

Automatic Continuity of Biseparating Maps

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ABSTRACT. We prove that a biseparating map between spaces of vector-valued continuous functions is usually automatically continuous. However, we also discuss special cases when it is not true.

1. Introduction

Assume A, B are spaces of continuous functions on a set X taking values in a normed space E . A linear map $T : A \rightarrow B$ is called separating if

$$\|f(\cdot)\| \|g(\cdot)\| \equiv 0 \quad \implies \quad \|Tf(\cdot)\| \|Tg(\cdot)\| \equiv 0,$$

and biseparating if T^{-1} exists and is separating as well. The concept has its source in the theory of topological lattices but is also an important generalization of multiplicative maps on Banach algebras with application to many other areas. It has been studied intensively by a number of authors, see e.g. [2, 3, 5, 6, 7, 8, 9, 12]. The most typical questions are whether in a given setting a separating linear bijection must be biseparating, and if such a map must be automatically continuous. Both questions have positive answers for separating linear bijections between $C(X)$ spaces with X compact [11], negative in general [1], and are open in some other cases.

In this note we show that any biseparating map between spaces of *bounded* vector valued functions is automatically continuous if and only if the underlying topological space X has no isolated points. For maps between spaces of unbounded functions the situation is more complicated but perhaps also more interesting. If X is metrizable, or connected, or locally compact, or separable, or satisfies some other simple restrictions, we again get the same condition - no isolated points. However, there are sets X with no isolated points that admit biseparating maps which are discontinuous in a very strong sense. Our results are valid in both real and complex cases.

2. Preliminaries

For a completely regular space X and a normed space E we denote by $C(X, E)$ and $C_b(X, E)$ the space of all continuous functions from X into E , and the space of all bounded continuous functions from X into E , respectively. If E is equal to

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the scalar field, we write $C(X)$ in place of $C(X, E)$. We equip $C_b(X, E)$ with the topology of uniform convergence on X , and $C(X, E)$ with the topology of uniform convergence on compact subsets of X . For normed spaces E, F we denote by $L^{-1}(E, F)$ the set of all linear bijections from E onto F .

For a function $f \in C(X)$ it may be often convenient to extend f to a continuous function on βX , the maximal compactification of X . In general, for $x \in \beta X \setminus X$, the value of $f(x)$ may be infinite. There are, however, completely regular spaces X with a point $x_0 \in \beta X \setminus X$ such that the value of $f(x_0)$ is finite for all $f \in C(X)$; the set of all points in βX with this property is called the realcompactification of X and is denoted by rX . Since we have $C(X) = C(rX)$, the natural domain for a continuous function on X is rX , not X . Hence we will often consider realcompactifications, or alternatively, we will assume that the completely regular spaces under consideration are realcompact. All compact sets are clearly realcompact, also all subsets of Euclidean spaces are realcompact, or even more generally all metrizable spaces of nonmeasurable cardinal are realcompact ([10], p.232).

We will often refer to the following result from [4].

THEOREM 1. *Suppose that X and Y are completely regular realcompact spaces, E, F are normed spaces, and that*

- *T is a linear biseparating map from $C(X, E)$ onto $C(Y, F)$,*
- *or E, F are infinite-dimensional and T is a linear biseparating map from $C_b(X, E)$ onto $C_b(Y, F)$.*

Then there exists a homeomorphism $h : Y \rightarrow X$ and a map $J : Y \rightarrow L^{-1}(E, F)$ such that $(Tf)(y) = (J(y))(f(h(y)))$ for every f in the domain of T , and $y \in Y$.

