Small Deformations of Topological Algebras

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Abstract. We investigate stability of various classes of topological algebras and individual algebras under small deformation of multiplication.

1. Introduction

By an ε-deformation (or perturbation) of a *Banach* algebra $(A, \cdot)$ we mean a second multiplication $\times$ defined on the same Banach space A such that the norm of the bilinear map $\times - \cdot$ is not greater than ε, that is

$$\|a \times b - a \cdot b\| \leq \varepsilon \|a\| \|b\|, \quad \text{for all } a, b \in A. \tag{1.1}$$

We always assume that a multiplication is associative, but not necessarily commutative.

Small deformation of Banach algebras have been investigated since early seventies by R. Rochberg [**22, 23, 24, 25, 26, 27, 28**], B. E. Johnson [**18, 19**], K. Jarosz [**10, 11, 12, 13, 14, 15, 16, 17**], and others. While some of the results are applicable to general Banach algebras, the main interest has been in the deformations of uniform algebras in connection with small deformations of holomorphic structures. There are three basic problems in the field:

1. *To characterize stable Banach algebras.*
   A Banach algebra is called *stable* if there is an $\varepsilon_0 > 0$ such that for any ε-deformation $\times$ with $\varepsilon < \varepsilon_0$, the algebras $(A, \cdot)$ and $(A, \times)$ are isomorphic.
2. *To characterize stable properties.*
   We say that a property P is *stable* if there exists an $\varepsilon_0 > 0$ such that for any algebra $(A, \cdot)$ having property P and any ε-deformation $\times$ of that algebra, with $\varepsilon < \varepsilon_0$, the algebra $(A, \times)$ also has property P.
3. *Characterizing continuous or differentiable structures on the space of all small deformations.*

For example Banach algebras $C(X), A(\mathbf{D}), H^\infty(\mathbf{D})$, and the properties '*Dirichlet*' or '$\partial A = ChA$' are stable [**10, 13, 18, 27**], while the algebras of analytic functions of one variable defined on nonsimply connected domains with nonempty interiors are not stable [**28**].

1991 *Mathematics Subject Classification.* Primary 46H05, 46H40; Secondary 46J05, 46A03.

Key words and phrases. topological algebras, small perturbations, small deformations, Banach algebras, stability.

Research partially supported by a grant from the National Research Council.

The concept of small deformations of algebras provides a very natural definition of a small deformation of a Riemann manifold Ω: we call Ω' a small deformation of Ω if an algebra $A_{\Omega'}$ of analytic functions on Ω' is isomorphic to a small deformation $(A_\Omega, \times)$ of an analogous algebra A_Ω. It turns out that for one-dimensional Riemann manifolds, this approach, with A_Ω equal to the algebra of functions analytic on Ω and continuous on $\overline{\Omega}$, is exactly equivalent to the theory of quasiconformal deformations [**28**]. On the other hand, almost nothing is known about small deformations of algebras of analytic functions of many variables [**15**]. The problem is of particular importance since an answer may provide a multidimensional quantitative version of the Riemann Mapping Theorem.

In this paper we extend the theory of small deformations to topological algebras. There are several ways to generalize the definition of a small deformation into the class of algebras equipped with a topology but without a norm. In the two sections following the *Definitions and Notation* we discuss two very natural extensions, we show however, that both have serious limitations. Under the first one, almost all non-Banach topological algebras are nonstable (we shall use term nonstrongly stable). Under the second one, most topological algebras, in particular all semisimple algebras, are stable (weak stability). Finally, in the main section (*Section 5*), we arrive with what we believe is the right definition, extending the concept of stability and small deformations into the class of topological algebras. We show, that in many cases, the theory is analogous to the theory of deformations of Banach algebras, while it may provide even better framework for working with small deformations of holomorphic manifolds. The paper does not provide a comprehensive theory of deformations of topological algebras, it is rather intended as an invitation to this mostly open area of research.

By a topological algebra we mean a topological vector space with an associative and separately[1] continuous multiplication. In general such an algebra may be quite pathological, e.g. all elements other than multiples of the identity may have unbounded spectrum, the multiplicative inversion $a \longmapsto a^{-1}$ may be discontinuous, the set of invertible elements may be nonopen, etc. In this paper we are particularly interested in small deformations of reasonably nice topological algebras, like topological function algebras, lmc F-algebras, and Q-algebras. We will also often assume that the algebras under consideration are complete, as several examples will show that without such an assumption, small deformations even of a normed algebra can be pathological.

2. Definitions and Notation

Since the terminology concerning topological algebras varies, we state for the record the definitions of some of the classes and properties of algebras we will refer to.

Definition 1. *By a* Fréchet algebra *we mean a complete metrizable topological algebra.*

Definition 2. *By a* Gelfand-Mazur algebra *we mean a topological algebra A such that for each closed two-sided regular ideal M which is maximal as a left or as*

[1]Some authors assume that multiplication in a topological algebra must be jointly continuous and refer to algebras with separately continuous multiplication as weak topological algebras [**4**], or pseudotopological algebras. Here, we adopt a more general definition; however, in most special cases we consider, like m-convex algebras, multiplications will automatically be jointly continuous.

a right ideal, the quotient algebra A/M is topologically isomorphic with the scalar field.

Definition 3. *By an m-*pseudoconvex algebra *we mean an algebra whose topology can be given by a family $\{p_\lambda : \lambda \in \Lambda\}$ of m-*pseudoconvex seminorms*, that is seminorms satisfying*

$$p_\lambda(ab) \le p_\lambda(a)\, p_\lambda(b), \quad \text{for all } a, b \in A, \lambda \in \Lambda,$$

and

$$p_\lambda(\mu a) = |\mu|^{k_\lambda} p_\lambda(a), \quad \text{for all } a \in A, \text{ and scalars } \mu,$$

*where $k_\lambda \in (0, 1]$. If $k_\lambda = 1$ for all $\lambda \in \Lambda$ we call the algebra m-*convex.

