e_2^*\| \\ &\geq \|(\alpha\delta_x \otimes e_1^* + \beta\delta_x \otimes e_2^*) \circ \Phi\| \\ &= \|\Phi^* (\alpha\delta_x \otimes e_1^* + \beta\delta_x \otimes e_2^*)\| \\ \left\|\alpha\delta_{y_1} \otimes e_1^{*\prime} + \beta\delta_{y_2} \otimes e_2^{*\prime}\right\| &= |\alpha| + |\beta| \ . \end{aligned}$$

Hence $\operatorname{span}\{e_1^*, e_2^*\}$ is isometric to the two-dimensional ℓ^1 space which contradicts the assumption **b**.

Assume now that the Banach space E satisfies the assumption **a**. Fix an $e \in \partial_e E_1$ and $f \in C(X)$ with $f(y_1) = 1$ and $f(y_2) = 0$. Let Ψ be a surjective isometry of $C(X, E)$ onto itself satisfying $\Psi(f \otimes e) = \Phi(f \otimes e)$. By Theorem 2 there is a homeomorphism ψ of X onto itself and $\tau : X \to \mathcal{I}(E)$ such that

$$\begin{aligned} f(y_i)\, e_i^{*\prime}(e) &= \left(\delta_{y_i} \otimes e_i^{*\prime}\right)(f \otimes e) = e_i^*(\Phi(f \otimes e)(x)) \\ &= e_i^*(\Psi(f \otimes e)(x)) = f(\psi(x)) e_i^*(\tau(x)(e)) \,. \end{aligned}$$

Hence putting $i = 2$ we get

$$f(\psi(x)) = 0.$$

Since $\tau(x)$ is an isometry, it maps extreme points onto extreme points and by our assumption $e_1^{*\prime}(e) \neq 0$ so $i = 1$ gives

$$f(\psi(x)) \neq 0.$$

The contradiction shows again that $y_1 = y_2$. Thus we can define $\phi : X \to X$ by $\phi(x) = \varphi(x, e^*)$, where e^* is an arbitrary element of $\partial_e E_1^*$. From (5.3) we have

$$\begin{aligned} & e^*(\Phi(g)(x)) = \Upsilon(x, e^*)(g(\phi(x))) \ , \\ (5.4) \qquad & \text{for} \quad x \in X, g \in C(X, E), e^* \in \partial_e E_1^* \,. \end{aligned}$$

We shall show that ϕ is a homeomorphism of X onto itself.

To see that ϕ is one-to-one suppose that $x_1 \neq x_2$. Let $g \in C(X, E)$ be a function that vanishes only at the point $\phi(x_1)$. By (5.4) $\Phi(g)(x_1) = 0$ and by Theorem 2 x_1 is the only point where $\Phi(g)$ is equal to zero. Hence $\Phi(g)(x_2) \neq 0$, so by (5.4) $g(\phi(x_2)) \neq 0$ and consequently $\phi(x_1) \neq \phi(x_2)$.

Assume $y_0 \notin \phi(X)$ and let $g \in C(X, E)$ be a norm one function such that $\|g(y)\| = 1$ only for $x = y_0$. By (5.4) $\|\Phi(g)(x)\| < 1$ for any $x \in X$, contrary to the assumption that Φ is an isometry. Hence ϕ is surjective.

For any fixed $e^* \in \partial_e E_1^*$ the map $\gamma : X \to \delta(X) \otimes \partial_e E_1^*$, defined by $\gamma(x) = \delta(x) \otimes e^*$, is continuous (the range set is equipped with the w^*-topology) and so is the projection map $p : \delta(X) \otimes \partial_e E_1^* \to X$ defined by

$p(\delta(x)\otimes e^*)=x$. Since Φ^* is w^*-continuous, we get that $\phi=p\circ\Phi^*\circ\gamma$ is continuous, and consequently a surjective homeomorphism.

