

Separating maps on spaces of continuous functions

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ABSTRACT. We investigate linear bijections $T : C(X) \rightarrow C(Y)$ such that

$$f \cdot g = 0 \implies Tf \cdot Tg = 0, \text{ for } f, g \in C(X).$$

1. Introduction

Let A, B be vector lattices and $T : A \rightarrow B$ be linear transformation. We call T separating or disjointness preserving if it maps elements with disjoint support onto elements with disjoint support. In some cases we have a complete description of all the separating maps. For example if A is equal to an algebra $C(X)$ of all continuous functions on a compact set X , and $B = C(Y)$, for a compact set Y , then any separating bijection $T : C(X) \rightarrow C(Y)$ has a very simple form

$$(1.1) \quad T(f) = \varkappa \cdot f \circ \varphi, \text{ for } f \in C(X),$$

where $\varphi : Y \rightarrow X$ is a homeomorphism and \varkappa is a non-vanishing continuous function [J1], see also [ABN2][FH]. However as soon as we drop the assumption that T is a bijection, and/or the assumption that A, B are Banach spaces the situation become much more complicated. In general a separating map may be discontinuous, and the inverse of a separating bijection may be non-separating, see for example [AK1][AK2][J1].

In this note we are interested in separating bijections $T : C(X) \rightarrow C(Y)$, for non-compact sets X, Y . The question whether the inverse of such a map must be automatically separating and consequently given by homeomorphism is perhaps the simplest and most natural among the open problem in this area. We prove that the answer is positive if X is a subset of the real line.

2. Definitions and basic properties

Let X, Y be Tychonov spaces and let $T : C(X) \rightarrow C(Y)$ be a linear map; we do not assume that X, Y are compact, neither we assume that T is continuous. For a function $f \in C(X)$ it may be often convenient to consider f also as a continuous function on βX , the maximal compactification of X ; the extended function takes values in $\mathbb{R} \cup \{\infty\}$. There exist Tychonov non compact spaces X with a point x_0 in $\beta X \setminus X$ such that the value of $f(x_0)$ is finite for all $f \in C(X)$; the set of all points in βX with this property is called the realcompactification of X and denoted by rX [E]. We have $C(X) = C(rX)$. As we investigate the possible extensions

of (1.1) the natural domain and codomain for the homeomorphism φ are the real compactifications of Y and X , respectively. Hence we will often assume that the sets X, Y are realcompact; in some cases we get this property automatically as for example any subset of the real line is realcompact.

For any $y \in Y$ we define

$$\begin{aligned} \widehat{y} &: C(Y) \rightarrow \mathbb{R}, & \text{by } \widehat{y}(g) &= g(y), \text{ and} \\ T^*(\widehat{y}) &: C(X) \rightarrow \mathbb{R}, & \text{by } T^*(\widehat{y})(f) &= T(f)(y). \end{aligned}$$

For a continuous function f we denote by $c(f)$ the cozero set of f , that is the set of all x from the domain of f such that $f(x) \neq 0$.

For a linear functional F on $C(X)$ and an open set U of X we say that U is a vanishing set for F if

$$c(f) \subset U \implies F(f) = 0, \quad \text{for all } f \in C(X).$$

We define $\text{supp}(F)$, the support of F , as the complement of the union of all vanishing sets for F .

We say that $T : C(X) \rightarrow C(Y)$ is separating, or disjointness-preserving if

$$f \cdot g = 0 \implies Tf \cdot Tg = 0, \quad \text{for } f, g \in C(X).$$

A bijection $T : C(X) \rightarrow C(Y)$ is biseparating if both T and T^{-1} are separating.

The following Lemma is the most fundamental technical tool concerning the separating maps on $C(X)$ spaces.

LEMMA 2.1. *Let X, Y be Tychonov spaces and let $T : C(X) \rightarrow C(Y)$ be an injective separating linear map. Then there is an open subset D of Y and a continuous function φ from D onto a dense subset of βX such that*

$$\text{supp}(T^*(\widehat{y})) = \{\varphi(y)\}, \quad \text{for } y \in D.$$

Let $f \in C(X)$, it follows directly from the definition of the support set that $T(f)(y) = 0$ for $y \notin \varphi^{-1}(\overline{c(f)})$. The next Lemma generalizes this observation.