If A is both a Fréchet algebra and m-(pseudo)convex, then its topology can be given by an increasing sequence of m-(pseudo)convex seminorms.

Definition 4. *By a* topological function algebra *we mean a closed subalgebra A of an algebra $C(X)$ of all continuous functions defined on a completely regular Hausdorff space X; we assume that the topology of A is that of uniform convergence on compact subsets of X.*

The algebra $C(X)$ is metrizable if and only if X is hemicompact; it is complete if and only if X is a k_R-space ([**5**] pp 63-65). Since every locally compact and σ-compact Hausdorff space X is hemicompact and a k_R-space, for such class of spaces X, the topological algebra $C(X)$ is lmc and Fréchet.

Definition 5. *By a Q-algebra we mean a topological algebra A such that the group G_A^q of its quasi-invertible elements is open.*

If an algebra A has a unit e then $a \in A$ is called quasi-invertible if and only if $e - a$ is invertible. If an algebra does not have a unit we call an element a quasi-invertible if $e - a$ is invertible in the algebra $A \bigoplus \{\lambda e : \lambda \in \mathbb{C}\}$ obtained from A by adding a unit element e to the algebra. Equivalently, a is quasi-invertible if there is an element a^0, called a quasi-inverse of a, such that $a + a^0 = aa^0$; in the algebra $A \bigoplus \{\lambda e : \lambda \in \mathbf{C}\}$, the inverse of $e - a$ is $e - a^0$.

In spite of the fact that Q-algebras may be noncomplete they share many of the fundamental properties of Banach algebras; in fact, several of these properties characterize the Q-property. Q-algebras, like for example $C^\infty[0, 1]$, algebras of rapidly decreasing functions, algebras of functions with compact support, and others, play crucial role in the distributions theory, pseudodifferential operators, etc. [**8**]

For a commutative topological algebra A we will denote by $\hom A$ the set of all nonzero continuous linear and multiplicative functionals on A, by $M(A)$ the set of all maximal regular ideals of A and by $m(A)$ the subset of $M(A)$ consisting of closed ideals. By the *Jacobson radical* of A we mean

$$\mathrm{Rad}A = \bigcap M(A),$$

by the *topological radical*

$$\mathrm{rad}A = \bigcap m(A), \quad \text{provided } m(A) \ne \emptyset,$$

and by *functional radical*

$$\mathrm{frad}A = \bigcap \{\ker \varphi : \varphi \in \hom A\}, \quad \text{provided } \hom(A) \ne \emptyset.$$

We have

$$\mathrm{Rad}A \subset \mathrm{rad}A \subset \mathrm{frad}A.$$

In general, the inclusions above are proper. The topological radical $\mathrm{rad}A$ is closed while $\mathrm{Rad}A$ may not have this property [**29**]. For the algebra $A = L(H)$ of continuous linear maps on an infinite dimensional Hilbert space we have $\mathrm{Rad}A = \mathrm{rad}A \neq \mathrm{frad}A = \emptyset$ [**6**]. However, if A is a Q-algebra [**21**] or satisfies other conditions [**3**] then $M(A) = m(A)$ so

$$\mathrm{Rad}A = \mathrm{rad}A,$$

and when A is commutative and Gelfand-Mazur with $m(A) \neq \emptyset$, then

$$\mathrm{rad}A = \mathrm{frad}A.$$

Definition 6. *A commutative topological algebra A is called*

- semisimple *if*
$$\mathrm{Rad}A = \{0\},$$
- topologically semisimple *if*
$$\mathrm{rad}A = \{0\},$$
and
- functionally semisimple *if*
$$\mathrm{frad}A = \{0\}.$$

Any functionally semisimple commutative topological algebra can be identified with an algebra of functions on $M(A)$:

$$A \ni a \longmapsto \widehat{a} : \hom A \to \mathbb{C},\ \widehat{a}(F) = F(a),$$

though the topology on A may be quite different from the topology of uniform convergence on $\hom A$.

Definition 7. *An element a of a topological algebra A is called* bounded *if for some nonzero complex number λ_a the set $\left\{ \left(\frac{a}{\lambda_a} \right)^n : n \in \mathbb{N} \right\}$ is bounded in A. A topological algebra in which all elements are bounded is called a* topological algebra with bounded elements*.*

If the topology of an algebra is given by a family of seminorms we also have a related uniform property.

Definition 8. *Let A be a topological algebra with the topology given by a fixed family $p_j, j \in J$. of seminorms. We say that A is an algebra with* uniformly bounded elements *if there is a positive number λ_a such that*

$$\sup_{j \in J} \sup_n p_j \left(\left(\frac{a}{\lambda_a} \right)^n \right) < \infty.$$

An example of a topological algebra with uniformly bounded elements whose topology can not be given by a norm is algebra $C_b(\mathbb{R})$ of all continuous bounded functions on the real line with the seminorms

$$p_n(f) = \sup_{-n \leq t \leq n} |f(t)|,\ f \in C_b(\mathbb{R}),\ n \in \mathbb{N}.$$

3. Strong Stability

While the original definition (1.1) of a small deformation of a Banach algebra is expressed by a formula involving the norm of the algebra, the concept of stability of the algebra, or stability of a particular property, can be described in topological terms.

Definition 9. *We call a topological algebra* $(A,\cdot)$ strongly stable *if there is a neighborhood* V *of* $\cdot \in L_2(A)$ *such that for each associative map* $\times \in V$ *the algebras* $(A,\cdot)$ *and* $(A,\times)$ *are topologically isomorphic.*

Here, by $L_2(A)$ we denote the space of all continuous bilinear maps $T: A\times A \to A$ equipped with the bounded-open topology; that is, the topology with a base of neighborhoods of zero defined by

$$\left\{T \in L_2(A) : T\left(B^2\right) \subset U\right\},$$

where B is a bounded subset of A and U a neighborhood of zero in A.

