Multiplicative functionals and entire functions

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ABSTRACT. Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let T, φ be continouos functionals, where T is linear, and let F be a nonlinear entire function. If $T \circ F = F \circ \varphi$ and $T(e) = 1$ then T is multiplicative.

1. INTRODUCTION

If T is a multiplicative functional on a complex Banach algebra $\mathcal{A}$ with a unit e then $T(e) = 1$, and for any invertible element x of $\mathcal{A}$ we have $T(x) \neq 0$. A. M. Gleason [5] and, independently, J. P. Kahane & W. Żelazko [7] proved that the above property characterizes multiplicative functionals. In fact, they proved even a stronger result:

Theorem 1. *If T is a continuous linear functional on a complex unital Banach algebra $\mathcal{A}$, such that $T(e) = 1$ and $T(\exp x) \neq 0$ for $x \in \mathcal{A}$, then T is multiplicative.*

The above statement can be rephrased in the following equivalent way.

Theorem 2. *If T is a continuous linear functional on a complex unital Banach algebra $\mathcal{A}$ with $T(e) = 1$, and there is a complex valued function φ on $\mathcal{A}$ such that*

$$T(\exp x) = \exp(\varphi(x)) \qquad \text{for } x \in \mathcal{A}, \tag{1}$$

then T is multiplicative.

R. Arens [1] asked if the exponential function in (1) can be replaced by any other entire function F, that is, whether

$$T \circ F = F \circ \varphi \tag{2}$$

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implies multiplicativity of T. Of course, the conjecture fails if F is surjective; in such a case we can take any linear map T and simply define $\varphi(x)$ to be one of the elements of $F^{-1}(T(F(x)))$. However, the function φ so defined may be discontinuous, unless F is linear, that is, of the form $F(z) = \alpha + \beta z$. Consequently, Arens amended his conjecture by requiring that φ be continuous and F not be a polynomial of degree at most 1.

In [1] Arens proved that (2) implies multiplicativity of T if φ is a polynomial of degree more than 1, or if $\mathcal{A}$ is a uniform algebra. Later, C. Badea [2] proved that (2) implies multiplicativity of T for any nonlinear $F(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$ for all $n = 0, 1, \ldots$. In this note we prove the conjecture for all nonlinear entire functions.

The Gleason-Kahane-Żelazko theorem has also been extended in several other directions; a number of problems remains open [6].

2. THE RESULT

Theorem 3. *Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let F be a nonlinear entire function, let T be a linear functional on $\mathcal{A}$, and let φ be a continuous complex valued function on $\mathcal{A}$. Suppose that*

$$T(F(x)) = F(\varphi(x)) \qquad \text{for each } x \in \mathcal{A}. \tag{3}$$

Then $T \equiv 0$ or $T/T(e)$ is multiplicative.

To show the result we need two simple lemmas; the proof of the first one is a minor modification of a part of the proof in [1], page 195.

Lemma 4. *For any nonlinear entire function g there is a real number R_0 such that for any $R > R_0$, and any $z_1, z_2 \in \mathbb{C}$ with $|z_1| = R, |z_2| = 2R$ there exists $w \in \mathbb{C}$ with $|w| \leq R^{2/3}$ and such that $g(w)$ is either z_1 or z_2.*

Proof of the lemma. Assume to the contrary that for any R_0 there are $R > R_0$ and z_1, z_2 with moduli R and $2R$, respectively, and such that for every w with $|w| \leq R^{2/3}$ $g(w)$ is neither z_1 nor z_2. Put $h(z) = \left(g(R^{2/3}z) - z_1\right) / (z_2 - z_1)$. Then for $|z| \leq 1$, $h(z)$ is neither 0 nor 1; moreover, $|h(0)| \leq 1$. By the Schottky's theorem [3], $|h(z)| \leq C$ for $|z| \leq \frac{1}{2}$, where the constant C is independent of R. Hence $\left|g(R^{2/3}z)\right| \leq 4CR$ for $|z| \leq \frac{1}{2}$.