LEMMA 2.2 ([**ABN2**, Lemma 7]). *For any $f \in C(X)$ we have*

$$(2.1) \quad T(f)(y) = 0 \quad \text{if } y \notin \text{int}(\varphi^{-1}(c(f))).$$

In general the map φ need not be surjective neither injective, however if T is biseparating φ must be a homeomorphism.

THEOREM 2.3 ([**ABN1**]). *Let X, Y be realcompact spaces and let $T : C(X) \rightarrow C(Y)$ be a biseparating linear map. Then there is a homeomorphism $\varphi : Y \rightarrow X$ and a non-vanishing continuous function \varkappa such that*

$$T(f) = \varkappa \cdot f \circ \varphi, \quad \text{for } f \in C(X).$$

The assumption that a map is biseparating is essential and the Theorem follows in a simple way from Lemma 2.1. For general vector lattices, and Banach lattices a separating bijection does not have to be biseparating. For $C(X)$ spaces, it is not known whether a separating bijection must be biseparating, except for some special classes of sets X .

THEOREM 2.4 ([**ABN2**]). *Let X, Y be Tychonov spaces and let $T : C(X) \rightarrow C(Y)$ be a separating linear bijection. If X is zero dimensional or Y is connected then T is biseparating.*

THEOREM 2.5. *Let X, Y be Tychonov spaces and let $T : C(X) \rightarrow C(Y)$ be a separating linear bijection. Assume X is a union of two non empty disjoint open sets U_1, U_2 . Then Y is a union of two disjoint nonempty open sets V_1, V_2 such that*

$$T(\{f \in C(X) : c(f) \subset U_j\}) = \{g \in C(Y) : c(g) \subset V_j\}, \text{ for } j = 1, 2.$$

PROOF. Put

$$V_j = \bigcup \{c(Tf) : c(f) \subset U_j\}, \text{ for } j = 1, 2.$$

Clearly V_1, V_2 are open and, since T is separating, disjoint. Assume there is $y \in Y \setminus (V_1 \cup V_2)$ and let $g \in C(Y)$ be such that $g(y) \neq 0$. We have

$$0 \neq g(y) = T(T^{-1}g)(y) = T(\chi_{U_1} \cdot T^{-1}g)(y) + T(\chi_{U_2} \cdot T^{-1}g)(y) = 0 + 0.$$

The contradiction shows that Y is a union of V_1 and V_2 . \square

3. The result

THEOREM 3.1. *Assume X is a subset of the real line, Y is a realcompact topological space, and $T : C(X) \rightarrow C(Y)$ is a separating linear bijection. Then T is biseparating, there is a non vanishing continuous scalar valued function \varkappa on Y and a homeomorphism $\varphi : Y \rightarrow X$ such that*

$$(3.1) \quad T(f) = \varkappa \cdot f \circ \varphi, \quad \text{for } f \in C(X).$$

PROOF. Let X be a subset of the real line, Y a topological space, $T : C(X) \rightarrow C(Y)$ a separating linear bijection, and let $\varphi : D \rightarrow X$ be the map given by Lemma 2.1. Put

$$X_1 = \{x \in X : \text{int}(\varphi^{-1}(\{x\})) \neq \emptyset\}.$$

LEMMA 3.2. *The closure of X_1 in \mathbb{R} has an empty interior.*

PROOF. Suppose that there is an open segment (a, b) contained in $\overline{X_1}$. Since the closure of any set of isolated points in \mathbb{R} has empty interior, it follows that there is a non isolated point x_1 in $(a, b) \cap X_1$. Since $\varphi^{-1}(\{x_1\})$ has a non empty interior there is $g_1 \in C(Y)$ such that

$$\overline{c(g_1)} \subset \varphi^{-1}(\{x_1\}), \quad g_1 \neq 0.$$

Put $f_1 = T^{-1}g_1$. By Lemma 2.1 the point x_1 belongs to the closure of $c(f_1)$. Since x_1 is a non isolated point $(a, b) \cap c(f_1)$ is an infinite open set so we can find

$$x_2 \in (a, b) \cap X_1 \cap c(f_1), \quad x_1 \neq x_2.$$

Let $g_2 \in C(Y)$ be such that

$$\overline{c(g_2)} \subset \varphi^{-1}(\{x_2\}), \quad g_2 \neq 0.$$

Again by Lemma 2.1 the point x_2 belongs to the closure of $c(f_2)$, where $f_2 = T^{-1}(g_2)$, so

$$c(f_1) \cap c(f_2) \cap G \setminus \{x_2\} \neq \emptyset, \text{ for any non empty neighbourhood } G \text{ of } x_2.$$