The above definition coincides with the classical one for Banach algebras where the bounded-open topology is just the norm operator topology. So it may seem to be the most natural generalization of stability from Banach algebras to topological algebras. However, in a typical topological algebra, bounded sets are quite small and rarely have nonempty interior. Consequently, the requirement in the definition that all associative maps $\times$ from V produce an isomorphic algebra is very strong. So strong, indeed, that even very *nice and rigid looking* topological non-Banach algebras are not strongly stable.

Proposition 1. *Let* A *be equal to the algebra* $C(\mathbb{C})$ *of all continuous functions defined on the complex plane* $\mathbb{C}$*, or to the algebra* $Hol(\mathbb{C})$ *of all holomorphic functions on* $\mathbb{C}$*, or to the algebra* $Hol(\mathbb{D})$ *of all holomorphic functions defined on the open unit disc* $\mathbb{D}$*. We equip* A *with the topology of uniform convergence on compact sets and define multiplication pointwise. Then* A *is not strongly stable.*

The algebras A above are functionally semisimple, metrizable, complete, and m-convex. In fact, such algebras often serve as standard examples of the simplest topological algebra outside the class of Banach algebras. We show that A is not strongly stable. Indeed, in any neighborhood of the original multiplication of A one can find a multiplication with very different properties. Similar arguments can be applied to many other algebras of continuous or holomorphic functions.

Proof. Let V be an open neighborhood of $\cdot$ in $L_2(A)$; we shall construct a new multiplication $\times \in V$ such that the algebras $(A,\cdot)$ and $(A,\times)$ are not isomorphic.

Let $R_0 = 1$ if $A = Hol(\mathbb{D})$, and let $R_0 = +\infty$ if A is equal to $C(\mathbb{C})$ or to $Hol(\mathbb{C})$. For a positive number $R < R_0$ we define

$$\|a\|_R = \sup\{|a(z)| : |z| \le R\}, \quad \text{for } a \in A.$$

Any bounded set in A is contained in a set of the form

$$B = \left\{a \in A : \|a\|_{R_n} \le k_n \quad \text{for } n = 1,2,3,...\right\} \tag{3.1}$$

for some sequence (k_n) of positive numbers and some sequence (R_n) of positive numbers such that $\lim R_n = R_0$. Any neighborhood of zero in A contains a set of the form

$$U = \{a \in A : \|a\|_R < r\}, \tag{3.2}$$

for some positive $R < R_0$, and a positive number r. That means we need to construct a new multiplication $\times$ on A such that

$$\|a \cdot b - a \times b\|_R < r, \quad \text{for all } a, b \in B, \tag{3.3}$$

for some $r > 0$ and some fixed set B as above.

Assume first that A is equal to $Hol(\mathbb{D})$ or to $Hol(\mathbb{C})$, and put

$$a \times b = \pi_N (a \cdot b), \quad \text{for } a, b \in A$$

where

$$\pi_N : A \to A \text{ is defined by } \pi_N \left(\sum_{n=0}^{\infty} \alpha_n z^n \right) = \sum_{n=0}^{N} \alpha_n z^n, \tag{3.4}$$

and the natural number N will be defined later. It is easy to check that $\times$ is a well defined associative multiplication on A.

Let

$$f = \sum_{n=1}^{\infty} \alpha_n z^n$$

be an arbitrary element of

$$A_0 \overset{df}{=} \{f \in A : f(0) = 0\}.$$

Notice that

$$f^{N+1}(z) = a_1^{N+1} z^{N+1} + (...) z^{N+2} + ...,$$

so

$$f \underbrace{\times ... \times}_{N+1 \text{ times}} f = \pi_N \left(f^{N+1}\right) = 0.$$

The above shows that A_0 is the radical of $(A, \times)$ - indeed, all the elements of A_0 are nilpotent. Since the original algebra $(A, \cdot)$ is functionally semisimple the algebras $(A, \cdot)$ and $(A, \times)$ are not isomorphic.

To show (3.3) let R_m be such an element of the sequence (R_n) which we used to define the bounded set B, that $R < R_m < R_0$. Let $a, b \in B$, and let

$$(a \cdot b)(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$$

We have

$$|\alpha_n| \leq \frac{1}{2\pi} \int_{|\xi| = R_m} \frac{|f(\xi)|}{|\xi|^{n+1}} d\xi \leq \frac{2\pi R_m \|f\|_{R_m}}{2\pi R_m^{n+1}} \leq \frac{k_m}{R_m^n},$$

hence for any $z \in \mathbb{C}$ with $|z| \leq R$ we have

$$|(a \cdot b - a \times b)(z)| = \left| \sum_{n=N+1}^{\infty} \alpha_n z^n \right| \leq \sum_{n=N+1}^{\infty} \frac{k_m}{R_m^n} R^n = \frac{k_m}{1 - R/R_m} \frac{R^{N+1}}{R_m^{N+1}}.$$

Since $\frac{R^{N+1}}{R_m^{N+1}} \to 0$ as $N \to \infty$ we can now fix N so that

$$|(a \cdot b - a \times b)(z)| \leq r,$$

completing the proof of (3.3).

Assume now that $A = C(\mathbb{C})$ and put

$$(a \times b) = \pi(a \cdot b), \text{ for } a, b \in A$$

where this time

$$\pi : A \to A \text{ is defined by } \pi(a)(z) = a(\varphi(z)),$$

where φ is a continuous retraction from $\mathbb{C}$ onto $\{z \in \mathbb{C} : |z| \leq R\}$. We have

$$\|a \cdot b - a \times b\|_R = 0$$

for all $a, b \in A$, not only for a, b in B.

