

# Automatic Continuity

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## Selected Examples

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ABSTRACT. We consider several examples of the following general problem:

Let  $T$  be a function between two Banach spaces (or Banach algebras, or lattices, or other structures) that satisfies certain algebraic conditions (for instance, it is linear, multiplicative, translation invariant, self-adjoint, commutes with other maps, etc.). Can we *automatically* conclude that  $T$  is continuous?

### 1. INTRODUCTION.

This brief informal note is not intended as a survey of a vast area of automatic continuity, we can not even attempt to point out here all the important problems in that area. We simply present several examples to show the beauty and broadness of the theory. In most cases the examples and theorems are not given in the most general form, and are listed without proofs. The selection is not based on the importance of a particular question but whether the problem can be presented at an elementary level without introducing a long list of definitions and advanced machinery. It is also biased by the author's personal preferences and prejudices.

### 2. LINEAR MAPS.

The following is perhaps the most elementary example of an automatic continuity result.

**Example 1.** *Let  $X$  and  $Y$  be normed vector spaces. If  $T : X \rightarrow Y$  is linear and  $\dim X < \infty$  then  $T$  is continuous.*

One may notice that the property that any linear map on  $X$  is continuous is equivalent to  $\dim X < \infty$ .

For the next example we need to recall that a linear map  $T$  on a Hilbert space  $H$  is called *self-adjoint*, or Hermitian, if  $\langle x, Ty \rangle = \langle Tx, y \rangle$  for all  $x, y \in H$ .

**Example 2.** *If  $T : H \rightarrow H$  is a linear self-adjoint map on a Hilbert space then  $T$  is continuous.*

Proof. Assume that  $x_n \rightarrow 0$ , and  $Tx_n \rightarrow y$ . We have

$$\begin{aligned} \langle Tx_n, y \rangle &\rightarrow \langle y, y \rangle = \|y\|^2 \\ \parallel & \\ \langle x_n, Ty \rangle &\rightarrow \langle 0, Ty \rangle = 0. \end{aligned}$$

Hence  $y = 0$ , so by the Closed Graph Theorem  $T$  is continuous.  $\square$

Notice that in the above two examples the norm of  $T$  could be arbitrary large, so while we could *automatically* conclude that  $T$  was continuous, we could conclude nothing about the degree of the continuity. Fortunately this is not always the case, as we see in the next section.

### 3. MULTIPLICATIVE AND ALMOST MULTIPLICATIVE MAPS

We start with classical results [21].

**Example 3.** *If  $T : A \rightarrow \mathbb{C}$  is a linear-multiplicative functional on a Banach algebra  $A$  then  $T$  is continuous **and**  $\|T\| = 1$ .*

The above result can be easily extended to linear-multiplicative maps into semisimple commutative Banach algebras. We call a commutative Banach algebra  $B$  *semisimple* if it has sufficiently many linear-multiplicative functionals; that is, if for any  $b \in B$  there is a linear-multiplicative functional  $G$  on  $B$  such that  $G(b) \neq 0$ .

**Example 4.** *If  $T : A \rightarrow B$  is a linear-multiplicative map from a Banach algebra  $A$  into a semisimple Banach algebra  $B$  then  $T$  is continuous.*

Proof. For any linear-multiplicative functional  $F : B \rightarrow \mathbb{C}$ ,  $F \circ T$  is a linear and multiplicative functional, so based on the previous example it is continuous. Hence, from the Closed Graph Theorem,  $T$  is continuous.  $\square$

In the latest, result we assume that

- $B$  is semisimple,
- $A$  is a Banach algebra (not a general topological algebra),
- $T$  is *exactly* multiplicative (as oppose to almost multiplicative as defined later).

We discuss now if the above assumptions can be weakened or removed.