Notice the result can also be applied if E, F are finite-dimensional and T is defined on $C_b(X, E)$ since in such case we have $C_b(X, E) \simeq C_b(\beta X, E) = C(\beta X, E)$. Based on the above Theorem, each time we are asked about continuity of a biseparating map T , we can compose the map with $f \mapsto f \circ h^{-1}$, and assume without loss of generality that $X = Y$ and that

$$(2.1) \quad (Tf)(x) = (J(x))(f(x)), \quad \text{for every } f \text{ and } x \in X.$$

Hence, if the normed spaces E, F are finite-dimensional any biseparating map T must be automatically continuous. On the other hand, if X contains an isolated point x_0 and E is infinite-dimensional then one can trivially define a discontinuous biseparating map on $C(X, E)$ or $C_b(X, E)$: take any discontinuous linear bijection J_0 from E onto itself and put

$$(Tf)(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ J_0(f(x_0)) & \text{if } x = x_0. \end{cases}$$

In the next section we show that, except for some very special sets X , this is basically the only way one can get a discontinuous biseparating map.

Recall that a point x_0 in a topological space is called a P -point if for each sequence (U_n) of neighborhoods of x_0 , the intersection $\bigcap_{n=1}^{\infty} U_n$ contains again a neighborhood of x_0 ; if a topological space is first countable, for example if it is metrizable, then only isolated points are the P -points. A completely regular topological space is called P -space if each point of the space is a P -point. Among the metric spaces, P -spaces coincide with the discrete spaces. We will need the following properties of the P -spaces ([10], p. 63).

PROPOSITION 1. *For a completely regular space X the following conditions are equivalent*

- X is a P -space,
- any real-valued continuous function on X is locally constant,
- any G_δ subset of X is open.

3. Results

THEOREM 2. *Assume X, Y are completely regular realcompact spaces, E, F are normed spaces, and $T : C_b(X, E) \rightarrow C_b(Y, F)$ is a biseparating map. If Y has no isolated points then T is continuous.*

PROOF. By Theorem 1 we may assume without loss of generality that $Y = X$ and that T is of the form (2.1). Suppose that T is not continuous and let $(f_n)_{n=1}^\infty$ be a sequence of elements in $C_b(X, E)$ such that

$$\|f_n\| < 1/n^2 \text{ and } \|Tf_n\| > n + \sum_{j=1}^{n-1} \|Tf_j\|, \quad \text{for } n \in \mathbb{N}.$$

Let $x_1 \in X$ be such that $\|(Tf_1)(x_1)\| \geq 1$. Since x_1 is not isolated, then there exists an $x_2 \neq x_1$ with $\|(Tf_2)(x_2)\| \geq 1 + \|Tf_1\|$. Continuing the process, we can inductively construct a sequence $(x_n)_{n=1}^\infty$ of different points in X such that

$$\|(Tf_n)(x_n)\| \geq n + \sum_{j=1}^{n-1} \|Tf_j\|, \quad \text{for } n \in \mathbb{N}.$$

Let $(U_n)_{n=1}^\infty$ be a sequence of open subsets of X and (g_n) a sequence of continuous bounded real-valued functions on X such that for any $n \in \mathbb{N}$

$$x_n \in U_n \text{ and } x_k \notin U_n, \quad \text{for } k < n,$$

and

$$g_n(x_n) = 1 = \|g_n\| \quad \text{and} \quad \text{supp } g_n \subset U_n.$$

Since $\|g_n f_n\| \leq \|f_n\| < 1/n^2$ we can define a function f in $C_b(X, E)$ by

$$f := \sum_{j \in \mathbb{N}} g_j f_j.$$

From (2.1) and since $g_n(x_n) = 1$ and $g_j(x_n) = 0$ for $j > n$ we get

$$\begin{aligned}
\|(Tf)(x_n)\| &= \|(J(x_n))(f(x_n))\| = \left\| (J(x_n)) \left(\sum_{j \in \mathbb{N}} g_j(x_n) f_j(x_n) \right) \right\| \\
&= \left\| (J(x_n)) \left(\sum_{j=1}^{n-1} g_j(x_n) f_j(x_n) + f_n(x_n) \right) \right\| \\
&= \left\| \sum_{j=1}^{n-1} g_j(x_n) (J(x_n)(f_j(x_n)) + (J(x_n))(f_n(x_n))) \right\| \\
&= \left\| \sum_{j=1}^{n-1} g_j(x_n) (T(f_j)(x_n) + (T(f_n))(x_n)) \right\| \\
&\geq \|(T(f_n))(x_n)\| - \sum_{j=1}^{n-1} \|(T(f_j)(x_n))\| \\
&\geq \left(n + \sum_{j=1}^{n-1} \|Tf_j\| \right) - \sum_{j=1}^{n-1} \|Tf_j\| \geq n.
\end{aligned}$$

