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Biseparating maps between operator algebras

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Abstract

We prove that a biseparating map between spaces $B(E)$, and some other Banach algebras, is automatically continuous and a multiple of an algebra isomorphism.

Keywords: Automatic continuity; Biseparating maps; Preserver problem

1. Introduction

Linear maps between Banach algebras, Banach lattices, or Banach spaces preserving certain properties have been of a considerable interest for many years. The most classical question concerns isometries, that is, maps that preserve the norm. More recently, maps that preserve spectrum, spectral radius, commutativity, normal elements, self-adjoint elements, nilpotents, idempotents, linear rank, disjointness, or other properties have been intensely investigated, see for example [5,6,8,9,11–15] and the references given there. Here we study the biseparating maps, that is maps preserving disjointness in both directions.

Definition 1. A linear map T between algebras $\mathcal{A}$, $\mathcal{B}$ is called separating if

$$ab = 0 \;\Rightarrow\; T(a)T(b) = 0, \quad \text{for all } a, b \in \mathcal{A};$$

it is called biseparating if $T^{-1} : \mathcal{B} \to \mathcal{A}$ exists and is also separating.

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Any algebra isomorphism, that is, a map that preserves both linear and multiplicative structures, is clearly separating, also any algebra isomorphism followed by a multiplication by a fixed element from the commutant of $\mathcal{B}$ is separating—we will call such a map a *standard separating map*. However in general a separating map may be very far from being multiplicative. For example if $\mathcal{A}$ is the disc algebra then the product of two elements from $\mathcal{A}$ is equal to zero only if one of them is already zero, consequently any linear map on $\mathcal{A}$ is separating. On the other hand any biseparating map $T : C(X) \to C(Y)$, where $C(X)$ is the space of all continuous functions defined on a completely regular set X, is of the form

$$Tf = \tau \cdot f \circ \varphi, \quad \text{for all } f \in C(X), \tag{1.1}$$

where τ is a nonvanishing continuous function and φ is a homeomorphism from rY, the realcompactification of Y, onto rX [3]. It is interesting to notice that for compact sets X, Y a separating invertible map $T : C(X) \to C(Y)$ is automatically biseparating and consequently of the form (1.1) [10]. However, whether this is also true for noncompact sets X, Y, is an open problem (see [4] for a partial solution). A reader interested in separating and biseparating maps in a more general setting may want to check a recent monograph [1].

In this note we show that any biseparating map between the algebras $B(E)$ of all continuous linear maps on a Banach space E, as well as between certain subalgebras of $B(E)$ and tensor products of such algebras is standard. The results apply both in the real and in the complex cases.

2. Results

For a Banach space E we will call a subalgebra $\mathcal{A}$ of $B(E)$ *standard* if it contains all finite-dimensional operators and the identity operator Id. Some authors also assume that a standard subalgebra is closed in the norm topology, here we do not make this assumption. Notice that a standard subalgebra must contain all continuous projections onto closed finite-codimensional subspaces of E. Indeed if P is such a projection then

$$P = Id - (Id - P),$$

where $Id - P$ is finite-dimensional.

Theorem 1. *Assume E_1, E_2 are Banach spaces and $\mathcal{A}_1, \mathcal{A}_2$ are standard subalgebras of $B(E_1)$ and of $B(E_2)$, respectively. If $T : \mathcal{A}_1 \to \mathcal{A}_2$ is biseparating then it is continuous and of the form*

$$T(A) = \alpha S \circ A \circ S^{-1}, \quad \textit{for } A \in \mathcal{A}_1,$$

where S is a continuous linear isomorphism from E_1 onto E_2 and α a nonzero scalar.