**3.1. Maps into nonsemisimple algebras.** Assume that  $T : A \rightarrow B$  is a linear-multiplicative and discontinuous map between Banach algebras  $A$  and  $B$  and define

$$p(a) = \max \{ \|a\|, \|T(a)\| \}, \text{ for } a \in A.$$

Then  $p(\cdot)$  is a new submultiplicative norm on  $A$ , nonequivalent with the original one. On the other hand if there is a nonequivalent submultiplicative norm  $p(\cdot)$  on  $A$ , then the identity map from  $(A, \|\cdot\|)$  into  $(A, p(\cdot))$  is a discontinuous linear map between Banach algebras  $(A, \|\cdot\|)$  and the completion of  $(A, p(\cdot))$ .

Hence the question concerning the continuity of linear-multiplicative maps is equivalent to the question concerning the existence of a second nonequivalent submultiplicative norm.

Let  $C(K)$  be the algebra of all continuous functions on a compact set  $K$ . The problem whether there exists a submultiplicative norm on  $C(K)$  not equivalent to the sup norm attracted a lot of attention for many years. Already in 1949 I. Kaplansky [15] proved that there is no complete nonequivalent norm on  $C(K)$ . The general question remained open for many years until finally in 1977 H. G. Dales [3] and J. Esterle [6] announced two independent proofs. They showed that under the continuum hypothesis there is a non-complete submultiplicative norm on  $C(K)$ .

In this context one may ask if there can be a submultiplicative norm on  $C(K)$ , for a noncompact  $K$ . It may happen that  $K$  is noncompact but still all scalar-valued continuous functions on  $K$  are bounded [8]. In such case we have the obvious sup norm. However, this is the only exception. If  $C(K)$  contains an unbounded function, then it does not admit a submultiplicative norm complete or not (see [11] in this volume). The general problem of characterizing algebras admitting submultiplicative norms is far from being solved.

**3.2. Multiplicative maps on topological algebras.** Let  $T : A \rightarrow \mathbb{C}$  be a linear-multiplicative functional on a topological algebra  $A$ . Is then  $T$  automatically continuous? Not surprisingly, the answer to such a very general question is NO, and an example is easy.

**Example 5.** Let  $A = C([0, 1])$  be the algebra of all continuous functions on the unit segment with the pointwise multiplication. We introduce on  $A$  not the usual sup norm topology but the topology given by a family of submultiplicative seminorms:

$$p_t(f) = |f(t)|, \quad 0 < t \leq 1.$$

(This is the topology of pointwise convergence on  $(0, 1]$  - notice that 0 is excluded.)  
Then

$$C([0, 1]) \ni f \longmapsto f(0)$$

is a discontinuous linear-multiplicative functional.

The example is nice and simple but we can not say the same about the algebra involved.

**Definition 6.** *An algebra  $A$  is a Fréchet algebra if its topology is given by a countable family of submultiplicative seminorms:*

$$p_n(fg) \leq p_n(f)p_n(g),$$

*and the algebra is complete.*

For example,  $A$  can be equal to  $C(\mathbb{C})$ , the algebra of all continuous functions on the complex plane  $\mathbb{C}$ , or to the algebra  $Hol(\mathbb{C})$  of holomorphic functions on  $\mathbb{C}$ , both equipped with the topology of uniform convergence on compact sets. Perhaps the most famous, still open problem of the theory of topological algebras is whether any linear-multiplicative functional on a Fréchet algebra is continuous (see for example [2], [5], [9]). There are several partial results; let us list just three.

- Any linear-multiplicative functional on a Fréchet algebra with finitely many generators is continuous ([9]);
- If the spectrum of a Fréchet algebra  $A$  is countable then any linear-multiplicative functional on  $A$  is continuous ([22]);
- Let  $X$  be an infinite dimensional Banach space and let  $Hol_b(X)$  be an algebra of holomorphic functions on  $X$  which are bounded on bounded subsets of  $X$ . If any multiplicative functional on  $Hol_b(X)$  is continuous then any multiplicative functional on any Fréchet algebra is continuous ([1]).