Since our space contains only bounded functions, the contradiction shows continuity of T . ■

THEOREM 3. *Assume X, Y are completely regular realcompact spaces and E, F are both infinite-dimensional normed spaces. Then*

- (i) *if the interior of the set of P -points of Y is empty then every biseparating map $T : C(X, E) \rightarrow C(Y, F)$ is continuous,*
- (ii) *if the interior of the set of P -points of Y is not empty then there is a discontinuous biseparating map $T : C(Y, F) \rightarrow C(Y, F)$,*
- (iii) *if $\dim E = \dim F$ then all points of Y are P -points if and only if for any linear bijection J from E onto F (continuous or not) the map $T(f) \stackrel{\text{df}}{=} J \circ f$ is well defined and biseparating from $C(Y, E)$ onto $C(Y, F)$.*

COROLLARY 1. *For completely regular realcompact spaces X, Y and normed spaces E, F there is a biseparating map between $C(X, E)$ and $C(Y, F)$ if and only if X and Y are homeomorphic and*

- *E and F are isomorphic, or*
- *$\dim E = \dim F$ and Y is a P -space.*

REMARK 1. *There is another useful way to think about these results. Notice that if J is a linear bijection from a normed space E onto itself, then we can phrase the usual definition of continuity of J as follows:*

J is continuous iff transformation $f \mapsto J(f)$ maps $C(\mathbb{N}^, E)$ into itself,*

where by \mathbb{N}^ we denote the one-point compactification of the set of natural numbers. Now, based on our results we can replace the set \mathbb{N}^* in the definition above with a set X if and only if X is not a P -space.*

PROOF OF THE THEOREM. (i) Assume that the interior of the set of P -points of Y is empty and $T : C(X, E) \rightarrow C(Y, F)$ is a biseparating map. As before by

Theorem 1 we may assume without loss of generality that $Y = X$ and that T is of the form (2.1).

First we are going to prove that $J(y) : E \rightarrow F$ is continuous for every $y \in X$. Assume that there is a $y_0 \in X$ such that $J(y_0)$ is discontinuous. If y_0 is not a P -point we put $x_0 \stackrel{df}{=} y_0$. If y_0 is a P -point we put $V_n = \{x \in X : \|J(x)\| > n\}$. Since the map $\|\cdot\| \circ J : X \rightarrow [0, +\infty]$ sending each $y \in X$ into $\|J(y)\|$ is continuous, all of the sets V_n are neighborhoods of y_0 , and so is $\bigcap_{n=1}^{\infty} V_n$. Moreover for any $x \in \bigcap_{n=1}^{\infty} V_n$ the map $J(x)$ is discontinuous. Since the interior of the set of P -points is empty, the set $\bigcap_{n=1}^{\infty} V_n$ contains a point which is not a P -point. We denote by x_0 one of such points.

Let (e_n) be a sequence in E such that

$$\|e_n\| < 1/n^2 \text{ and } \|J(x_0)(e_n)\| > n, \quad \text{for } n \in \mathbb{N}.$$

Let \mathbf{e}_n be the constant function equal to e_n . Since x_0 is not a P -point there is a sequence of open neighborhoods U_n of x_0 with

$$U_n \subset \{x \in X : \|(T\mathbf{e}_n)(x)\| = \|J(x)(e_n)\| > n\}$$

and such that the intersection of all U_n is not a neighborhood of x_0 ; we may also assume that $\overline{U_{n+1}} \subset U_n$, for $n \in \mathbb{N}$. It follows that x_0 belongs to the closure of the following set:

$$A := \bigcup_{n=0}^{\infty} U_{n+1} - U_{n+2}.$$

For each $n \in \mathbb{N}$, let $g_n \in C_b(X)$ be such that

$$\|g_n\| = 1 = g_n(x), \text{ for } x \in U_{n+1}, \text{ and } \text{supp } g_n \subset U_n.$$