Notice that any function $a \in A$ which is equal to zero on the set $\{z \in \mathbf{C} : |z| \leq R\}$ is a nilpotent. Since the original algebra $A = C(\mathbb{C})$ is functionally semisimple the algebras $(A, \times)$ and $(A, \cdot)$ are not isomorphic. One could also notice that the maximal ideal spaces of the two algebras are quite different: the maximal ideal space of $(A, \cdot)$ is homeomorphic to $\mathbb{C}$ while that of $(A, \times)$ is homeomorphic to the closed disc of radius R. ∎

Proposition 2. *Let A be equal to the algebra $C^{\infty}([0,1])$ of all infinitely differentiable functions on the unit segment, or to the algebra $A^{\infty}(\overline{\mathbb{D}})$ of all C^{∞}-functions defined on the closed unit disc $\overline{\mathbb{D}}$ which are holomorphic on the interior of the disc. We equip A with a topology of uniform convergence of all the derivatives*

$$p_j(f) = \sup_{z \in \mathbf{X}} \left| f^{(j)}(z) \right|, \; j \in \mathbb{N},$$

where X is equal to $[0,1]$ or $\overline{\mathbb{D}}$, respectively, and define the multiplication pointwise. Then A is not strongly stable.

Notice that $C^{\infty}([0,1])$ and $A^{\infty}(\overline{\mathbb{D}})$ are complete, metrizable Q-algebras and their topology can be defined by a family of m-convex seminorms.

Proof. Assume first that $A = A^{\infty}(\overline{\mathbb{D}})$. We need to show that for any non-negative integer K and arbitrary positive numbers r and R there is a new multiplication $\times$ on A such that

$$\left\| (a \times b - ab)^{(j)} \right\|_{\infty} \leq r, \text{ for all } j = 0, 1, ..., K, \text{ and all } a, b \in A \tag{3.5}$$

$$\text{with } \left\| a^{(s)} \right\|_{\infty} \leq R \text{ and } \left\| b^{(s)} \right\|_{\infty} \leq R, \text{ where } s = 0, 1, ..., K+2, \tag{3.6}$$

but the algebras A and $(A, \times)$ are not topologically isomorphic. Here we denote by $a^{(j)}$ the j-th derivative of $a \in A$, and by $\|\cdot\|_{\infty}$ the sup norm on the unit disc.

Put, as in (3.4),

$$(a \times b) = \pi_N(a \cdot b), \text{ for } a, b \in A, \tag{3.7}$$

where

$$\pi_N : A \to A \text{ is defined by } \pi_N\left(\sum_{n=0}^{\infty} \alpha_n z^n\right) = \sum_{n=0}^{N} \alpha_n z^n, \tag{3.8}$$

and the integer N shall be fixed later. We notice that if $a, b \in A$ satisfy the conditions set in (3.5) then

$$\begin{aligned}\left\|(ab)^{(K+2)}\right\|_{\infty} &\leq \sum_{s=0}^{K+2}\binom{K+2}{s}\left\|a^{(s)}\right\|_{\infty}\left\|b^{(K+2-s)}\right\|_{\infty} \\ &\leq \sum_{s=0}^{K+2}\binom{K+2}{s}R^{K+2} = (2R)^{K+2} \overset{df}{=} C_0.\end{aligned}$$

For any

$$a(z)\,b(z) = f(z) = \sum_{n=0}^{\infty}\alpha_n z^n \in A,$$

with

$$\left\|f^{(K+2)}\right\|_{\infty} \leq C_0,$$

applying the Cauchy integral formula for the n-th derivative to

$$f^{(K+2)} = \sum_{n=K+2}^{\infty}\alpha_n\frac{n!}{(n-K-2)!}z^{n-K-2},$$

for any $n \geq K+2$, we have

$$\left|\frac{\alpha_n n!}{(n-K-2)!}\right| = \left|\frac{1}{2\pi}\int_{|\xi|=1}\frac{f^{(K+2)}(\xi)}{\xi^{n+1}}d\xi\right| \leq C_0,$$

so

$$|\alpha_n| \leq \frac{C_0\,(n-K-2)!}{n!}.$$

Hence, for any $j = 0, 1, ...K$ we get

$$\begin{aligned}\left\|(a\times b - ab)^{(j)}\right\|_{\infty} &= \left\|(f-\pi_N(f))^{(j)}\right\|_{\infty} \\ &= \left\|\left(\sum_{n=N+1}^{\infty}\alpha_n z^n\right)^{(j)}\right\|_{\infty} \\ &= \left\|\sum_{n=N+1}^{\infty}\alpha_n\frac{n!}{(n-j)!}z^{n-j}\right\|_{\infty} \\ &\leq C_0\sum_{n=N+1}^{\infty}\frac{n!}{(n-j)!}\frac{(n-K-2)!}{n!} \\ &\leq C_0\sum_{n=N+1}^{\infty}\frac{1}{(n-K)(n-K-1)} \\ &= \frac{C_0}{N-K}.\end{aligned}$$

Since the last expression tends to zero with $N \to \infty$, we can now fix N so that

$$\left\|(a\times b - ab)^{(j)}\right\|_{\infty} \leq r \text{ for } j = 0, 1, ..., K$$

for all a, b satisfying (3.5). Hence $\times$ is a small deformation of the original multiplication however the algebras A and $(A, \times)$ are obviously not isomorphic since the first algebra has a unit while the second one does not.

If $A = C^{\infty}[0,1]$ we need to modify only slightly the above construction by replacing π_N with the following:

$$\pi_N : A \to A \text{ is defined by } \pi_N(a)(t) = \sum_{n=-N}^{N} a_n e^{2\pi int},$$

where $a_n = \int_0^1 a(t) e^{-2\pi it} dt$ are the Fourier coefficients of a, and keeping the formula (3.7) for the definition of new multiplication.

As before, the new algebra $(A, \times)$ contains nilpotents, e.g. $e^{2\pi it}$, so it is not isomorphic with A; similar computations will also show that $\times$ is a small perturbation of the original multiplication, that is (3.5-3.6) are valid provided N is large enough. ∎

4. Weak Stability

The approach to small deformations presented in the previous section, while natural, is not fully satisfactory, as even very simple topological algebras outside the class of Banach algebras fail the property. The main reason is that in the class of topological algebras neighborhoods of zero are usually not bounded and bounded sets are normally small. Hence the last definition did not guarantee that the new multiplication is *uniformly* close to the original one. If the topology of an algebra A can be defined by a family $\{p_\lambda : \lambda \in \Lambda\}$ of m-pseudoconvex seminorms, one can define a class of deformations that are uniformly close to the original one and define the corresponding stability property.