### 3.3. Almost multiplicative functionals.

**Definition 7.** *A linear functional  $F : A \rightarrow \mathbb{C}$  defined on a Banach algebra  $A$  is  $\varepsilon$ -multiplicative if*

$$|F(ab) - F(a)F(b)| \leq \varepsilon \|a\| \|b\|, \text{ for } a, b \in A.$$

**Proposition 8.** ([12]) *Any  $\varepsilon$ -multiplicative functional  $F$  is continuous, and  $\|F\| \leq 1 + \varepsilon$ .*

The above Proposition answers the automatic continuity question, however several related problems remain open. Notice that if  $T : A \rightarrow \mathbb{C}$  is linear-multiplicative  $S : A \rightarrow \mathbb{C}$  is a linear map with a small norm,  $\|S\| \leq \varepsilon$  then  $T + S$  is  $(3\varepsilon + \varepsilon^2)$ -multiplicative (direct computation). So a functional close to a multiplicative one is almost multiplicative. The obvious question is whether the converse is true: *Is any*

*almost multiplicative map close to a multiplicative one?* It is an easy exercise to check that the answer is YES for the  $C(K)$  algebras. In 1986 B. E. Johnson proved that the disc algebra  $A(\mathbb{D})$  and some other Banach algebras also have the property [14]. It was then conjectured that all uniform algebras, that is, subalgebras of  $C(K)$  have the property. However, very recently S. J. Sidney constructed an example of a uniform algebra without the property [19]. It is still an open problem whether  $H^\infty(\mathbb{D})$  has the property. The question may be phrased as whether  $H^\infty(\mathbb{D})$  has an “almost corona”, that is, almost multiplicative functionals that are far from any multiplicative functionals. See [10] in this volume for more information on almost multiplicative functionals.

**3.4. Disjointedness preserving maps.** In the previous section we defined almost multiplicative maps as maps  $F$  for which  $F(ab)$  is close to  $F(a)F(b)$  for any  $a, b$ . There are several other ways to weaken the multiplicatively condition.

**Definition 9.** *We say that a linear map  $T : A \rightarrow B$  is disjointedness preserving if for any  $a, b \in A$  we have*

$$ab = 0 \quad \Rightarrow \quad TaTb = 0.$$

Notice that the definition applies not only to Banach algebras but also to Banach lattices. In general disjointedness preserving maps may be discontinuous.

**Example 10.** ([13]) *For any infinite compact set  $K$  and for any compact set  $L$  there is a discontinuous disjointedness preserving linear map*

$$T : C(K) \rightarrow C(L).$$

*For example there is even a surjective disjointedness preserving map from the separable space  $c_0$  of all sequences convergent to zero, onto the non-separable space  $l^\infty$  of all bounded sequences.*

However, there are also some positive results.

**Example 11.** ([13]) *If  $T : C(K) \rightarrow C(L)$  is a disjointness preserving linear bijection then  $T$  is continuous.*

The above result follows from a general description [13] of disjointedness preserving maps  $T : C(K) \rightarrow C(L)$ . Roughly speaking we can always divide  $L$  into three subsets. On the first part  $T$  is just the zero map, on the second part  $T$  is given by a composition with a continuous map from this subset of  $L$  into  $K$ . The third part of  $L$  is finite and  $T$  is discontinuous at every point of this part.

The problem of extending the above results to other Banach lattices is in general open; there are, however, some partial result, see for example [7].

## 4. TRANSLATION INVARIANT MAPS

Let  $X$  be a Banach space of functions on a locally compact group  $G$  such that  $f_\alpha \in X$  whenever  $f \in X$  and  $\alpha \in G$ ; where  $f_\alpha(x) \stackrel{\text{df}}{=} f(\alpha + x)$ . For example  $X = L^p(G), C(G), H^p, M(G), \dots$ . Let  $F$  be a translation invariant linear functional on  $X$ , that is, a linear functional satisfying

$$F(f_\alpha) = F(f) \quad f \in X, \alpha \in G.$$

The question we want to address in this section is whether any translation invariant map is automatically continuous. In most cases it would then easily follow that  $F$  is of the form

$$F(f) = \text{const} \int_G f.$$

The above question is a special case of a much more general problem: If  $\mathfrak{U}$  is a set of continuous linear maps on a Banach space  $X$  and  $T : X \rightarrow X$  is a linear map such that

$$U \circ T = T \circ U, \quad U \in \mathfrak{U},$$

could we then automatically conclude that  $T$  is continuous?