Consequently there is a constant C_1 such that for arbitrary large r

$$|g(u)| \leq C_1 |u|^{1.5} \qquad \text{for } |u| = r. \tag{4}$$

By the Cauchy integral formula $\left|g^{(n)}(0)\right| \leq \frac{1}{2\pi} \int_{|u|=r} \frac{|g(u)|}{u^{n+1}}$, so (4) shows that $g^{(n)}(0) = 0$ for $n > 1$. This proves that g is a polynomial of degree at most 1. □

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we define the maximum modulus M_f and the maximum term μ_f as usual by

$$M_f(R) = \max\{|f(z)| : |z| = R\} \qquad \text{and} \qquad \mu_f(R) = \max\{|a_n|R^n : n = 1, 2, ...\}.$$

Notice that from the Cauchy integral formula for any $n \in \mathbb{N}$ and R we have

$$|a_n| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{|f(z)|}{R^{n+1}} \leq M_f(R)/R^n,$$

so

$$\mu_f(R) \leq M_f(R) \qquad \text{for any } R. \tag{5}$$

Lemma 5. *Let f be an entire function and g a nonlinear entire function. Then there is an R_0 such that*

$$M_{f\circ g}(R^{2/3}) \geq M_f(R) \qquad \textit{for } R > R_0.$$

Proof of the lemma. Let R_0 be the constant given for the function g by the previous lemma. Let $R > R_0$ and let z_1, z_2 with moduli R and $2R$, respectively, be such that $M_f(R) = |f(z_1)|$ and $M_f(2R) = |f(z_2)|$. By the previous lemma there is a w with $|w| \leq R^{2/3}$ such that $g(w) = z_1$, in which case $M_{f\circ g}(R^{2/3}) \geq |f(g(w))| = M_f(R)$; or $g(w) = z_2$, in which case $M_{f\circ g}(R^{2/3}) \geq |f(g(w))| = M_f(2R) \geq M_f(R)$. $\square$

Proof of the theorem. By [10] a linear functional on a unital Banach algebra is multiplicative if it is multiplicative on any commutative subalgebra, so without loss of generality we can assume that $\mathcal{A}$ is commutative.

We first show that, as a consequence of (3), T is continuous. To this end take $\lambda_0 \in \mathbb{C}$ with $F'(\lambda_0) \neq 0$. There is a neighborhood U of λ_0 such that $F_{|U}$ is a homeomorphism onto a neighborhood of $F(\lambda_0)$. Hence, for any $y \in \mathcal{A}$ from an open neighborhood of $F(\lambda_0 e)$, we have

$$T(y) = F(\varphi((F_{|U})^{-1}(y))),$$

so T is continuous at the point $F(\lambda_0 e)$ and consequently continuous at any point.

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of the function F. Notice that formally the meaning of the symbol F on both sides of (3) is different - on the right hand side F is a holomorphic function defined on the complex plane $\mathbb{C}$; on the left F is defined by the same power series, but with the Banach algebra $\mathcal{A}$ as the domain. Assume $F(z_0 + z) = \sum_{n=0}^{\infty} b_n z^n$ is a power series expansion of the same function around a point z_0, so that $\sum_{n=0}^{\infty} a_n (z_0 + z)^n = \sum_{n=0}^{\infty} b_n z^n$ for $z \in \mathbb{C}$. It is easy to check that these two expansions define the same function on $\mathcal{A}$, that is, $\sum_{n=0}^{\infty} a_n (z_0 e + x)^n = \sum_{n=0}^{\infty} b_n x^n$ for $x \in \mathcal{A}$.

We select z_0 in such a way that $b_1 \neq 0 \neq b_2$ and put

$$G(z) = F(z_0 + z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad \psi(x) = \varphi(z_0 e + x) - z_0.$$

From (3) we have

$$T(G(x)) = T(F(z_0 e + x)) = F(\varphi(z_0 e + x)) = F(z_0 + (\varphi(z_0 e + x) - z_0)) = G(\psi(x)),$$

that is,

$$T \circ G = G \circ \psi. \tag{6}$$

For $x \in \mathcal{A}$ we define

$$\psi_x(\lambda) = \psi(\lambda x) \qquad \text{and} \qquad f(\lambda) = T(G(\lambda x)) \qquad \text{for } \lambda \in \mathbb{C}.$$