Put

$$g := \sum_{n=1}^{\infty} (g_n - g_{n+2})e_{n+1};$$

since $\|g_n - g_{n+2}\| \leq 2$ and $\|e_n\| < 1/n^2$ the series is convergent and $g \in C(X, E)$. For any $k \in \mathbb{N}$ and any $x_{k+1} \in U_{k+1} - U_{k+2}$ we have

$$(g_n - g_{n+2})(x_{k+1}) = \begin{cases} 1 - 1 = 0 & \text{if } n < k \\ 1 - 0 = 1 & \text{if } n = k \\ 0 - 0 = 0 & \text{if } n > k \end{cases}.$$

Hence

$$\begin{aligned} \|(Tg)(x_{k+1})\| &= \left\| (J(x_{k+1})) \left(\sum_{n=1}^{\infty} (g_n - g_{n+2})(x_{k+1}) e_{n+1} \right) \right\| \\ &= \|(J(x_{k+1}))(e_{k+1})\| = \|(T\mathbf{e}_{k+1})(x_{k+1})\| > k + 1. \end{aligned}$$

Since x_0 belongs to the closure of $\bigcup_{n=0}^{\infty} U_{n+1} - U_{n+2}$ we get $\|(Tg)(x_0)\| = \infty$, and this contradiction proves the continuity of $J(y)$ for every $y \in X$.

Suppose now that T is not continuous. Then there is a compact subset K of Y and a sequence (f_n) in $C(X, E)$ with $p_K(f_n) < 1/n^2$ and $p_K(Tf_n) > n$.

Let $y_n \in K$ be such that $\|(Tf_n)(y_n)\| > n$, for each $n \in \mathbb{N}$. Notice that the set $\{y_n \in K : n \in \mathbb{N}\}$ is infinite since otherwise there would be a point $y_{n_0} \in K$ with $\|(Tf_n)(y_{n_0})\| > n$ for infinitely many $n \in \mathbb{N}$, contrary to the continuity of $J(y_{n_0})$. Without loss of generality we may assume that $y_n \neq y_m$ when $n \neq m$,

and take a sequence $(U_n)_{n=1}^\infty$ of pairwise disjoint open subsets of X such that $y_n \in U_n \subset \{x \in X : \|f_n(x)\| < 1/n^2\}$, for $n \in \mathbb{N}$. Let $(g_n)_{n=1}^\infty$ be a sequence in $C_b(X)$ such that $\text{supp } g_n \subset U_n$ and $g_n(y_n) = 1$, for $n \in \mathbb{N}$. It is easy to see that

$$f \stackrel{\text{df}}{=} \sum_{k=1}^{\infty} g_k f_k \in C(X, E)$$

while, by 2.1, $\|(Tf)(y_n)\| = \|(Tf_n)(y_n)\| > n$ so Tf is not bounded on K . Since this is not possible we conclude that T must be continuous.

(ii) Assume now that the interior \mathcal{I} of the set of P -points of Y is not empty and let h_0 be a continuous real-valued function on Y such that $h_0(x) = 1$ for $x \in X \setminus \mathcal{I}$, and $h_0(z_0) = 0$ at some point $z_0 \in \mathcal{I}$. Put $\mathcal{A} = h_0^{-1}(0)$. By Proposition 1 \mathcal{A} is an open subset of \mathcal{I} , and consequently of X ; clearly as a zero set \mathcal{A} is also closed. Let J be a discontinuous linear bijection from F onto itself and define $T : C(Y, F) \rightarrow C(Y, F)$ by

$$(Tf)(x) = \begin{cases} f(x) & \text{if } x \notin \mathcal{A} \\ J(f(x)) & \text{if } x \in \mathcal{A} \end{cases}.$$