Suppose $g(\phi(x))=0$. By (5.4) for any $e^*\in\partial_e E_1^*$ we have $e^*(\Phi(g)(x))=\Upsilon(x,e^*)(g(\phi(x)))=0$, so $\Phi(g)(x)=0$. Thus

(5.5) $\{g\in C(X,E):g(\phi(x))=0\}\subset\{g\in C(X,E):\Phi(g)(x)=0\}$.

For any $y\in X$ we define $S_y:E\to E$ by $S_y(e)=(\Phi(g))(\phi^{-1}(y))$, where $g\in C(X,E)$ is such that $g(y)=e$. Note that if $g_1(y)=g_2(y)$ then $(g_1-g_2)(y)=0$, so by (5.5) $\Phi(g_1-g_2)(\phi^{-1}(y))=0$, therefore $(\Phi(g_1))(\phi^{-1}(y))=(\Phi(g_2))(\phi^{-1}(y))$, which means that S_y is well defined. Clearly $\|S_y\|\leq 1$ and by (5.4) we get

(5.6) $\Phi(g)(x)=S_{\phi(x)}(g(\phi(x)))$, for $g\in C(X,E), x\in X$.

Fix $y_0\in X$ and $e\in E$. Let $f\in C(X)$ be such that $\|f\|=|f(y_0)|=1>|f(y)|$ for $y\neq y_0$. By Theorem 2 the norm of $\Phi(f\otimes e)$ is equal to one at exactly one point of X, and by (5.6) this can only be the point $\phi^{-1}(y_0)$. Let $\Psi:C(X,E)\to C(X,E)$ be a surjective isometry such that $\Phi(f\otimes e)=\Psi(f\otimes e)$. By Theorem 2 there is a surjective isometry $\tau:E\to E$ such that $S_{y_0}(e)=\Phi(f\otimes e)\left(\phi^{-1}(y_0)\right)=\Psi(f\otimes e)\left(\phi^{-1}(y_0)\right)=\tau(e)$. Hence S_{y_0} is a locally surjective isometry, since by our assumption E is algebraically reflexive it follows that S_{y_0} is surjective. From (5.6) we now get that Φ is surjective. □

Notice that in the proof of the above theorem we have only used the fact that Φ is a local isometry at the points of $C(X)\otimes E$. As the next result shows, it is possible to remove all the conditions on E if we make a stronger assumption about the local properties of the isometry in question.

Theorem 9. *Let X be first countable compact Hausdorff space, E a Banach space, and Φ a linear map from $C(X,E)$ into itself. Assume that Φ^* maps the set of the extreme points onto itself and that for any $f\in C(X,E)$ there is a homeomorphism ψ_f of X and a unimodular continuous scalar-valued function $\varkappa_f$ on X with $\Phi(f)=\varkappa_f f\circ\psi_f$. Then Φ is surjective.*

Proof. Since Φ^* preserves the set of extreme points, there are maps $\psi:\partial_e E_1^*\times X\to X$ and $\varphi:X\times\partial_e E_1^*\to\partial_e E_1^*$ such that $e^*(\Phi(f)(x))=(\varphi_x(e^*))(f(\psi_{e^*}(x)))$, for $e^*\in\partial_e E_1^*$. On the other hand, according to our assumptions we have

(5.7) $e^*(\Phi(f)(x))=e^*(\varkappa_f(x)f\circ\psi_f(x))=\varkappa_f(x)e^*(f\circ\psi_f(x))$.

Hence

$$(\varphi_x(e^*))(f(\psi_{e^*}(x)))=\varkappa_f(x)e^*(f\circ\psi_f(x)) \text{ , for } e^*\in\partial_e E_1^*, x\in X .$$

For $f = \mathbf{1} \otimes e$ we get $(\varphi_x(e^*))(e) = \varkappa_{\mathbf{1}\otimes e}(x)\, e^*(e)$. Since $\varkappa_{\mathbf{1}\otimes e}$ is unimodular, it follows that $\ker e^* = \ker \varphi_x(e^*)$ for any $x \in X$. Consequently, the functionals e^* and $\ker \varphi_x(e^*)$ are proportional and we get

$$e^*(\Phi(f)(x)) = \tau_x e^*(f(\psi_{e^*}(x))), \tag{5.8}$$

where $|\tau_x| = 1$. Applying (5.7) and (5.8) to a function $f = g \otimes e$, where $g \in C(X)$, we get

$$e^*(e)(\tau_x g \circ \psi_{e^*}(x) - \varkappa_{g\otimes e}(x)\, g \circ \psi_{g\otimes e}(x)) = 0.$$