From (6) we have

$$f(\lambda) = T(G(\lambda x)) = \sum_{n=0}^{\infty} b_n T(x^n) \lambda^n = G(\psi_x(\lambda)) \tag{7}$$

so ψ_x is analytic as a continuous solution of a holomorphic relation. Define

$$h(\lambda) = \sum_{n=0}^{\infty} |b_n T(x^n)| \lambda^n \qquad \text{for } \lambda \in \mathbb{C}.$$

We now prove by contradiction that ψ_x is a linear function. So assume ψ_x is not linear. By Lemma 5 there is an R_0 such that

$$M_h(R^{2/3}) \geq M_f(R^{2/3}) = M_{G \circ \psi_x}(R^{2/3}) \geq M_G(R) \qquad \text{for } R > R_0.$$

For any n we have $|T(x^n)| \leq K^n$, where $K = \max\{\|x\|, \|T\|\,\|x\|\}$, so

$$\mu_h(R) \leq \mu_G(KR) \qquad \text{for any } R.$$

By [8] p. 10 there exist arbitrary large values of r such that

$$M_h(r) < \mu_h(r) \log \mu_h(r).$$

Since $\log M_G(R)$ is a convex function of $\log R$ we have

$$M_G(R^{3/4}) \leq (M_G(1))^{1/4} (M_G(R))^{3/4} = c(M_G(R))^{3/4}.$$

From the last four inequalities and from (5) there are arbitrary large values of R such that

$$\begin{aligned} M_G(R) &\leq M_h(R^{2/3}) \\ &< \mu_h(R^{2/3}) \log \mu_h(R^{2/3}) \\ &\leq \mu_G(KR^{2/3}) \log \mu_G(KR^{2/3}) \\ &\leq M_G(KR^{2/3}) \log M_G(KR^{2/3}) \\ &\leq M_G(R^{3/4}) \log M_G(R^{3/4}) \\ &\leq c(M_G(R))^{3/4}(\log c + \frac{3}{4} \log M_G(R)). \end{aligned}$$

Hence M_G is bounded and consequently G is constant. This contradicts our assumption and proves that ψ_x is linear, that is, $\psi_x(\lambda) = \alpha + \beta_x \lambda$; notice that $\alpha = \psi(0)$ and $\beta_x = \psi(x) - \psi(0)$, so that α does not depend on x.

We have

$$\sum_{n=0}^{\infty} b_n T(x^n)\lambda^n = T(G(\lambda x)) = G(\psi(\lambda x)) = G(\alpha + \beta_x \lambda) = \sum_{n=0}^{\infty} b_n(\alpha + \beta_x \lambda)^n,$$

hence, comparing the coefficient of the first and the second power of λ we get

$$b_1 T(x) = \sum_{n=1}^{\infty} b_n n \alpha^{n-1} \beta_x = \beta_x G'(\alpha), \tag{8}$$

$$b_2 T(x^2) = \sum_{n=2}^{\infty} b_n \frac{n(n-1)}{2} \alpha^{n-2} \beta_x^2 = \frac{\beta_x^2}{2} G''(\alpha). \tag{9}$$

Assume that $T \neq 0$, and let $x_0 \notin \ker T$. Notice that regardless of the value of $T(e)$, for all sufficiently large t,

$$te + x_0 \notin \ker T, \tag{10}$$

also for any $t > \|x_0\|$, the element $e + x_0/t$ has logarithm in $\mathcal{A}$, so

$$te + x_0 \in \left\{x^2 : x \in \mathcal{A}\right\}. \tag{11}$$

Recall that defining G we have selected z_0 such that b_1, b_2 were not zero. So by (8) for $x = x_0$ we get $G'(\alpha) \neq 0$. By (9), (10) and (11) for $x^2 = te + x_0$ we get $G''(\alpha) \neq 0$. Consequently, for any $x \in \mathcal{A}$

$$x \in \ker T \iff \beta_x = 0 \iff x^2 \in \ker T. \tag{12}$$

Since for any $x, y \in \mathcal{A}$ we have $xy = \frac{(x+y)^2 - (x-y)^2}{4}$ it follows that $\ker T$ is a subalgebra of $\mathcal{A}$. By [9] (see also [4] p. 23) there are only three types of subalgebras of codimension one of a unital commutative Banach algebra:

- $e \notin \ker T$ and $\ker T$ is a maximal ideal, so $T/T(e)$ is a multiplicative functional,
- $e \in \ker T$ and
    - T is a difference between two multiplicative functionals, or
    - T is point derivation.