Proof. We first need to introduce some notation. For $i = 1, 2$ we put

$$B^{\perp} = \{A \in \mathcal{A}_i \backslash \{0\}:\ A \circ B = 0\}, \quad \text{for } B \in \mathcal{A}_i,$$

$$\mathcal{M}_i = \{B^{\perp}:\ B \in \mathcal{A}_i\},$$

$$\mathcal{M}_i^1 = \mathcal{M}_i \backslash \{B^{\perp} \in \mathcal{M}_i:\ \exists C \in \mathcal{A}_i,\ \emptyset \neq C^{\perp} \subsetneq B^{\perp}\}.$$

For $e \in E_i$ and $e^* \in E_i^*$ we denote by $e \otimes e^*$ the one-dimensional operator on E_i defined by

$$(e \otimes e^*)(e') = e^*(e')e, \quad \text{for } e' \in E_i.$$

Since the definitions of $\mathcal{M}_1$ and $\mathcal{M}_1^1$ involve only the structures that are preserved by T, T maps these sets on $\mathcal{M}_2$ and $\mathcal{M}_2^1$, respectively.

Notice that $B^\perp$ is trivial if and only if range(B) is dense, and $B^\perp$ consists of operators of dimension one if and only if the closure $\overline{\text{range}\, B}$ of the range of B is one-codimensional. Since $\mathcal{A}_i$ contains all projections onto closed one-codimensional subspaces, we have

$$\mathcal{M}_i^1 = \bigl\{\{A \in \mathcal{A}_i \colon \ker A = E_0\} \colon E_0 = \overline{E_0} \subset E_i,\ \dim E_i/E_0 = 1\bigr\}.$$

Hence $\bigcup \mathcal{M}_i^1$ is simply equal to the set of all the one-dimensional operators, so T maps a one-dimensional operator onto a one-dimensional operator.

Fix a linear continuous functional e^* on E_1. For any $e \in E_1$ we have

$$T(e \otimes e^*) = f \otimes f_{e^*}^*$$

for some $f \in E_2$ and $f_{e^*}^* \in E_2^*$. If we change the point e but keep the same functional e^*, the operator $e_1 \otimes e^*$ will still belong to the same element of $\mathcal{M}_1^1$. Hence $T(e_1 \otimes e^*) = f_1 \otimes f_1^*$ belongs to the same element of $\mathcal{M}_2^1$; this means that $\ker f_1^* = \ker f_{e^*}^*$ and the functionals $f_{e^*}^*, f_1^*$ are proportional. So there is a map $S_{e^*} : E_1 \to E_2$ such that

$$T(e \otimes e^*) = S_{e^*}(e) \otimes f_{e^*}^*, \quad \text{for all } e \in E_1.$$

Since T is linear, so must be S_{e^*}; since $\ker T$ is trivial, the same must be true about $\ker S_{e^*}$. Finally, since T is surjective, it maps an element of $\mathcal{M}_1^1$ onto an entire element of $\mathcal{M}_2^1$ so S_{e^*} is surjective.

Assume e_1^* is another continuous linear functional on E_1, not proportional to e^*, and let $S_{e_1^*}, f_{e_1^*}^*$ be such that

$$T\bigl(e \otimes e_1^*\bigr) = S_{e_1^*}(e) \otimes f_{e_1^*}^*, \quad \text{for all } e \in E_1.$$

Since $\ker e^* \neq \ker e_1^*$, then $e \otimes e^*$ and $e \otimes e_1^*$ belong to distinct elements of $\mathcal{M}_1$, for any $e \in E_1 \backslash \{0\}$, so $\ker f_{e^*}^* \neq \ker f_{e_1^*}^*$. Suppose the linear maps S_{e^*} and $S_{e_1^*}$ are not proportional and let e_0 be such that $S_{e^*}(e_0)$ and $S_{e_1^*}(e_0)$ are linearly independent. Then the operator

$$e_0 \otimes e^* + e_0 \otimes e_1^*$$

is one-dimensional, while

$$T\bigl(e_0 \otimes e^* + e_0 \otimes e_1^*\bigr) = S_{e^*}(e_0) \otimes f_{e^*}^* + S_{e_1^*}(e_0) \otimes f_{e_1^*}^*,$$

is two-dimensional. The contradiction shows that there is a linear bijection $S : E_1 \to E_2$ and a map $\Psi : E_1^* \to E_2^*$ such that

$$T(e \otimes e^*) = S(e) \otimes \Psi(e^*), \quad \text{for all } e \in E_1, e^* \in E_1^*. \tag{2.1}$$

As in the case of S_{e^*}, since T is a linear bijection so must be Ψ.