Definition 10. *Let $(A, \cdot)$ be an m-pseudoconvex algebra with a unit e, and let $\{p_\lambda : \lambda \in \Lambda\}$ be the family of **all** m-pseudoconvex continuous seminorms on A. We call $(A, \cdot)$ weakly stable if there is an $\varepsilon > 0$ such that for any associative multiplication $\times$ on A with*

$$p_\lambda(a \cdot b - a \times b) \le \varepsilon p_\lambda(a) p_\lambda(b), \quad \text{for all } a, b \in A, \lambda \in \Lambda, \tag{4.1}$$

the algebras $(A, \cdot)$ and $(A, \times)$ are topologically isomorphic.

Notice that the new algebra $(A, \times)$ is also m-pseudoconvex, indeed from (4.1) we get

$$p_\lambda(a \times b) \le (1 + \varepsilon) p_\lambda(a) p_\lambda(b), \quad \text{for } a, b \in A, \lambda \in \Lambda,$$

so

$$q_\lambda \stackrel{df}{=} (1 + \varepsilon) p_\lambda,$$

are m-pseudoconvex seminorms on $(A, \times)$.

Again, the definition is very natural, however this time most complete topological algebras are weakly stable, while noncomplete algebras may still behave pathologically.

Example 1. *Let A be the algebra of all polynomials of a variable t with pointwise multiplication and the sup norm on the unit segment:*

$$\|p\| = \sup\{|p(t)| : 0 \le t \le 1\}, \quad \textit{for } p, q \in A.$$

Fix $\varepsilon > 0$ and define a new multiplication $\times$ on A by

$$(p \times q)(t) = (1 + \varepsilon t)\, p(t)\, q(t), \quad \textit{for } p, q \in A.$$

It is easy to check that the new multiplication satisfies (4.1) however the algebras A and $(A, \times)$ are not isomorphic, since A has a unit while $(A, \times)$ does not.

PROPOSITION 3. *Any unital semisimple complete commutative m-pseudoconvex Hausdorff algebra is weakly stable.*

PROOF. Assume A is a unital commutative semisimple complete m-convex algebra; we denote by e the unit of A. Let $\times$ be another multiplication on A that satisfies (4.1) with an $\varepsilon < 1$. Put

$$M : A \to A \text{ by } M(a) = e \times a$$

and

$$T = \sum_{n=0}^{\infty} (Id - M)^n : A \to A,$$

where Id is the identity map on A.

For any m-pseudoconvex seminorm p we may assume that $p(e) = 1$ [**31**] and we have

$$p((Id - M)(a)) = p(a - e \times a) \leq \varepsilon p(a),$$

$$p\left((Id - M)^2(a)\right) = p((Id - M)(a) - e \times (Id - M)(a)) \leq \varepsilon p((Id - M)(a)) \leq \varepsilon^2 p(a),$$

.....

$$p((Id - M)^n(a)) \leq \varepsilon^n p(a), \text{ for all } n.$$

So, since A is complete, the above series above defining the map T is convergent. We have

$$\begin{aligned} M \circ T = T \circ M &= \sum_{n=0}^{\infty} (Id - M)^n \circ M = \sum_{n=0}^{\infty} (Id - M)^n \circ (Id - (Id - M)) \\ &= \sum_{n=0}^{\infty} (Id - M)^n - \sum_{n=1}^{\infty} (Id - M)^n = Id \end{aligned}$$

so $T = M^{-1}$.

Let F be a continuous linear and multiplicative functional on A, and put

$$p_F(a) = |F(a)|, \text{ for } a \in A.$$

The map p_F is a continuous m-pseudoconvex seminorm on A. Let $a \in A$; since $p_F(a - F(a)e) = 0$, and since we assumed in Definition 10 that the condition (4.1) holds true for all continuous m-pseudoconvex seminorms, we get

$$F((a - F(a)e) \times b) = 0,$$

hence

$$F(a \times b) = F(a)\, F(e \times b), \quad \text{for all } a, b \in A. \tag{4.2}$$

Replacing in (4.2) a with $e \times a$ we get

$$\begin{aligned} F(M(a \times b)) &= F((e \times a) \times b) = F(e \times a)\, F(e \times b) \\ &= F(M(a))\, F(M(b)) = F(M(a) \cdot M(b)). \end{aligned}$$

Since A is complete we have Rad$A$ = rad$A$ [**3**]. Because it is also a Gelfand-Mazur algebra we have rad$A$ = frad$A$ ([**1**] p. 125, or [**2**] Th. 1) hence A is functionally semisimple. As $F(M(a \times b)) = F(M(a) \cdot M(b))$ for all multiplicative functionals F, it follows that

$$M(a \times b) = M(a) \cdot M(b)$$

so, since both M and $M^{-1} = T$ are continuous, M is a topological isomorphism from $(A, \times)$ onto $(A, \cdot)$. ∎

5. Stability

Since the last two definitions were not fully satisfactory, providing too weak or too strong property, let us examine again the original definition of small deformation of a Banach algebra. The definition involves not only the topology but a specific norm of the algebra. If we replace the original norm $\|\cdot\|$ on A by an equivalent norm $p(\cdot)$, then (1.1) becomes

$$p(a \times b - a \cdot b) \leq \varepsilon C p(a) p(b), \text{ for all } a, b \in A,$$

where the constant C can be arbitrarily large or arbitrarily small. Hence small deformations of a Banach algebra are related not only to particular algebraic and topological structures on A, but also to a *specific* norm on that space, rather than a family of all equivalent norms. Consequently it will only be natural to define small perturbations not for an abstract topological algebra, but for a concrete topological algebra equipped with a given set of seminorms.