Assume T is equal to the difference between two multiplicative functionals Φ_1, Φ_2 and let $x \in \mathcal{A}$ be such that $\Phi_1(x) = 1 = -\Phi_2(x)$. Then $T(x) \neq 0 = T(x^2)$, which violates (12). Assume now T is equal to a point derivation at Φ; from the definition of point derivation $T(x^2) = 0$ for any $x \in \ker \Phi$, by (12), and since $e \notin \ker \Phi$ we have $\ker \Phi \subsetneq \ker T$, hence $\ker T = \mathcal{A}$. The contradictions prove that $T = 0$ or $T/T(e)$ is multiplicative. $\square$

3. REMARKS, GENERALIZATIONS, AND OPEN PROBLEMS

The result of the previous section can be easily extended to linear maps between two commutative complex Banach algebras $\mathcal{A}$ and $\mathcal{B}$. If $T : \mathcal{A} \to \mathcal{B}$ is a bounded linear map such that $T \circ F = F \circ \varphi$, where F is a nonlinear entire function and φ a continuous map from $\mathcal{A}$ into $\mathcal{B}$, then we can apply the theorem to all pairs $\Phi \circ T$, for each linear multiplicative functional Φ, and conclude that T is multiplicative modulo $T(e)$ and the radical of $\mathcal{A}$ (compare [1], §3).

Let $\mathcal{A}$ be an m-convex topological algebra and assume a linear functional T on $\mathcal{A}$ satisfies the usual condition $T \circ F = F \circ \varphi$. Since $\mathcal{A}$ is an inductive limit of a net of Banach algebras, and any continuous linear functional on $\mathcal{A}$ is also continuous on some of these algebras, standard arguments extend the result to m-convex algebras.

The result is not valid in general if F is an analytic function defined on a proper subset of the plane and the equation $T \circ F(x) = F \circ \varphi(x)$ is assumed to hold only for elements x whose spectrum is contained in the domain of F. It may be interesting to decide for what pairs of functions (F, φ) the equation $T \circ F(x) = F \circ \varphi(x)$ implies multiplicativity. For example, by comparing the coefficients of the power series expansions, we can show the following.

Proposition 6. *Let A be a complex Banach algebra with a unit e, let F be a nonlinear analytic function defined on an open connected and nonempty set U, let T be a linear functional on A. Suppose that*

$$T(F(x)) = F(T(x)) \qquad \text{for each } x \in A \text{ with } \sigma(x) \subset U.$$

Then $T \equiv 0$ or T is multiplicative.

However, the most interesting related open problem is perhaps the following one.

Conjecture 7. *Let A be a complex Banach algebra with a unit e, let F be a nonsurjective entire function, let T be a linear functional on A with $T(e) = 1$. Suppose that*

$$T(F(x)) \in F(\mathbb{C}) \qquad \text{for each } x \in A. \tag{13}$$

Then T is multiplicative.

By the Weierstrass Factorization Theorem [3] any nonsurjective entire function F is of the form

$$F(z) = c + \exp g(z).$$

By Theorem 1 the Conjecture is true for $g(z) = z$. C. Badea [2] proved that it holds for $g(z) = z + z^2$. Below we prove that it is also valid if g is any polynomial of degree three. It will be clear that the proof can be applied to many other polynomials for example to any nonzero polynomial of the form $g(z) = az^n + bz^{n+1}$ for some $n \in \mathbb{N}$. However the author does not know if the result is true for arbitrary nonconstant polynomials.

Theorem 8. *Let A be a complex Banach algebra with a unit e, let g be a polynomial of degree three, and T a linear functional on A with $T(e) = 1$. Suppose that*

$$T(\exp g(x)) \neq 0 \qquad \text{for each } x \in A. \tag{14}$$

Then T is multiplicative.