Since for arbitrary $e_1^*, e_2^* \in \partial_e E_1^*$ we can always find $e \in E$ such that $e_1^*(e) \neq 0 \neq e_2^*(e)$, the above formula gives

$$g\left(\psi_{e_1^*}(x)\right) = 0 \Leftrightarrow g(\psi_{g\otimes e}(x)) = 0 \Leftrightarrow g\left(\psi_{e_2^*}(x)\right) = 0,$$

and consequently $\psi_{e_1^*}(x) = \psi_{e_2^*}(x)$. We now conclude from (5.8) that Φ is given by a composition with a homeomorphism followed by a multiplication by a unimodular function. □

In the complex case we can get a much stronger result by removing the assumption that Φ^* preserves the extreme points.

Theorem 10. *Let X be first countable compact Hausdorff space, E a complex Banach space, and Φ a linear map from $C(X, E)$ into itself such that for any $f \in C(X, E)$ there is a homeomorphism ψ_f of X and a unimodular continuous scalar-valued function $\varkappa_f$ on X with $\Phi(f) = \varkappa_f f \circ \psi_f$. Then Φ is surjective.*

Notice that in the special case, when $\dim E = 1$ and X is first countable, we get from the above proposition that $C(X)$ is algebraically reflexive.

Proof. Fix a nonzero vector $e \in E$ and $x \in X$. For any nonvanishing function $g \in C(X)$ the function $\Phi(g \otimes e)$ does not vanish, so $\Phi(g \otimes e)(x)$ has the same property which means that the kernel of the functional $g \longmapsto \Phi(g \otimes e)(x)$ consists of noninvertible elements. By Theorem 1 there is $\psi_e(x) \in X$ such that

$$\{g \in C(X) : \Phi(g \otimes e)(x) = 0\} = \{g \in C(X) : g(\psi_e(x)) = 0\}.$$

Hence

$$\Phi(g \otimes e)(x) = \varkappa_e(x)\, g(\psi_e(x)) \otimes e\ , \text{ for } g \in C(X), x \in X\ , \tag{5.9}$$

where $\varkappa_e$ is a scalar-valued function on X. By our assumptions Φ is an isometry, so it is easy to check that $\varkappa_e$ is unimodular and that both ψ_e and $\varkappa_e$ must be continuous.

If for vectors $e_1, e_2 \in E$ we had $\psi_{e_1}(x) \neq \psi_{e_2}(x)$ then we could construct a function $f = g_1 \otimes e_1 + g_2 \otimes e_2$ not vanishing on X but such that $\Phi(f)(x) = 0$ which would contradict our assumption. Since the linear combinations of the functions of the form $g \otimes e$ are dense in $C(X, E)$, by (5.9) the value of Φf at x depends only on the value of f at $\psi(x)$ and there are linear maps $S_x : E \to E$ such that

$$\Phi f(x) = S_x(f(\psi(x))) \ \text{, for } f \in C(X, E), x \in X \text{ .}$$

Again by (5.9) and our assumption, any vector in E is an eigenvector of S_x so S_x is a multiple of the identity map. □

Remark 1. In all of the proofs of this section we used the assumption that X is first countable only to show that the map ψ is injective. Without that assumption our arguments show that ψ is injective on the set $\psi^{-1}(X_{G_\delta})$. Using this observation one can show that any isometry Φ of the l^∞ direct sum $\bigoplus_{j\in J} E$ is surjective provided E and/or Φ satisfy the assumptions of one of the theorems of this section and J is of nonmeasurable cardinality. This is in spite of the fact that $\bigoplus_{j\in J} E$ is not identical with $C(\beta J, E)$ for an infinite-dimensional space E. The next proposition shows that if $\mathrm{card} J = \aleph_0$ we can get a much stronger result.