The map T is obviously biseparating but we need to check that it is well defined, that is, that Tf is continuous for any continuous f . It is of course true on the clopen subset $X \setminus \mathcal{A}$. Let $f \in C(Y, F)$ and let $z \in \mathcal{A}$. By Proposition 1 the function

$$\mathcal{A} \ni x \mapsto \|f(x) - f(z)\|$$

is constant on a neighborhood of z , hence f , and so Tf , are both locally constant and consequently continuous.

(iii) As above we assume without loss of generality that $X = Y$ and that h is the identity map. Suppose that $\dim E = \dim F$ and that there is a point $x_0 \in X$ which is not a P -point. We will see that there exists a linear bijection J from E onto F such that the map $T(f) \stackrel{\text{df}}{=} J \circ f$ is not well defined.

Since J is not continuous, then there exists a sequence (e_n) of norm-one elements of E such that $\|Je_n\| > n^3$, for $n \in \mathbb{N}$. On the other hand, since x_0 is not a P -point, there exists a sequence $(U_n)_{n=1}^\infty$ of neighborhoods of x_0 such that the intersection of all U_n is not a neighborhood of x_0 . We may assume that $\overline{U_{n+1}} \subset U_n$, for $n \in \mathbb{N}$. Consequently x_0 belongs to the closure of at least one of the following sets:

$$A_1 := \bigcup_{n=1}^{\infty} U_{2n} - U_{2n+2} \quad \text{or} \quad A_2 := \bigcup_{n=1}^{\infty} U_{2n-1} - U_{2n+1}.$$

Without loss of generality we may assume that $x_0 \in \overline{A_2}$.

For each $n \in \mathbb{N}$, let $g_n \in C_b(X)$, $0 \leq g_n \leq 1$, be such that

$$g_n(U_{2n-1} - U_{2n+1}) = 1 \text{ and } \text{supp } g_n \subset U_{2n-2} - U_{2n+2}.$$

It is clear that the map

$$f \stackrel{\text{df}}{=} \sum_{k=1}^{\infty} \frac{1}{k^2} g_k f_k$$

belongs to $C(X, E)$. Also, according to the description of T and to the fact that $x_0 \in \overline{A_2}$, we deduce that $\|(Tf)(x_0)\| = \infty$, which is absurd. Consequently, X is a P -space.

To prove the other implication we just repeat the above argument in **(ii)** with $\mathcal{A} = X$. ■

As for the proof of the Corollary, notice that if Y is not a P -space, then it contains a point y_0 which is not a P -point, and consequently, as in the proof of Theorem 3 **(i)**, $J(y_0)$ is continuous (and bijective). On the other hand, by Theorem 1, $(T^{-1}g)(x) = (Kx)(g(h^{-1}(x)))$, for some $K \in L^{-1}(F, E)$. In particular $K(h(y))$ is the inverse of $J(y)$ (see the proof of Theorem 2.4 in [4]), which implies in particular that $h(y)$ is not a P -point and, consequently, E and F are isomorphic.

The P -sets without isolated points are rather unusual, they can not be locally compact or metrizable. An example of such a set can be found in [10], below we provide another one.

EXAMPLE 1. Let Ω be an uncountable set and put $X = \{0, 1\}^\Omega$. We say that a function f on X depends on countably many coordinates if there is a countable subset Ω_0 of Ω such that

$$f(x) = f(x') \text{ whenever } x(\omega) = x'(\omega) \text{ for all } \omega \in \Omega_0.$$