Let us summarize our construction - we have two distinct points x_1 and x_2 in X and two functions f_1, f_2 in $C(X)$ such that

1. the support of $T(f_1) = g_1$ is contained in $\varphi^{-1}(\{x_1\})$,
2. the support of $T(f_2) = g_2$ is contained in $\varphi^{-1}(\{x_2\})$, and

3. $c(f_1) \cap c(f_2) \cap G \setminus \{x_2\} \neq \emptyset$, for any neighborhood G of x_2 .

We may also assume without loss of generality that

4. $x_1 < x_2$.

We shall divide the rest of the proof of the Lemma into two cases.

a: $c(f_1) \cap c(f_2) \cap (x_1, x_2) \neq \emptyset$, and

b: $c(f_1) \cap c(f_2) \cap (x_1, x_2) = \emptyset$ and $c(f_1) \cap c(f_2) \cap (x_2, \infty) \neq \emptyset$.

In the first case let x_3 be any element of the set $c(f_1) \cap c(f_2) \cap (x_1, x_2)$. Multiplying f_1 by a non zero scalar we may assume that $f_1(x_3) = f_2(x_3)$. We put $X_- = X \cap (-\infty, x_3)$, $X_+ = X \cap (x_3, \infty)$ and we define a continuous function f_0 on X by

$$f_0(x) = \begin{cases} f_2(x), & x \in X_- \\ f_1(x), & x \in X_+ \cup \{x_3\}. \end{cases}$$

We show that $Tf_0 = 0$ which will contradict the assumption that T is injective. Let $y \in Y$. Since X_-, X_+ are disjoint open sets and T is separating we have two possibilities

$$(3.2) \quad T(f)(y) = 0 \text{ for any function } f \text{ such that } c(f) \subset X_-, \text{ or}$$

$$(3.3) \quad T(f)(y) = 0 \text{ for any function } f \text{ such that } c(f) \subset X_+.$$

Without loss of generality we may assume 3.3. Hence $\varphi(y) \neq x_2$, and consequently $T(f_2)(y) = g_1(y) = 0$ and since $c(f_0 - f_1) \subset X_+$, we have $T(f_0 - f_1)(y) = 0$, so

$$T(f_0)(y) = T(f_0)(y) - T(f_0 - f_1)(y) = T(f_1)(y) = 0.$$

If $\varphi(y) = x_1$, then $f_2 = f_0$ on an open neighborhood X_+ of $\varphi(y)$, so by Lemma 2.1, and since $g_2 = 0$ off $\varphi^{-1}(x_2)$ we get $T(f_0)(y) = T(f_2)(y) = g_2(y) = 0$.

Suppose now that $c(f_1) \cap c(f_2) \cap (x_1, x_2) = \emptyset$ and $c(f_1) \cap c(f_2) \cap (x_2, \infty) \neq \emptyset$. Since $f_1(x_2) \neq 0$ it follows that $f_2(x_2) = 0$. Multiplying f_2 by an appropriate scalar we may assume that f_2 is bigger than f_1 at some point $x_4 \in X \cap (x_2, \infty)$. If the portion of X between x_2 and x_4 is connected we can find $x_3 \in X \cap (x_2, x_4)$ such that $f_1(x_3) = f_2(x_3)$. If $X \cap (x_2, x_4)$ is disconnected we let x_3 be any point of $(x_2, x_4) \setminus X$. As before we put $X_- = X \cap (-\infty, x_3)$, $X_+ = X \cap (x_3, \infty)$ and we define a continuous function f_0 on $C(X)$ by

$$f_0(x) = \begin{cases} 0, & x \in X_- \\ f_2(x) - f_1(x), & x \in X_+ \cup (X \cap \{x_3\}). \end{cases}$$