Proposition 1. *Let E be a Banach space with a trivial centralizer. Then E is algebraically reflexive if and only if the l^∞ direct sum $\bigoplus_{n\in\mathbb{N}} E$ is algebraically reflexive.*

Proof. One implication is obvious: if $T : E \to E$ is a nonsurjective locally surjective isometry then $(e_n)_{n=1}^\infty \longmapsto (Te_n)_{n=1}^\infty$ defines a nonsurjective locally surjective isometry of $\bigoplus_{n\in\mathbb{N}} E$.

Assume $\Phi : \bigoplus_{n\in\mathbb{N}} E \to \bigoplus_{n\in\mathbb{N}} E$ is a locally surjective isometry and that E is algebraically reflexive. Any surjective isometry $S : \bigoplus_{n\in\mathbb{N}} E \to \bigoplus_{n\in\mathbb{N}} E$ is of the form

$$S((e_n)_{n=1}^\infty) = \left(S_n e_{\pi(n)}\right)_{n=1}^\infty, \tag{5.10}$$

where $S_n : E \to E$ are surjective isometries and π is a permutation of $\mathbb{N}$ [6]. Hence $\Phi\left(\chi_{\{n\}} \otimes e\right) = \chi_{\{\varphi(n,e)\}} \otimes \psi_n(e)$, where $\chi_{\{k\}} : \mathbb{N} \to \{0, 1\}$ is a characteristic function of the set $\{k\}$ and ψ_n a map from E into itself. Since ∂E_1 is connected and Φ is continuous, it follows that for a fixed value of $n \in \mathbb{N}$ the set $\left\{\chi_{\{\varphi(n,e)\}} \otimes \psi_n(e) : n \in \mathbb{N}, e \in \partial E_1\right\}$ is connected. However

$$\left\|\chi_{\{\varphi(n,e)\}} \otimes \psi_n(e) - \chi_{\{\varphi(n,e')\}} \otimes \psi_n\left(e'\right)\right\| = 1$$

when $\varphi(n, e) \neq \varphi(n, e')$ so the function φ does not depend on the second coordinate and we get

$$\Phi\left(\chi_{\{n\}} \otimes e\right) = \chi_{\{\varphi(n)\}} \otimes \psi_n(e) \ \text{, for } n \in \mathbb{N}, e \in E.$$

The map ψ_n is a locally surjective isometry of E so by our assumption must be surjective. Put $\varphi(\mathbb{N}) = \Omega \subset \mathbb{N}$, fix $e_0 \in \partial E_1$ and define $\mathbf{f} = (f_n)_{n=1}^{\infty} \in \bigoplus_{n\in\mathbb{N}} E$ by $f_n = \frac{e_0}{n}$. Since $\mathbf{f}$ does not vanish on $\mathbb{N}$ by (5.10) the same is true for $\Phi(\mathbf{f})$. On the other hand $\mathbf{f}$ is a limit of a sequence of linear combinations of elements of the form $\chi_{\{n\}} \otimes e$ so the support of $\Phi(\mathbf{f})$ is contained in Ω. Hence φ and consequently Φ are surjective. $\square$

6. Local isometries of $A(K, E)$

In this section we discuss local isometries of spaces $A(K, E)$ of affine continuous functions taking values in a Banach space E and defined on a Choquet simplex K; we equip $A(K, E)$ with the supremum norm. It is easy to show that if $\partial_e\ K$ is a closed subset of K then the restriction map is an isometry of $A(K, E)$ onto $C(\partial_e K, E)$, thus these spaces are a natural generalization of the ones considered in the previous sections. When K is metrizable, $A(K)$ has the metric approximation property and $A(K, E)$ can also be identified as the injective tensor product space $A(K) \otimes_\epsilon E$ [22]. It can be deduced from the main theorem of [11] that if Ψ is a surjective isometry of such a space, with E^* strictly convex, then there is an affine homeomorphism ψ of K and a weight function $\tau : \partial_e K \to \mathcal{I}(E)$ such that for any $k \in \partial_e K$ and $a \in A(K, E)$ we have $\Psi(a)(k) = \tau(k)(a(\psi(k)))$. We also note that since strtict convexity of E^* implies smoothness of E , it follows from Corollary 4.23 of ([6]) that E has trivial centralizer.