We now show that S is continuous. For any $e_1, e_2 \in E_1$ and $e_1^*, e_2^* \in E_1^*$ we have

$$\bigl(e_1 \otimes e_1^*\bigr) \circ \bigl(e_2 \otimes e_2^*\bigr) = e_1^*(e_2)\bigl(e_1 \otimes e_2^*\bigr),$$

and by (2.1)

$$T\big(e_1 \otimes e_1^*\big) \circ T\big(e_2 \otimes e_2^*\big) = \big(S(e_1) \otimes \Psi\big(e_1^*\big)\big) \circ \big(S(e_2) \otimes \Psi\big(e_2^*\big)\big)$$
$$= \big(\Psi\big(e_1^*\big)\big)\big(S(e_2)\big)\big(S(e_1) \otimes \Psi\big(e_2^*\big)\big).$$

So

$$e_1^*(e_2) = 0 \quad \text{iff} \quad \big(\Psi\big(e_1^*\big)\big)\big(S(e_2)\big) = 0,$$

hence

$$\ker \Psi\big(e_1^*\big) = S\big(\ker e_1^*\big), \quad \text{for all } e_1^* \in E_1^*$$

or

$$\ker\big(\Psi\big(e_1^*\big) \circ S\big) = S^{-1}\big(\ker \Psi\big(e_1^*\big)\big) = \ker e_1^*. \tag{2.2}$$

Hence $\Psi(e_1^*) \circ S$ has a closed kernel, and consequently is continuous for any $e_1^* \in E_1^*$. Since Ψ is surjective, this means that S becomes continuous when composed with any continuous functional. So, by the Closed Graph Theorem $S : E_1 \to E_2$ is continuous itself. We now can define a map $\widetilde{T} : \mathcal{A}_2 \to B(E_1)$ by

$$\widetilde{T}(A) = S^{-1} \circ A \circ S, \quad \text{for } A \in \mathcal{A}_2.$$

By (2.1)

$$\widetilde{T} \circ T(e \otimes e^*) = S^{-1} \circ \big(S(e) \otimes \Psi(e^*)\big) \circ S = e \otimes \Psi(e^*) \circ S = e \otimes S^* \circ \Psi(e^*)$$

and by (2.2)

$$\ker \Psi(e^*) \circ S = \ker e^*, \quad \text{for all } e^* \in E_1^*,$$

so each $e^* \in E_1^*$ is an eigenvector of $S^* \circ \Psi$, hence $S^* \circ \Psi = \alpha Id$, and

$$\widetilde{T} \circ T(e \otimes e^*) = \alpha e \otimes e^*, \quad \text{for all } e \in E_1,\ e_1^* \in E^*. \tag{2.3}$$

Since $\widetilde{T} \circ T$ is linear it follows that

$$\widetilde{T} \circ T(A) = \alpha A$$

for all finite-dimensional operators A. Notice that $\alpha \neq 0$ since both T and $\widetilde{T}$ are invertible.

Assume that $\widetilde{T} \circ T \neq \alpha Id$ and let $A_0 \in \mathcal{A}_1$, $\tilde{e} \in E_1$ be such that $\widetilde{T} \circ T(A_0) \neq \alpha A_0$ and $\alpha A_0(\tilde{e}) \neq (\widetilde{T} \circ T(A_0))(\tilde{e})$. Put

$$B_0 = A_0 - A_0(\tilde{e}) \otimes \tilde{e}^*,$$

where $\tilde{e}^* \in E_1^*$ is such that $\tilde{e}^*(\tilde{e}) = 1$. We have

$$B_0 \circ (\tilde{e} \otimes \tilde{e}^*) = \big(A_0 - A_0(\tilde{e}) \otimes \tilde{e}^*\big)(\tilde{e}) \otimes \tilde{e}^* = 0,$$

while by (2.3)

