Norms on $C(X)$

Krzysztof Jarosz*
Southern Illinois University at Edwardsville, IL 62026, and
Bowling Green State University, Bowling Green, OH 43403
E-mail: kjarosz@siue.edu

Abstract. Let $C(X)$ be the algebra of all scalar valued continuous functions on a topological space X. Then $C(X)$ has a submultiplicative (incomplete) norm if and only if all functions in $C(X)$ are bounded.

For a topological space X we denote by $C(X)$ the algebra of all continuous scalar valued functions on X.

If X is a compact Hausdorff space then $C(X)$ is a Banach algebra if equipped with the obvious sup-norm. It is a classical result due to I. Gelfand [8] that $C(X)$, as well as any commutative semi-simple Banach algebra, has a unique norm in the sense that any other complete and submultiplicative norm induces the same topology. We call a norm $\|\cdot\|$ submultiplicative, or a normed algebra norm if

$$\|f \cdot g\| \leq \|f\| \, \|g\| \text{ for any } f, g.$$

In 1949 I. Kaplansky [6] proved that any other submultiplicative norm on $C(X)$ (whether complete or not) is at least as large as the sup-norm on X. Whether $C(X)$ admits a submultiplicative non-complete norm had then been a long standing open problem . In 1977 H. G. Dales and J. Esterle [2] announced two independent proofs of the existence - under the continuum hypothesis - of a noncomplete submultiplicative norm on $C(X)$. The proofs were later published in [1] and [4].

If X is not compact and supports an unbounded scalar valued function, then it can not have a Banach algebra norm since the spectrum of any element in a Banach algebra is compact. The answer to whether $C(X)$ may have an incomplete submultiplicative norm on $C(X)$ is less obvious. In general the spectrum of an element of a normed algebra may be unbounded. For example if A is an algebra of polynomials, or an algebra of entire functions, then the spectrum of any non-constant element of A is unbounded, yet there is a submultiplicative norm on A: $\|f\| = \sup \{|f(z)| : |z| < 1\}$. B. Yood [7] recently proved that if X supports an unbounded continuous complex valued function, if in addition X is locally compact, and if there is a continuous function h on X, such that $h^{-1}(w)$ is compact for any complex number w, then $C(X)$

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does not have a submultiplicative norm. In this note we prove that the same is true without any additional assumptions on X. The result is valid in both the real and in the complex cases. We notice that the assumption that the norm is submultiplicative is crucial. For any X, $C(X)$ has a non-submultiplicative norm, even a complete one.

Theorem 1. *$C(X)$ has a submultiplicative norm if and only if $C(X)$ does not contain an unbounded function.*

Notice that the last condition is equivalent to the assumption that realcompactification of X is compact [5].

Proof. For a general topological space X, continuous scalar valued functions may not separate points of X, and we can introduce an equivalence relation on X :

$$x_1 \sim x_2 \text{ iff } f(x_1) = f(x_2) \text{ for any continous scalar valued function } f \text{ on } X.$$

The quotient space $X_{/\sim}$ is completely regular ([5], p.41) and the algebras $C(X)$ and $C(X_{/\sim})$ are isomorphic. Hence, without loss of generality, we may assume that X is completely regular.

Suppose that $\|\cdot\|$ is a normed algebra norm on $C(X)$. We denote by $C_b(X)$ the algebra of all bounded continuous functions on X, and by $\overline{C(X)}$ the completion of $C(X)$ in the norm $\|\cdot\|$. Let Y be the subset of X consisting of all points x such that the corresponding linear-multiplicative functional $\delta_x : f \longmapsto f(x)$ is $\|\cdot\|$-continuous. Any continuous linear-multiplicative functional on a normed algebra has norm one. Hence if $x \in \overline{Y}$, then for any $f \in C(X)$ we have $|f(x)| \leq \sup\{|f(x')| : x' \in Y\} \leq \|f\|$, so $x \in Y$. This shows that Y is a closed subset of X.

Assume $Y \neq X$, let $x_0 \in X \backslash Y$, and let $\{U_\alpha\}$ be a net of open neighborhoods of x_0 contained in $X \backslash Y$ and such that for any open neighborhood U of x_0 there is an α_0 such that $U_\alpha \subset U$ for all $\alpha \succ \alpha_0$. For each α let $f_\alpha \in C_b(X)$ be such that $f_\alpha \equiv 1$ on $X \backslash U_\alpha$, and $f_\alpha \equiv 0$ on some open neighborhood V_α of x_0. The existence of such a function follows for example from the fact that any completely regular space is a subset of a normal space [3]. Suppose that f_α has an inverse element f_α^{-1} in the Banach algebra $\overline{C(X)}$. Let $g_\alpha \in C_b(X)$ be a nonzero function with the supp(g_α) contained in V_α, so that $f_\alpha \cdot g_\alpha \equiv 0$. We have $g_\alpha = f_\alpha^{-1} \cdot f_\alpha \cdot g_\alpha = f_\alpha^{-1} \cdot 0 = 0$. The contradiction proves that f_α is a noninvertible element of $\overline{C(X)}$, so f_α is an element of a maximal ideal M_α of $\overline{C(X)}$, hence there is a linear-multiplicative functional F_α on $\overline{C(X)}$ such that $F_\alpha(f_\alpha) = 0$. Since any linear-multiplicative functional on $C_b(X)$ is of the form δ_x for some x in βX, the Čech-Stone compactification of X, there is an $x_\alpha \in \beta X$ such that $F_\alpha(f) = f(x_\alpha)$ for $f \in C_b(X)$. Here we use the same symbol f to denote a bounded continuous function on X and the corresponding continuous function on βX. Since $F_\alpha(f_\alpha) = 0$, it follows that $x_\alpha \in cl_{\beta X}(U_\alpha)$ where $cl_{\beta X}(\cdot)$ denotes the closure in βX. Hence $x_\alpha \to x_0$. Taking a suitable subnet, we may also assume, without loss of generality, that the net F_α is convergent in the weak $*$ topology of the dual space $\left(\overline{C(X)}\right)^*$ to a linear-multiplicative functional F_0 on $\overline{C(X)}$. Hence $F_0(f) = f(x_0)$, for $f \in C_b(X)$. Assume that the functionals F_0 and δ_{x_0} do not coincide on $C(X)$. Then there is an $h \in C(X)$ such that $F_0(h) = 0$ and $h(x_0) = 1$. Let $k \in C_b(X)$ be such that $k(x_0) = 1$ and $h \cdot k \in C_b(X)$. We have

$$1 = h(x_0)\, k(x_0) = \delta_{x_0}(h \cdot k) = F_0(h \cdot k) = F_0(h) \cdot F_0(k) = 0 \cdot F_0(k) = 0.$$

The contradiction proves that $F_0 = \delta_{x_0}$ on $C(X)$. However F_0 is a continuous functional on $\overline{C(X)}$ and we selected x_0 so that δ_{x_0} be discontinuous on $C(X)$. The last contradiction shows that $Y = X$.

We proved that for any x the corresponding linear-multiplicative functional δ_x is continuous on $C(X)$, so it can be extended to a continuous functional on $\overline{C(X)}$. Hence for any $f \in C(X)$ the spectrum of f in the Banach algebra $\overline{C(X)}$ contains $f(X)$, and since the spectrum of any element in a Banach algebra is compact, it proves that all functions in $C(X)$ are bounded.

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