Theorem 11. *Assume K is a metrizable Choquet simplex and E an algebraically reflexive Banach space with E^* is strictly convex . Then any locally surjective isometry $\Phi : A(X, E) \to A(X, E)$ whose adjoint Φ^* preserves the set of the extreme points of the unit ball is surjective.*

Proof (sketch). The theorem can be proven by combining the arguments used in the proof of Theorem 8 and of [23] and using the description of surjective isometries of $A(X, E)$ mentioned before.

We first notice that

$$\partial_e A(K, X)_1^* = \{\delta_k \otimes e^* : k \in \partial_e K, e^* \in \partial_e E_1^*\}$$

and that E statisfies the condition b of Theorem 8. Since Φ^* preserves the set of the extreme points of the unit ball we have $\Phi^*(\delta_k \otimes e^*) = \delta_{k'} \otimes \Upsilon_k^*(e^*)$, where as in the proof of Theorem 8, we can check that the point $k' \in \partial_e K$ depends only on the choice of $k \in \partial_e K$ but not on the choice of $e^* \in \partial_e E_1^*$. Still following the same line of arguments we verify that the function $\phi : \partial_e K \to \partial_e K$ defined by $k \longmapsto k'$ is a bijection. Since K is metrizable, ϕ is a Borel isomorphism (see [4]). When E is the scalar field, since Φ^* can also be assumed to preserve the state space, the function ϕ has an affine continuous extension to a homeomorphism from K onto itself which concludes our proof. If E is an arbitrary Banach space satisfying our assumptions then, since K is a metrizable Choquet simplex, for any $k \in K$ there is a unique probability measure μ on K with $\mu(\partial_e K) = 1$ representing δ_k. Define $\phi(k) = \mu \circ \phi^{-1}$. It is fairly routine to verify that ϕ is continuous, affine, and surjective. The fact that ϕ is injective on K can be proved by using arguments similar to the ones given in [23]. Thus, we conclude that Φ is surjective. □

In general the adjoint of a locally surjective isometry may not preserve the set of extreme points; to get the simplest example one may take a non-surjective isometry of a Hilbert space. We do not know if such a situation may happen for function spaces considered here. The following example illustrates a situation where the adjoint of a locally surjective isometry of an $A(K,E)$ space must preserve the set of extreme points.

Example 1. Let $A = \left\{ g \in C(\mathbb{N} \cup \{\infty\}) : g(\infty) = \sum_{n=1}^{\infty} \mu_n g(n) \right\}$, where $\mu = (\mu_n)_{n=1}^{\infty}$ is a probability measure on $\mathbb{N}$, that is $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n > 0$. The state space of A, which we denote by K, is a Choquet simplex with $\partial_e K = \{\delta_n : n \in \mathbb{N}\}$. Let E be an algebraically reflexive Banach space with trivial centralizer and note that $A(K,E)$ is isometric to $\left\{ f \in C(\mathbb{N} \cup \{\infty\}, E) : f(\infty) = \sum_{n=1}^{\infty} \mu_n f(n) \right\}$. We claim that $A(K,E)$ is algebraically reflexive, in particular for any locally surjective isometry map $\Phi : A(K,E) \to A(K,E)$ the adjoint map Φ^*preserves the set of extreme points. One can verify this claim following almost exactly the proof of Proposition 1. The only difference is that we need to replace the functions that are supported by a single point of the set $\mathbb{N}$ by functions supported by two points - a function of the form $\chi_{\{n\}} \otimes e$ may not belong to $A(K,E)$ however for any $n_1, n_2 \in N$ and any $e \in E$ we can always find numbers α_1, α_2 not both equal to zero and such that $\alpha_1 \chi_{\{1\}} \otimes e + \alpha_2 \chi_{\{2\}} \otimes e \in A(K,E)$.

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