We show that $Tf_0 = 0$. Let $y \in Y$. Since X_-, X_+ are disjoint open sets and T is separating we have again two possibilities - (3.2) and (3.3). In the second case we immediately get $T(f_0)(y) = 0$ since $c(f_0) \subset X_+$. In the first case, if $x_1 \neq \varphi(y) \neq x_2$ then $g_1(y) = 0 = g_2(y)$ and since $c(f_0 + f_1 - f_2) \subset X_-$ we have

$$\begin{aligned} T(f_0)(y) &= T(f_0)(y) - T(f_0 + f_1 - f_2)(y) \\ &= T(f_2)(y) - T(f_1)(y) = g_2(y) - g_1(y) = 0. \end{aligned}$$

If $\varphi(y) = x_1$ or $\varphi(y) = x_2$ then $f_0 = 0$ on X_- so, by Lemma 2.1 we have $T(f_0)(y) = 0$. \square

LEMMA 3.3. *For any $f \in C(X)$ and any $y_0 \in Y$ such that $\varphi(y_0) \in X \setminus X_1$ we have*

$$f(\varphi(y_0)) = 0 \implies (Tf)(y_0) = 0.$$

PROOF. Let $y_0 \in \varphi^{-1}(X \setminus X_1)$ and let $f \in C(X)$ be such that $f(x_0) = 0$, where $x_0 = \varphi(y_0)$. We need to show that $T(f)(y_0) = 0$. Since any continuous real valued function is a linear combination of two non negative continuous functions we may assume that $f \geq 0$. Furthermore we may assume that

$$(3.4) \quad f(x_0) = 0 \text{ and } f(x) > 0 \text{ for } x \neq x_0.$$

Indeed, if f_0 is any function that satisfies (3.4) and f is any non negative function on X with $f(x_0) = 0$ then $f = (f + f_0) - f_0$ and both $f + f_0$ and f_0 satisfy (3.4).

Since $\varphi(y_0) \notin X_1$, the interior of $\varphi^{-1}(\varphi(y_0))$ is empty, by (3.4), $\varphi^{-1}(\varphi(y_0)) = \varphi^{-1}(c(f_0))$, so by Lemma 2.2 we get $(Tf)(y_0) = 0$. \square

By Theorem 2.3, to finish the proof of our Theorem we need to show that T is biseparating. Let $f, g \in C(X)$ be such that $c(f) \cap c(g) \neq \emptyset$. We will show that $c(Tf) \cap c(Tg) \neq \emptyset$.

Assume first that the set $c(f) \cap c(g)$ contains a segment $[a, b]$. Since $[a, b]$ is compact the set (a, b) is open in βX [E]. Since the image of φ is dense in βX , by Lemma 3.2 there is $y_0 \in Y$ with $\varphi(y_0) \in (a, b) \setminus X_1 \subset c(f) \cap c(g) \setminus X_1$. Since T is surjective we can find $k \in C(X)$ such that $T(k)(y_0) \neq 0$; by Lemma 3.3, $k(\varphi(y_0)) \neq 0$. Let α be a nonzero scalar such that $\alpha k(\varphi(y_0)) + f(\varphi(y_0)) = 0$. Applying again Lemma 3.3 we get $\alpha Tk(\varphi(y_0)) + Tf(\varphi(y_0)) = 0$, so $Tf(y_0) \neq 0$. The same arguments prove that $Tg(y_0) \neq 0$. Hence $c(Tf) \cap c(Tg) \neq \emptyset$.

Assume now that the set $c(f) \cap c(g)$ does not contain any interval. Let U_1 be a nonempty clopen subset of $c(f) \cap c(g)$. By the Theorem 2.5 $T|_{C(U_1)}$ is a separating bijection from $C(U_1)$ onto $C(V_1)$ for some $V_1 \subset Y$. By Theorem 2.4, since U_1 is zero-dimensional, $T|_{C(U_1)}$ is biseparating and consequently $c(T|_{C(U_1)}(\chi_{U_1} \cdot f)) \cap c(T|_{C(U_1)}(\chi_{U_1} \cdot g))$ is nonempty. As $c(Tf) = c(T|_{C(U_1)}(\chi_{U_1} \cdot f)) \cup c(T(\chi_{X \setminus U_1} \cdot f))$ it follows that $c(Tf) \cap c(Tg) \neq \emptyset$. \square

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