There is a large number of papers devoted to this and related problems, see for example [4] or [20] and the references there. We mention here only few results concerning spaces of functions on the unit circle  $S$ .

- Any translation invariant linear map on  $X = L^2(S)$  is continuous ([16]),
- There are discontinuous translation invariant linear maps on  $X = C(S)$  or  $X = L^\infty(S)$  ([18]),
- Any translation invariant linear map on  $X = L^p(S)$ ,  $1 < p < \infty$  is continuous ([17]).

We very briefly sketch the basic idea behind the proofs. We first notice that  $F$  is translation invariant if and only if

$$M \stackrel{\text{df}}{=} \text{span} \{f - f_\alpha : f \in X, \alpha \in S\} \subseteq \ker F.$$

Since any continuous translation invariant linear map on  $L^p(S)$  or  $C(S)$  is a scalar multiple of the Lebesgue integral it follows that there is a discontinuous, translation invariant linear functional on  $X$  if and only if

$$\text{codim}_X(M) > 1.$$

For any integer  $n$  we find the Fourier transform  $(\widehat{f - f_\alpha})(n)$  of  $f - f_\alpha$  at  $n$

$$\begin{aligned} (\widehat{f - f_\alpha})(n) &= \hat{f}(n) - \hat{f}_\alpha(n) \\ &= \int f(\theta)e^{-2\pi ni\theta}d\theta - \int f(\theta + \alpha)e^{-2\pi ni\theta}d\theta \\ &= \int f(\theta)e^{-2\pi ni\theta}d\theta - \int f(\theta)e^{-2\pi ni(\theta - \alpha)}d\theta \\ &= (1 - e^{2\pi ni\alpha})\hat{f}(n). \end{aligned}$$

Hence there is a discontinuous, translation invariant linear functional on  $X$  if and only if

$$\text{codim}_{\hat{X}} \left( \text{span} \left\{ \left( (1 - e^{2\pi ni\alpha}) \hat{f}(n) \right)_{n=-\infty}^{\infty} : \hat{f} \in \hat{X}, \alpha \in \mathbb{R} \right\} \right) > 1.$$

If  $\alpha_k, k = 1, 2, \dots, s$  are all rational numbers and  $f_k \in X$  then any linear combination  $\sum_{k=1}^s \lambda_k \left( (1 - e^{2\pi ni\alpha_k}) \hat{f}_k(n) \right)_{n=-\infty}^{\infty}$  is equal to zero for infinitely many  $n$ . It immediately follows that

$$\text{codim}_{\hat{X}} \left( \text{span} \left\{ \left( (1 - e^{2\pi ni\alpha}) \hat{f}(n) \right)_{n=-\infty}^{\infty} : \hat{f} \in \hat{X}, \alpha \in \mathbb{Q} \right\} \right) = \infty,$$

so there are discontinuous linear maps on  $X$  commuting with all rational translations. If not all of the numbers  $\alpha_k$  are rational then the sequence

$$\sum_{k=1}^s \lambda_k \left( (1 - e^{2\pi ni\alpha_k}) \hat{f}_k(n) \right)_{n=-\infty}^{\infty}$$

need not be equal to zero for any  $n \neq 0$ . Still

$$\inf_{n \neq 0} \left\{ \sum_{k=1}^s |1 - e^{2\pi ni\alpha_k}| \right\} = 0.$$

So, in some sense, as  $n \rightarrow \pm\infty$ ,  $\sum_{k=1}^s \lambda_k \left( (1 - e^{2\pi ni\alpha_k}) \hat{f}_k(n) \right)_{n=-\infty}^{\infty}$  converges to zero, faster than individual sequences  $\left( \hat{f}_k(n) \right)_{n=-\infty}^{\infty}$ . Now one needs to combine results from a geometric number theory and from harmonic analysis to decide whether this ‘improvement’ in convergence leads, for a particular Banach space  $\hat{X}$ , to a subspace of  $\hat{X}$  of codimension one or greater than one.

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