Definition 11. *Let $(A, \cdot)$ be an algebra and $\Xi = \{p_\lambda : \lambda \in \Lambda\}$ a family of m-pseudoconvex seminorms. Assume that the set Ξ separates the points of A, so that $(A, \cdot, \Xi)$ is an m-convex Hausdorff algebra. We call $(A, \cdot, \Xi)$ stable if there is an $\varepsilon > 0$ such that for any associative multiplication $\times$ on A with*

$$p(a \cdot b - a \times b) \leq \varepsilon p(a) p(b), \quad \text{for } a, b \in A, \quad \text{and } p \in \Xi, \tag{5.1}$$

the algebras $(A, \cdot)$ and $(A, \times)$ are topologically isomorphic.

We call a property P *stable (or stable in a given class of topological algebras) if for any algebra $(A, \cdot, \Xi)$ having the property* P *there is an $\varepsilon > 0$ such that any ε-deformation $\times$ of that algebra defined as above, the new algebra $(A, \times, \Xi)$ also has the same property* P.

We notice that, as in the case of the weak stability, the new algebra $(A, \times)$ is also m-pseudoconvex, with m-convex seminorms defined by

$$q_\lambda \overset{df}{=} (1 + \varepsilon) p_\lambda, \quad \text{for } \lambda \in \Lambda.$$

5.1. Basic stable properties. The next Proposition shows that, for a large class of topological algebras, a small perturbation of an algebra with a unit is again a unital algebra.

Proposition 4. *For the commutative sequentially complete m-pseudoconvex algebras the property "algebra has a unit" is stable.*

Proof. Assume A is a unital commutative sequentially complete m-pseudoconvex algebra; we denote by e the unit of A. Let $\times$ be another multiplication on A that satisfies (5.1) with an $\varepsilon < 1$.

Put

$$M : A \to A \text{ by } M(a) = e \times a.$$

As in the proof of Proposition 3 we can check that

$$p_\lambda\left((Id - M)^n (a)\right) \leq \varepsilon^n p_\lambda(a), \text{ for all } n, \text{ and } \lambda \in \Lambda, \tag{5.2}$$

and that

$$M^{-1} = \sum_{n=0}^{\infty} (Id - M)^n,$$

where Id is the identity map on A.

Since M is surjective there is an element $e_0 = M^{-1}(e)$ in A such that

$$M(e_0) = e \times e_0 = e.$$

Hence, for any b in A we get

$$M(e_0 \times b - b) = e \times e_0 \times b - e \times b = e \times b - e \times b = 0,$$

since M is injective it follows that

$$e_0 \times b - b = 0, \text{ for all } b \in A,$$

so e_0 is a unit of $(A, \times)$. ∎

Remark 1. *Based on the last Proposition we may always assume that for a commutative sequentially complete m-pseudoconvex algebra a new multiplication has the same unit as the original one. To justify this we need to show that there is a third multiplication $*$ on A such that the algebras $(A, \times)$ and $(A, *)$ are topologically isomorphic, $*$ is a small deformation of $(A, \cdot)$, that is there is an $\varepsilon' = O(\varepsilon)$ such that*

$$p(a \cdot b - a * b) \leq \varepsilon' p(a) p(b), \quad \text{for } a, b \in A, \quad \text{and } p \in \Xi, \tag{5.3}$$

and the algebras $(A, \cdot)$ and $(A, *)$ have the same unit.

To this end assume that $\times$ is a new multiplication on A satisfying (5.1) *with $\varepsilon < 1$, denote by e the unit of the original multiplication on A, and by e_0 the unit of the $\times$-multiplication. Put*

$$\Phi : A \to A, \text{ by } \Phi(a) = a \cdot e_0.$$

Arguing exactly as for the map M in the proof of the last Proposition one can check that Φ is an invertible map from A onto itself with

$$\Phi^{-1} = \sum_{n=0}^{\infty} (Id - \Phi)^n.$$

Moreover, as before

$$\begin{aligned} p_\lambda\left(\Phi^{-1}(a)\right) &= p_\lambda\left(\sum_{n=0}^{\infty} (Id - \Phi)^n (a)\right) \\ &\leq \sum_{n=0}^{\infty} \varepsilon^n p_\lambda(a) = \frac{p_\lambda(a)}{1 - \varepsilon}, \quad \text{for all } a \in A, \text{ and } \lambda \in \Lambda. \end{aligned} \tag{5.4}$$

We define a third multiplication $*$ on A by

$$a * b = \Phi^{-1}\left(\Phi(a) \times \Phi(b)\right), \quad \text{for all } a, b \in A.$$

Notice that

- Φ *is an algebra isomorphism from* $(A,*)$ *onto* $(A,\times)$*, so the algebras* $(A,*)$ *and* $(A,\times)$ *are isomorphic,*
- *since*

$$e_0=\Phi^{-1}(e)=\sum_{n=0}^{\infty}(Id-\Phi)^n e,$$

then from (5.4) *we get*

$$p_\lambda(e_0)\le\frac{1}{1-\varepsilon},\ \textit{and}\quad p_\lambda(e-e_0)\le\frac{\varepsilon}{1-\varepsilon},\quad \textit{for all }\lambda\in\Lambda,$$

so, by (5.4) *it follows that for all* $a,b\in A$*, and* $\lambda\in\Lambda$ *we have*

$$\begin{aligned}p_\lambda(a\cdot b-a*b)&=p_\lambda\left(a\cdot b-\Phi^{-1}((a\cdot e_0)\times(b\cdot e_0))\right)\\&\le\frac{1}{1-\varepsilon}p_\lambda(\Phi(a\cdot b-(a\cdot e_0)\times(b\cdot e_0)))\\&=\frac{1}{1-\varepsilon}p_\lambda(a\cdot b\cdot e_0-(a\cdot e_0)\times(b\cdot e_0))\\&=\frac{1}{1-\varepsilon}p_\lambda(a\cdot(b\cdot e_0)-a\times(b\cdot e_0)+a\times(b\cdot e_0)-(a\cdot e_0)\times(b\cdot e_0))\\&\le\frac{1}{1-\varepsilon}\left[\varepsilon p_\lambda(a)p_\lambda(b\cdot e_0)+p_\lambda((a-a\cdot e_0)\times(b\cdot e_0))\right]\\&\le\frac{1}{1-\varepsilon}\left[\varepsilon p_\lambda(a)(p_\lambda(b)p_\lambda(e_0)+(1+\varepsilon)p_\lambda(a)p_\lambda(e-e_0)p_\lambda(b)p_\lambda(e_0))\right]\\&\le\varepsilon' p_\lambda(a)p_\lambda(b),\end{aligned}$$

with $\varepsilon'\overset{df}{=}\frac{1}{1-\varepsilon}p_\lambda(e_0)\left[\varepsilon+(1+\varepsilon)p_\lambda(e-e_0)\right]\le\frac{1}{(1-\varepsilon)^2}\left(\varepsilon+\frac{(1+\varepsilon)\varepsilon}{1-\varepsilon}\right)=\frac{2\varepsilon}{(1-\varepsilon)^3}$.