Proof. The derivative of g must be equal to zero at some point z_0. Replacing g with $g(z+z_0) - g(z_0)$ we may assume without loss of generality that $g(0) = g'(0) = 0$, so

$$g(z) = a_2 z^2 + a_3 z^3, \quad \text{where } a_3 \neq 0.$$

Fix an $x \in A$ and put

$$\begin{aligned} f(\lambda) &= T(\exp g(\lambda x)) \\ &= T\left(\exp\left(a_2\lambda^2 x^2\right)\exp\left(a_3\lambda^3 x^3\right)\right) \\ &= T\left(\left(e + a_2\lambda^2 x^2 + \frac{1}{2!}\left(a_2\lambda^2 x^2\right)^2 + \ldots\right)\left(e + a_3\lambda^3 x^3 + \frac{1}{2!}\left(a_3\lambda^3 x^3\right)^2 + \ldots\right)\right) \\ &= 1 + a_2 T(x^2)\lambda^2 + a_3 T(x^3)\lambda^3 + \frac{1}{2}a_2^2 T(x^4)\lambda^4 + .. \end{aligned} \tag{15}$$

For any complex number λ with sufficiently large modulus we have

$$|f(\lambda)| \leq \|T\| \, \|\exp g(\lambda x)\| \leq \|T\| \exp \|g(\lambda x)\| \leq \|T\| \exp\left(\|x\|^3 \left(|a_3| + 1\right) |\lambda|^3\right).$$

Hence the entire function f is of order not greater than 3, and by our assumption does not assume value zero. By the Hadamard's Factorization Theorem [3] and the Weierstrass' Factorization Theorem f is of the form

$$f(\lambda)=\exp h(\lambda), \qquad \text{where} \quad h(\lambda)=\sum_{k=0}^{3} b_k\lambda^k, \qquad \text{for} \qquad \lambda\in\mathbb{C}.$$

Since $f(0)=1$, we have $b_0=0$, and

$$\begin{aligned} f(\lambda) &= \exp\left(\sum_{k=1}^{3} b_k\lambda^k\right)=\prod_{k=1}^{3}\exp\left(b_k\lambda^k\right)= \qquad (16)\\ &= 1+b_1\lambda+\left(b_2+\frac{1}{2}b_1^2\right)\lambda^2+\left(b_3+b_1b_2+\frac{1}{6}b_1^3\right)\lambda^3+\left(\frac{1}{2}b_2^2+b_1b_3\right)\lambda^4+\dots\,. \end{aligned}$$

The coefficients b_k may depend, of course, on all of the coefficients a_k, as well as on x. From (15) we have $b_1=0$, so (16) gives

$$f(\lambda)=1+b_2\lambda^2+b_3\lambda^3+\frac{1}{2}b_2^2\lambda^4...,$$

and

$$\begin{aligned} a_2T(x^2) &= b_2\\ a_3T(x^3) &= b_3 \qquad (17)\\ a_2^2T(x^4) &= b_2^2. \end{aligned}$$

Assume first that $a_2\neq 0$. From (17)

$$\left(T(x^2)\right)^2=T(x^4), \qquad \text{for any } x\in A. \qquad (18)$$

If y is any element of A such that $\|y\|<1$, then $e+y$ is a square of an element of A, and by (18) we have

$$\begin{aligned} 1+2Ty+T\left(y^2\right) &= T(e+2y+y^2)=T\left((e+y)^2\right)\\ &= (T(e+y))^2=(1+Ty)^2\\ &= 1+2Ty+(Ty)^2\,. \end{aligned}$$

Hence $T\left(y^2\right)=T\left(y^2\right)$, so T is multiplicative.

Assume now that $a_2=0$. In this case we need to compute and compare the sixths coefficients in (15) and (16). They are $\frac{1}{2}a_3^2T(x^6)$ and $\frac{1}{2}b_3^2$, respectively. Hence

$$a_3^2T(x^6)=b_3^2,$$

so since $a_3\neq 0$, from (17) we get

$$\left(T(x^3)\right)^2=T(x^6), \qquad \text{for any } x\in A.$$

As in the previous case, we conclude that T is multiplicative. □

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