$$\begin{aligned}
\big(\widetilde{T} \circ T(B_0)\big) \circ \big(\widetilde{T} \circ T(\tilde{e} \otimes \tilde{e}^*)\big) &= \alpha\big(\widetilde{T} \circ T(B_0)\big) \circ (\tilde{e} \otimes \tilde{e}^*) \\
&= \alpha\big(\widetilde{T} \circ T(B_0)\big)(\tilde{e}) \otimes \tilde{e}^* \\
&= \alpha\big(\widetilde{T} \circ T(A_0) - \alpha(A_0)(\tilde{e}) \otimes \tilde{e}^*\big)(\tilde{e}) \otimes \tilde{e}^* \\
&= \alpha\big(\widetilde{T} \circ T(A_0)(\tilde{e}) - \alpha(A_0)(\tilde{e})\big) \otimes \tilde{e}^* \\
&\neq 0
\end{aligned}$$

which is a contradiction since $\widetilde{T} \circ T$ is biseparating. Hence $\widetilde{T} \circ T = \alpha Id$ so

$$TA = \alpha \widetilde{T}^{-1} A = \alpha S \circ A \circ S^{-1}, \quad \text{for } A \in \mathcal{A}_1. \qquad \square$$

The next theorem extends the last result to the algebras of continuous operator valued functions. For a Hausdorff set X and a normed algebra $\mathcal{A}$ we denote by $C(X, \mathcal{A})$ the algebra of all continuous $\mathcal{A}$-valued functions on X with the obvious multiplication defined by

$$F \cdot G(x) = F(x) \cdot G(x), \quad \text{for } x \in X \text{ and } F, G \in C(X, \mathcal{A});$$

$C_b(X, \mathcal{A})$ is the subalgebra of $C(X, \mathcal{A})$ consisting of norm bounded functions. We equip $C_b(X, \mathcal{A})$ with the sup norm topology and $C(X, \mathcal{A})$ with the topology of uniform convergence on compact subsets of X. We denote by $C(X)$ the Banach algebra of all scalar valued continuous functions on X and by $C^{-1}(X)$ the set of invertible elements of $C(X)$; we use notation $C_b(X)$ for the Banach algebra of all bounded scalar valued continuous functions on X.

For a function $f \in C(X)$ it may be often convenient to extend f to a continuous function on βX, the maximal compactification of X. In general, for $x \in \beta X \backslash X$, the value of $f(x)$ may be infinite. There are however completely regular spaces X with a point $x_0 \in \beta X \backslash X$ such that the value of $f(x_0)$ is finite for all $f \in C(X)$; the set of all points in βX with this property is called the realcompactification of X and is denoted by rX. Since we have $C(X) = C(rX)$ the natural domain for a continuous function on X is rX, not X. Hence we will often consider realcompactifications, or alternatively we will assume that the completely regular spaces under consideration are realcompact. All compact sets are clearly realcompact, also all Lindelöf spaces, and all metrizable spaces of nonmeasurable cardinal are realcompact ([7, p. 232]).

For Banach spaces E_1, E_2, $B(E_1, E_2)$ is the space of all linear continuous maps from E_1 into E_2 equipped with the norm topology and $B^{-1}(E_1, E_2)$ is the subset (possibly empty) of $B(E_1, E_2)$ consisting of invertible isomorphisms.

Since $C(X, \mathcal{A})$ is an algebra, Definition 1 describes the meaning of a separating map from $C(X_1, \mathcal{A}_1)$ into $C(X_2, \mathcal{A}_2)$. However for the spaces of vector valued continuous functions there is a possible alternative but not equivalent natural definition: here we will call $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ *strictly separating* if

$$\begin{aligned} &\|F_1(\cdot)\| \|F_2(\cdot)\| \equiv 0 \implies \|\big(T(F_1)\big)(\cdot)\| \|\big(T(F_2)\big)(\cdot)\| \equiv 0, \\ &\quad \text{for } F_1, F_2 \in C(X_1, \mathcal{A}_1). \end{aligned}$$

That definition can be applied also if $\mathcal{A}$ is not an algebra but just a normed linear space. We will later refer to [2] where a general form of a strictly biseparating map is given.