Hence $*$ *is an* ε'*-deformation of* $(A,\cdot)$.

- the algebras $(A,\cdot)$ and $(A,*)$ have the same unit e.

Proposition 5. *Assume* $(A,\cdot,\Xi)$ *is a topological algebra with the topology given by the family* $\Xi=\{p_\lambda:\lambda\in\Lambda\}$ *of* m*-pseudoconvex seminorms, and let* $\times$ *be an* ε*-deformation of* $(A,\cdot,\Xi)$*. Then* $\ker p_\lambda$, $\lambda\in\Lambda$ *are closed two-sided ideals in both algebras* $(A,\cdot,\Xi)$ *and* $(A,\times,\Xi)$*, and for any* $\lambda\in\Lambda$ *and* $a_1,a_2,b_1,b_2\in A$ *we have*

$$p_\lambda(a_1-a_2)=0=p_\lambda(b_1-b_2)\quad\Longrightarrow\quad p_\lambda(a_1\times b_1-a_2\times b_2)=0.$$

Proof. Fix $\lambda\in\Lambda$ and assume $a_1,a_2,b_1,b_2\in A$ are such that

$$p_\lambda(a_1-a_2)=0=p_\lambda(b_1-b_2).$$

We have

$$\begin{aligned}p_\lambda((a_1-a_2)\times b_1)&\le p_\lambda((a_1-a_2)\times b_1-(a_1-a_2)b_1)+p_\lambda(a_1-a_2)p_\lambda(b_1)\\&\le\varepsilon p_\lambda(a_1-a_2)p_\lambda(b_1)=0,\end{aligned}$$

so

$$p_\lambda(a_1\times b_1-a_2\times b_1)=0.$$

By symmetry we also have

$$p_\lambda(a_2\times b_1-a_2\times b_2)=0.$$

Hence

$$p_\lambda (a_1 \times b_1 - a_2 \times b_2) = 0.$$

∎

Remark 2. *Assume* $(A, \cdot, \Xi)$ *is a topological algebra with the topology given by the family* $\Xi = \{p_\lambda : \lambda \in \Lambda\}$ *of* m*-convex seminorms, let* $\times$ *be an* ε*-deformation of* $(A, \cdot, \Xi)$, *and let* $\overline{A_\lambda}$ *be the completion of the quotient algebra* $A_{/\ker p_\lambda}$. *Based on the above proposition we can define two multiplications on each of the Banach space* $\overline{A_\lambda}$ *- one,* $\cdot_\lambda$*, induced by the original multiplication, and another one,* $\times_\lambda$*, induced by* $\times$. *We have*

$$\|f \cdot_\lambda g - f \times_\lambda g\|_\lambda \leq \varepsilon \|f\|_\lambda \|g\|_\lambda, \text{ for all } f, g \in \overline{A_\lambda},$$

where $\|\cdot\|_\lambda$ *is the quotient norm. Hence* $\left(\overline{A_\lambda}, \times_\lambda\right)$ *is an* ε*-deformation of the Banach algebra* $\left(\overline{A_\lambda}, \cdot_\lambda\right)$.

5.2. Stable algebras. The next Theorem states that the simplest topological function algebras, that is the algebras of all continuous functions, are stable.

Theorem 1. *Let* $A = C(X)$ *be the algebra of all continuous functions defined on a completely regular Hausdorff* k*-space* X *with the usual pointwise multiplication, and let* $\mathcal{K}$ *be a cover of* X *consisting of compact sets. Put*

$$\|f\|_K = \sup\{|f(z)| : z \in K\}, \quad \text{for } f \in A, K \in \mathcal{K}.$$

Then the algebra $(A, \{\|\cdot\|_K : K \in \mathcal{K}\})$ *is stable.*

Proof. Assume $\times$ is another multiplication on $C(X)$ such that

$$\|fg - f \times g\|_K \leq \varepsilon \|f\|_K \|g\|_K, \text{ for all } f, g \in C(X), K \in \mathcal{K}. \tag{5.5}$$

It follows that

$$\|fg - f \times g\|_X \leq \varepsilon \|f\|_X \|g\|_X, \text{ for all } f, g \in C_b(X),$$

where $(C_b(X), \|\cdot\|_X)$ is the Banach algebra of all bounded continuous functions on X equipped with the sup norm. Since $(C_b(X), \|\cdot\|_X)$ is isometrically isomorphic with stable Banach algebra $C(\beta X)$ of all continuous functions on the Čech-Stone compactification βX of X, there is ([**10**], Ex. 17.3)

$$\Psi : C(\beta X) \to C(\beta X)$$

with

$$\|\Psi\| \leq c\varepsilon,$$

where c is a constant, and such that

$$T \overset{df}{=} Id + \Psi$$

is an algebra isomorphism from $(C(\beta X), \times)$ onto $C(\beta X)$, that is,

$$T(f \times g) = T(f)T(g), \text{ for } f, g \in C_b(X). \tag{5.6}$$

Hence for any $x \in X$ there is a regular Borel measure ν_x on βX such that

$$\operatorname{var}(\nu_x) \leq c\varepsilon$$

and

$$Tf(x) = f(x) + \int_{\beta X} f d\nu_x, \quad \text{for } f \in C(\beta X).$$

Let $K_0 \in \mathcal{K}$ be such that $x \in K_0$. We show that the support of ν_x is contained in K_0. Assuming the contrary there is an $f_0 \in C(\beta X)$ with $f_0 = 0$ on K_0 and such that

$$\int_{\beta X} f_0 d\nu_x = 1 \text{ and } \|f_0\|_X \leq 1 + \frac{1}{|\nu_x|(X \backslash K_0)}.$$

