NOWHERE LOCALLY UNIFORMLY CONTINUOUS FUNCTIONS

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September 1, 1995

Abstract. Suppose X is a metric space that is nowhere locally compact. Then there is a bounded continuous real-valued function on X that is nowhere locally uniformly continuous.

We also provide an explicit example of such a function on an arbitrary separable infinite dimensional normed space.

Introduction

A function $f : X \to Y$ between metric spaces is *locally uniformly continuous at a point* x if there is an open neighborhood U of x on which f is uniformly continuous. A. Izzo [1] proved that on every infinite-dimensional separable normed space X there exists a bounded continuous real-valued function that is nowhere locally uniformly continuous. A very similar proof can be repeated for any separable metric space that is nowhere locally compact. In this note we prove, using different methods, that the separability assumption can be omitted. We also provide an explicit example of a continuous, nowhere locally continuous function on an arbitrary separable infinite dimensional normed space.

We use the standard notation. For a bounded continuous function $f : X \to \mathbb{R}$ we put $\|f\| = \sup\{|f(x)| : x \in X\}$, and for $x \in X$, $r > 0$, $B_x(r) = \{y \in X : d(x, y) < r\}$.

Function on metric spaces

Theorem . *Suppose X is a metric space that is nowhere locally compact. Then the set of bounded nowhere locally uniformly continuous functions is dense in the space of bounded continuous real-valued functions with the supremum norm.*

Lemma . *Let X be a nowhere locally compact metric space. Then for any open ball $B_{x_0}(r)$ in X, there is a continuous bounded real-valued function f on X, and there are two sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in $U = B_{x_0}(\frac{r}{2})$ such that*

$$\begin{aligned} f &\equiv 0 \text{ off } U, \\ \|f\| &= 1 = f(x_n) = -f(y_n), \text{ for } n \in \mathbb{N}, \text{ and} \\ d(x_n, y_n) &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Proof of the Lemma. Since X is not locally compact at x_0 there is a sequence $(x_n)_{n=1}^{\infty}$ in U with no convergent subsequence. Since X is nowhere locally compact, no point of X is isolated, and for any $n \in \mathbb{N}$ given x_n there is a y_n in U such that $0 < d(x_n, y_n) < \frac{1}{n}$. The set $S = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ is a discrete and closed subset of X. Hence the function that is 1 at each x_n, -1 at y_n, and 0 on $X \setminus U$ is continuous. By the Tietze extension theorem this function extends, with the same norm, to a continuous function on X. □

Proof of the Theorem. Let Ω_1 be the set of unions of pairwise disjoint open unit balls in X. Ω_1 is partially ordered by inclusion and by the Zorn's Lemma it has a maximal element. Let $\mathcal{U}_1$ be one of the maximal sets. We apply the Lemma to each of the balls from $\mathcal{U}_1$ to get a family of continuous functions. The supports of these functions are contained in the balls of radius $\frac{1}{2}$, and the corresponding unit balls are disjoint. Let f_1 be the sum of all the functions from that family. Obviously f_1 is continuous.

Let $\mathcal{U}_2$ be a maximal subset of X such that:

$\mathcal{U}_2$ is a union of pairwise disjoint open balls in X and for each ball $B_x(r)$ from $\mathcal{U}_2$,

$$r = \min\left\{\frac{1}{4}, \max\left\{\rho > 0 : |f_1(a) - f_1(b)| \le \frac{1}{4^2} \text{ for any } a, b \in B_x(\rho)\right\}\right\}.$$

Notice that since f_1 is continuous at x, the *maximum* in the above formula is positive, and that, if $r < \frac{1}{4}$, then

$$\max\{|f_1(a) - f_1(b)| : \ a, b \in B_x(2r)\} \ge \frac{1}{4^2}.$$

We apply the Lemma to each of the balls from $\mathcal{U}_2$ to get a family of continuous functions with supports in the disjoint (smaller) balls. Let f_2 be the sum of all the functions from that family. Obviously f_2 is continuous separately on each of the balls from $\mathcal{U}_2$, and zero on the complement of $\bigcup \mathcal{U}_2$. Let $(b_n)_{n=1}^{\infty}$ be a sequence of points from the distinct balls: $B_{a_n}(r_n)$ from $\mathcal{U}_2$, and such that $f(b_n) \neq 0$, for all $n \in \mathbb{N}$. It follows that $b_n \in B_{a_n}\left(\frac{r_n}{2}\right)$. Assume that the sequence $(b_n)_{n=1}^{\infty}$ is convergent to $b_0 \in X$. Then $r_n \to 0$, and any neighborhood of b_0 contains all but finitely many $B_{a_n}(2r_n)$; so the variation of f_1 is at least $\frac{1}{4^2}$, in any neighborhood of b_0. This contradicts the continuity of f_1, and in turn, proves the continuity of f_2. Notice that since $\mathcal{U}_2$ is maximal, the functions f_1, f_2 have the following property:

> Any ball of radius $2 \cdot \frac{1}{4}$ contains a ball V (from $\mathcal{U}_2$) such that for any $\varepsilon > 0$, there are two points x, y in V with $d(x, y) < \varepsilon$ and $f_2(x) = -f_2(y) = \|f_2\| = 1$, and for any two points a, b in V, $|f_1(a) - f_1(b)| \le \frac{1}{4^2}$.