Theorem 2. *Assume E_1, E_2 are Banach spaces and $\mathcal{A}_1$, $\mathcal{A}_2$ are standard subalgebras of $B(E_1)$ and of $B(E_2)$, respectively. Assume further that X_1, X_2 are realcompact spaces and*

- *$T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ is a biseparating map*

or

- *$T : C_b(X_1, \mathcal{A}_1) \to C_b(X_2, \mathcal{A}_2)$ is a biseparating map and $\mathcal{A}_1, \mathcal{A}_2$ are both infinite-dimensional.*

Then T is continuous and of the form

$$\big(T(F)\big)(x) = \alpha(x) S_x \circ F\big(\varphi(x)\big) \circ S_x^{-1}, \quad \textit{for } x \in X_2 \textit{ and } F \in \operatorname{domain}(T),$$

where $x \mapsto S_x$ is a continuous map from X_2 into $B^{-1}(E_1, E_2)$, φ is a homeomorphism from X_2 onto X_1, and $\alpha \in C^{-1}(X_2)$.

Proof. Again we first need to introduce some notation. For $i = 1, 2$ we denote $\widetilde{C}(X_i, \mathcal{A}_i)$ either $C(X_i, \mathcal{A}_i)$ or $C_b(X_i, \mathcal{A}_i)$ depending on the domain of the map T; for $F \in \widetilde{C}(X_i, \mathcal{A}_i)$ we put

$$\begin{aligned}
c(F) &= \big\{x \in X_i \colon\ F(x) \neq 0\big\},\\
L(F) &= \big\{G \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ G \cdot F = 0\big\},\\
R(F) &= \big\{G \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ F \cdot G = 0\big\},\\
\mathcal{AI}_i &= \big\{H \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ L(H) \subset R(H)\big\},\\
\mathcal{C}(F) &= \big\{G \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ \forall H \in \mathcal{AI}_i [F \cdot H = 0 \Longrightarrow G \cdot H = 0]\big\}.
\end{aligned}$$

Notice that the sets $L(F)$, $R(F)$, $\mathcal{AI}_i$, and $\mathcal{C}(F)$ have been defined solely using the properties that are preserved by T hence

$$\begin{aligned}
&T\big(L(F)\big) = L\big(T(F)\big), \qquad T\big(R(F)\big) = R\big(T(F)\big),\\
&T(\mathcal{AI}_1) = \mathcal{AI}_2, \qquad T\big(\mathcal{C}(F)\big) = \mathcal{C}\big(T(F)\big).
\end{aligned}$$

We show that

$$F_1 \cdot F_2 = 0 \iff c(F_1) \cap c(F_2) = \emptyset, \quad \text{for } F_1 \in \widetilde{C}(X_i, \mathcal{A}_i),\ F_2 \in \mathcal{AI}_i. \tag{2.4}$$

The implication $\Longleftarrow$ is obviously true for all functions in $\widetilde{C}(X_i, \mathcal{A}_i)$. Assume $F_1 \cdot F_2 = 0$ and $x_0 \in c(F_1) \cap c(F_2)$. Since both $F_1(x_0)$ and $F_2(x_0)$ are nonzero maps there is a continuous one-dimensional linear map A on E_i such that $A(F_1(x_0)) \notin \ker F_2(x_0)$. Put

$$G(x) = A \circ F_1(x), \quad \text{for } x \in X_i.$$

We have

$$G \cdot F_2 = 0 \quad \text{while } F_2 \cdot G(x_0) = F_2(x_0) \circ \big(A\big(F_1(x_0)\big)\big) \neq 0$$

so $F_2 \notin \mathcal{AI}_i$, which concludes the proof of (2.4).

By (2.4)

$$\begin{aligned}
\mathcal{C}(F) &= \big\{G \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ \forall H \in \mathcal{AI}_i\big[c(F) \cap c(H) = \emptyset \Longrightarrow c(G) \cap c(H) = \emptyset\big]\big\}\\
&= \big\{G \in \widetilde{C}(X_i, \mathcal{A}_i) \colon\ c(G) \subset \mathit{int}\big(\overline{c(F)}\big)\big\},
\end{aligned}$$

for arbitrary open sets K, L we have $K \cap L = \emptyset$ if and only if $\mathit{int}(\overline{K}) \cap \mathit{int}(\overline{L}) = \emptyset$, so we get

$$c(F_1) \cap c(F_2) = \emptyset \iff \mathcal{C}(F_1) \cap \mathcal{C}(F_2) = \{0\}, \quad \text{for } F_1, F_2 \in \widetilde{C}(X_i, \mathcal{A}_i).$$