Notice that f depends on countably many coordinates if and only if $f = f_0 \circ p_{\Omega_0}$ where Ω_0 is a countable subset of Ω , p_{Ω_0} is a natural projection from $\{0, 1\}^\Omega$ onto $\{0, 1\}^{\Omega_0}$ and f_0 is a function on $\{0, 1\}^{\Omega_0}$. We denote by $C_{\aleph_0}(X)$ and $C_{\aleph_0}^b(X)$ the set of all scalar valued functions that depend on countably many variables, and bounded functions with that property, respectively. We introduce on X the coarsest topology such that all functions in $C_{\aleph_0}(X)$ are continuous. That topology τ of X has a base that can be described as follows

$$\left\{ p_{\Omega_0}^{-1}(A) : A \subset \{0, 1\}^{\Omega_0}, \text{card}(\Omega_0) \leq \aleph_0 \right\}.$$

Since a uniform limit of a sequence of functions depending on countably many coordinates each, has also such a property, $C_{\aleph_0}^b(X)$ is a self-adjoint uniform algebra so X can be seen as a subset of the maximal ideal space of that algebra. Since any subset of a compact Hausdorff space is completely regular, so is X .

Suppose that f is a continuous real-valued function on X and fix $x_0 \in X$. Without loss of generality, we assume that $x_0(\omega) = 0$, for $\omega \in \Gamma \subset \Omega$, and $x_0(\omega) = 1$, for $\omega \in \Omega \setminus \Gamma$. Since, for any $n \in \mathbb{N}$, the set $U_n = \{x \in X : |f(x) - f(x_0)| < \frac{1}{n}\}$ is open, it contains an open set of the form

$$V_n = \{x \in X : x(\Omega_{1,n}) \subset \{0\}, x(\Omega_{2,n}) \subset \{1\}\},$$

for some countable sets $\Omega_{1,n} \subset \Gamma, \Omega_{2,n} \subset \Omega \setminus \Gamma$. Put $\Omega_1 = \bigcup_{n=1}^{\infty} \Omega_{1,n}, \Omega_2 = \bigcup_{n=1}^{\infty} \Omega_{2,n}$, clearly Ω_1, Ω_2 are countable and disjoint. Now

$$U_0 = \{x \in X : x(\Omega_1) \subset \{0\}, x(\Omega_2) \subset \{1\}\},$$

is a neighborhood of x_0 and f is constant on U_0 .

References

- [1] Y. A. Abramovich and A. K. Kitover, *A solution to a problem on invertible disjointness preserving operators*, Proc. Amer. Math. Soc. 126 (1998), 1501–1505.
- [2] Y. A. Abramovich and A. K. Kitover, *Inverses of Disjointness Preserving Operators*, Memoires of AMS 679 (2000).
- [3] J. Araujo, *Separating maps and linear isometries between some spaces of continuous functions*, J. Math. Anal. Appl. 226 (1998), 23–39.

- [4] J. Araujo, *Realcompactness and Banach-Stone theorems*, Submitted. Available at <http://www.matesco.unican.es/~araujo/isom3.dvi> and at <http://xxx.lanl.gov/pdf/math/0010292>.
- [5] J. Araújo, E. Beckenstein, and L. Narici, *When is a separating map biseparating?* Arch. Math. (Basel) 67 (1996), 395–407.
- [6] J. Araujo, E. Beckenstein, and L. Narici, *Biseparating maps and homeomorphic realcompactifications*, J. Math. Anal. Appl. 192 (1995), 285-265.
- [7] J. Araujo and K. Jarosz, *Separating maps on spaces of continuous functions*, Proc. of The 3rd Conference on Function Spaces, Cont. Math. 232 (1999) AMS, 33-38.
- [8] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs 24, The Clarendon Press, Oxford University Press, New York, 2000.
- [9] J. J. Font and S. Hernández, *On separating maps between locally compact spaces*. Arch. Math. 63 (1994), 158–165.
- [10] L. Gilman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.
- [11] K. Jarosz, *Automatic continuity of separating linear isomorphisms*, Bull. Canadian Math. Soc. 33 (1990), 139-144
- [12] J-S. Jeang and N-C. Wong, *Weighted composition operators of $C_0(X)$'s*. J. Math. Anal. Appl. 201 (1996), 981–993.

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