In general for any $n \in \mathbb{N}$, let $\mathcal{U}_{n+1}$ be a maximal subset of X that is a union of pairwise disjoint open balls in X such that for each ball $B_x(r)$ from $\mathcal{U}_{n+1}$ we have,

$$r = \min\left\{\frac{1}{4^n}, \max\left\{\rho > 0 : \max_{1\le j\le n}\{|f_j(a) - f_j(b)|\} \le \frac{1}{4^{n+1}} \text{ for any } a, b \in B_x(\rho)\right\}\right\}.$$

As before, using the Lemma, we construct a continuous function f_{n+1} on X such that:

> Any ball of radius $\frac{2}{4^n}$ contains a ball V (from U_{n+1}) such that for any $\varepsilon > 0$, there are two points x, y in V with $d(x, y) < \varepsilon$ and $f_{n+1}(x) = -f_{n+1}(y) = \|f_{n+1}\| = 1$, and for any two points a, b in V, and for any $j = 1, ..., n$, we have $|f_j(a) - f_j(b)| \le \frac{1}{4^{n+1}}$.

Put $f = \sum_{n=1}^{\infty} \frac{f_n}{4^n}$. We show that f is nowhere locally uniformly continuous. Let $x_0 \in X$ and let $G = B_{x_0}(r)$ be a neighborhood of x_0. Let $4^k > r$. By the above listed property of $(f_n)_{n=1}^{\infty}$, for $n + 1 = k$, there is a subset V of G such that for any $\varepsilon > 0$ there are two points x, y in V with $d(x, y) < \varepsilon$ and

$$\begin{aligned}
|f(x) - f(y)| &\ge \frac{1}{4^k}|f_k(x) - f_k(y)| - \sum_{j=1}^{k-1}\frac{1}{4^j}|f_j(x) - f_j(y)| - \sum_{j=k+1}^{\infty}\frac{1}{4^j}|f_j(x) - f_j(y)| \\
&\ge \frac{2}{4^k} - \sum_{j=1}^{k-1}\frac{1}{4^j}\frac{1}{4^k} - \sum_{j=k+1}^{\infty}\frac{2}{4^j} \\
&\ge \frac{2}{4^k} - \frac{1}{3\cdot 4^k} - \frac{2}{3\cdot 4^k} \\
&= \frac{1}{4^k}.
\end{aligned}$$

Since k is fixed it shows that f is not uniformly continuous on G.

We proved the existence of a bounded continuous nowhere locally uniformly continuous function. It remains to show that such functions are dense. Let g be a bounded continuous function on X and let $\varepsilon > 0$. By Theorem 4 of [1] there is a locally uniformly continuous function $\widetilde{g}$ such that $\|g - \widetilde{g}\| \le \frac{\varepsilon}{2}$. Put $h = \widetilde{g} + \varepsilon\frac{f}{2}$, where f is, as before, a continuous nowhere locally uniformly continuous function of norm one on X. We have $\|g - h\| \le \varepsilon$, where h is continuous and nowhere locally uniformly continuous. $\square$

Functions on normed spaces

In this section, for a given separable infinite dimensional normed space X, we give a specific example, of a continuous, nowhere uniformly continuous real valued function on X. Obviously such function can not be defined solely in terms of the norm; we shall define it in terms of a "coordinate system" this is in terms of a complete biorthogonal system in X. To simplify the construction we first provide a formula for such a function on l^∞, the nonseparable Banach space of all bounded sequences, then move the construction into an arbitrary infinite dimensional separable normed space.

Let $\mathcal{L}$ be the subset of l^∞ consisting of all sequences with integer terms. For any $\alpha \in \mathbb{R}$, let $\alpha\mathcal{L} = \{\alpha x : x \in \mathcal{L}\}$ and let $E(\alpha)$ be the integer part α. For $a = (a_n)_{n=1}^\infty \in l^\infty$ let $E(a) = (E(a_n))_{n=1}^\infty$.