Since $T(\mathcal{C}(F)) = \mathcal{C}(T(F))$, for $F \in C(X_i, \mathcal{A}_i)$, the above proves that T is strictly bi-separating:

$$\|F_1(\cdot)\| \|F_2(\cdot)\| \equiv 0 \iff \|\big(T(F_1)\big)(\cdot)\| \|\big(T(F_2)\big)(\cdot)\| \equiv 0, \\ \text{for } F_1, F_2 \in \widetilde{C}(X_i, \mathcal{A}_i). \tag{2.5}$$

We need the following result from [2].

Theorem 3. *Assume N_1, N_2 are normed spaces, X_1, X_2 are realcompact spaces, and $T : \widetilde{C}(X_1, N_1) \to \widetilde{C}(X_2, N_2)$ is a linear bijection satisfying* (2.5) (*where N_1, N_2 are assumed to be infinite-dimensional in the case when $\widetilde{C}(X_1, N_1) = C_b(X_1, N_1)$, $\widetilde{C}(X_2, N_2) = C_b(X_2, N_2)$*). *Then there is a bijective homeomorphism $\varphi : X_2 \to X_1$ and a map J from X_2 into the set of linear bijections from N_1 onto N_2 such that*

$$\big(T(F)\big)(x) = \big(J(x)\big)\big(F\big(\varphi(x)\big)\big), \quad \textit{for } x \in X_2, \textit{ and } F \in \widetilde{C}(X_1, N_1). \tag{2.6}$$

To finish the proof of Theorem 2 let $A_1, A_2 \in \mathcal{A}_1$ be such that $A_1 \circ A_2 = 0$, and denote by $\mathbf{A}_1, \mathbf{A}_2$ the constant functions on X_2 equal to A_1, and to A_2, respectively. By Theorem 3 for any $x \in X_2$

$$0 = T(\mathbf{A}_1) \cdot T(\mathbf{A}_2)(x) = \big(J(x)\big)(A_1) \circ \big(J(x)\big)(A_2),$$

so $J(x) : \mathcal{A}_1 \to \mathcal{A}_2$ is separating. By the same arguments applied to T^{-1} we conclude that $J(x)$ is biseparating. By Theorem 1

$$\big(J(x)\big)(A) = \alpha_x S_x \circ A \circ S_x^{-1}, \quad \text{for } A \in \mathcal{A}_1,\ x \in X_2,$$

where $S_x \in B^{-1}(E_1, E_2)$. Hence, by (2.6)

$$\big(T(F)\big)(x) = \alpha_x S_x \circ \big(F\big(\varphi(x)\big)\big) \circ S_x^{-1}, \quad \text{for } x \in X_2, \text{ and } F \in \widetilde{C}(X_1, \mathcal{A}_1),$$

to check that $x \mapsto \alpha_x$ is a continuous function it is enough to put into the above formula F equal, at every point of X_1, to the identity map on E_1. $\square$

Remark 1. Notice that in Theorem 2, when $\mathcal{A}_1$ is finite-dimensional, every continuous bounded map from X_1 into $\mathcal{A}_1$ can be extended to a continuous map from the Stone–Čech compactification βX_1 of X_1 into $\mathcal{A}_1$. We deduce that, for finite-dimensional $\mathcal{A}_1$ and $\mathcal{A}_2$, every biseparating map between $C_b(X_1, \mathcal{A}_1)$ and $C_b(X_2, \mathcal{A}_2)$ can be considered as a biseparating map between $C(\beta X_1, \mathcal{A}_1)$ and $C(\beta X_2, \mathcal{A}_2)$, that is, in this case the homeomorphism φ given in this theorem maps βX_2 onto βX_1.

Conjecture 1. Assume T is a biseparating map between C^* algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Then T is continuous and

$$T(a) = A_0 \circ \Psi(a), \quad \text{for } a \in \mathcal{A}_1,$$

where A_0 is in the commutant of $\mathcal{A}_2$.

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