Let f be a continuous function from l^∞ into $[0,1]$ such that

$$f(\frac{1}{10}e_n) = 1, \text{ for } n \in \mathbb{N},$$

and

$$f(a) = 0 \text{ if } \left\| a - \frac{1}{10}e_n \right\| \geq \frac{1}{n+10} \text{ for all } n \in \mathbb{N},$$

and such that f is uniformly continuous on each ball $B_{e_n}(\frac{1}{10})$, $n \in \mathbb{N}$. For example we may put

$$f(a) = \max\left\{0, \max_{n\in\mathbb{N}}\left\{1 - (n+10)\left\|a - \frac{1}{10}e_n\right\|\right\}\right\}.$$

We define a continuous function g on l^∞ by $g(a) = f\left(a - E(a + \frac{1}{2})\right)$. Notice that

1. g is continuous, nonnegative, and is bounded by 1,
2. g is uniformly continuous on any ball of radius $\frac{1}{10}$ since any such ball intersects at most one ball with radius $\frac{1}{10}$, centered at $\left(\frac{1}{10} + k\right)e_n$, for $k, n \in \mathbb{N}$,
3. for any $\varepsilon > 0$ and for any ball B with radius one and center at a point a_0 from $\mathcal{L}$ there is an n and $\alpha, \beta \in \left[0, \frac{1}{10}\right]$ with $|\alpha - \beta| < \varepsilon$ and $g(a_0 + \alpha e_n) = 1$, $g(a_0 + \beta e_n) = 0$.

Put

$$h(a) = \sum_{j=1}^{\infty} \frac{1}{10^j} g(10^j a), \qquad \text{for } a \in l^\infty. \tag{1}$$

We show that h is nowhere locally uniformly continuous. To prove this it is enough to show that for any k and any $c \in \frac{1}{10^k}\mathcal{L}$, h is not uniformly continuous on the ball U of radius $\frac{1}{10^k}$ around c. By the #2 above, for $j = 1, ..., k-1$, the functions $a \mapsto \frac{1}{10^j}g(10^j a)$ are uniformly continuous on U. By #1, for any $a, b \in U$ we have

$$\left\| \sum_{j=k+1}^{\infty} \frac{1}{10^j} g(10^j a) - \sum_{j=k+1}^{\infty} \frac{1}{10^j} g(10^j b) \right\| \leq \sum_{j=k+1}^{\infty} \frac{1}{10^j} = \frac{1}{9 \cdot 10^k},$$

and by #3 for any $\varepsilon > 0$ there are two points a, b in U with $\|a - b\| < \varepsilon$ and

$$\left\| \frac{1}{10^k} g(10^k a) - \frac{1}{10^k} g(10^k b) \right\| = \frac{1}{10^k} > \frac{1}{9 \cdot 10^k}.$$

It follows that h is not uniformly continuous on U.

Let now X be an arbitrary infinite dimensional separable normed space. Let $(x_n, x_n^*)_{n=1}^{\infty}$, be a bounded by 2, total and fundamental biorthogonal system in X [2]. That is, for any n, m in $\mathbb{N}$, we have $x_n^*(x_m) = \delta_{nm}$, and $1 = x_n^*(x_n) = \|x_n^*\| \le \|x_n\| \le 2$. Moreover for any $x \in X$, $x_n^*(x) = 0$ for all $n \in \mathbb{N}$ implies $x = 0$, and for any $x^* \in X^*$, $x^*(x_n) = 0$ for all $n \in \mathbb{N}$ implies $x^* = 0$. We define a norm one linear map $J : X \to l^\infty$ by $J(x) = (x_n^*(x))_{n=1}^{\infty}$ and we put

$$H(x) = h \circ J(x), \qquad x \in X,$$

where h is the function on l^∞ given by (1).

We show that H is nowhere locally uniformly continuous. Notice that for any two points a, b in the set

$$\mathcal{A} = J^{-1} \left(\bigcup_{j=1}^{\infty} \frac{1}{10^j} \mathcal{L} \right)$$

both $\frac{1}{2}(a + b)$, and $2a$ are contained in $\mathcal{A}$. Hence the norm closure of $\mathcal{A}$ is a linear subspace of X containing all x_n, so $\overline{\mathcal{A}} = X$. Let $y \in X$ be such that $J(y) \in \frac{1}{10^k}\mathcal{L}$ for some k, and let V be the ball in X of radius $\frac{1}{10}$ centered at y. As before, for $j = 1, ..., k-1$, the functions $a \mapsto \frac{1}{10^j} g(10^j J(a))$ are uniformly continuous on V, for any $x, y \in V$, we have

$$\left\| \sum_{j=k+1}^{\infty} \frac{1}{10^j} g(10^j J(x)) - \sum_{j=k+1}^{\infty} \frac{1}{10^j} g(10^j J(y)) \right\| \le \sum_{j=k+1}^{\infty} \frac{1}{10^j} = \frac{1}{9 \cdot 10^k},$$

and for any $\varepsilon > 0$ there is an n and $\alpha, \beta \in \left[0, \frac{1}{10}\right]$ with $|\alpha - \beta| < \varepsilon$ and

$$\left\| \frac{1}{10^k} g(10^k J(y + \alpha x_n) - \frac{1}{10^k} g(10^k J(y + \beta x_n) \right\| = \frac{1}{10^k} > \frac{1}{9 \cdot 10^k}.$$

Hence H is not uniformly continuous on V.

The author would like to thank A. J. Izzo for his comments on a preliminary version of this paper.

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