Let $g_0 \in C(\beta X)$ be such that

$$\|g_0\|_X = 1$$

and

$$g_0 = 1 \text{ on } K \text{ and } g_0 = 0 \text{ on } \{t \in X : |f_0(t)| > \varepsilon\},$$

so that

$$\|f_0 g_0\|_X \leq \varepsilon.$$

By Proposition 5, since $\|f_0\|_{K_0} = 0$ we have

$$f_0 \times g_0 = 0 \text{ on } K_0,$$

and

$$\|f_0 \times g_0\|_X \leq \|f_0 g_0\|_X + \varepsilon \|f_0\|_X \|g_0\|_X \leq \varepsilon + \varepsilon \|f_0\|_X \leq \varepsilon \left(2 + \frac{1}{|\nu_x|(X \backslash K_0)}\right).$$

Hence

$$\begin{aligned} |T(f_0 \times g_0)(x)| &= \left|(f_0 \times g_0)(x) + \int_{\beta X} (f_0 \times g_0) d\nu_x\right| \\ &= \left|\int_{\beta X \backslash K_0} (f_0 \times g_0) d\nu_x\right| \\ &\leq |\nu_x|(X \backslash K_0) \|f_0 \times g\|_X \\ &\leq |\nu_x|(X \backslash K_0) \varepsilon \left(2 + \frac{1}{|\nu_x|(X \backslash K_0)}\right) \\ &\leq \varepsilon (2 |\nu_x|(X \backslash K_0) + 1) \\ &\leq \varepsilon (2c\varepsilon + 1). \end{aligned}$$

On the other hand

$$\begin{aligned} |T(f_0 \times g_0)(x)| &= |T(f_0)(x) T(g_0)(x)| \\ &= \left|\int_{\beta X} f_0 d\nu_x\right| \left|\left(1 + \int_{\beta X} g_0 d\nu_x\right)\right| \\ &\geq (1 - \operatorname{var}(\nu_x)) \\ &\geq 1 - c\varepsilon, \end{aligned}$$

so

$$\varepsilon (2c\varepsilon + 1) \geq 1 - c\varepsilon,$$

which is impossible provided ε is small enough. We proved that

$$\operatorname{supp}(\nu_x) \subset \bigcap \{K : x \in K \in \mathcal{K}\} \qquad \text{for any } x \in X.$$

Since any continuous function is bounded on a compact set, $\int_{\beta X} f d\nu_x$ is well defined for any $f \in C(X)$, and we can extend the map

$$Id + \Psi = T : C_b(X) \to C_b(X)$$

to a map defined on all of $C(X)$. We shall use the same symbol T to denote the extended map

$$T(f)(x) = f(x) + \int_{\beta X} f d\nu_x, \quad f \in C(X). \tag{5.7}$$

To end the proof we shall show that T is an algebra isomorphism from $(C(X), \times)$ onto $C(X)$; we need to check that T is an algebra homomorphism, that it is injective, and that the range of T consists of all continuous functions on X and only continuous functions.

Let $f, g \in C(X)$. To show that

$$T(f \times g) = T(f) T(g),$$

fix $x \in X$ and a set $K \in \mathcal{K}$ with $x \in K$. Denote by f_1, g_1, h_1 arbitrary bounded continuous functions on X that coincide, on K, with f, g, and $f \times g$, respectively. By Proposition 5, since $\|f - f_1\|_K = \|g - g_1\|_K = 0$, we have $\|f \times g - f_1 \times g_1\|_K = 0$, so since the support of ν_x is contained in K, by (5.6) we get

$$\begin{aligned} T(f \times g)(x) &= (f \times g)(x) + \int_{\beta X} (f \times g) d\nu_x \\ &= (f_1 \times g_1)(x) + \int_{\beta X} (f_1 \times g_1) d\nu_x \\ &= T(f_1 \times g_1)(x) = T(f_1)(x) T(g_1)(x) \\ &= \left(f_1(x) + \int_{\beta X} f_1 d\nu_x\right)\left(g_1(x) + \int_{\beta X} g_1 d\nu_x\right) \\ &= \left(f(x) + \int_{\beta X} f d\nu_x\right)\left(g(x) + \int_{\beta X} g d\nu_x\right) \\ &= T(f)(x) T(g)(x). \end{aligned}$$

Now, to show that T is injective, assume f is a nonzero continuous function on X such that $T(f) = 0$. Select any compact set K from $\mathcal{K}$ such that f is not equal constantly to zero on K, and let $y \in K$ be such that $\|f\|_K = |f(y)| > 0$. Since $var(\nu_x) \leq c\varepsilon$, we get

$$0 = Tf(y) = \left| f(y) + \int_K f d\nu_x \right| \geq |f(y)| - c\varepsilon \|f\|_K = \|f\|_K (1 - c\varepsilon) > 0.$$

The contradiction shows that T is injective.

It is clear that for any $f \in C(X)$ the function $T(f)$ is continuous on compact sets from $\mathcal{K}$. Since according to our assumptions X is a k-space, it follows that $T(f)$ is continuous.

Define

$$S(f) = \sum_{n=0}^{\infty} (-\Psi)^n (f);$$

the map S is an inverse of T on the Banach algebra $C_b(X) = C(\beta X)$. It is easy to check that for any $f \in C(X)$, the function $S(f)$ is well defined, that the value of $S(f)(x)$ depends only on the behavior of f on K, for any $x \in K \in \mathcal{K}$, so $S(f) \in C(X)$ and consequently $S = T^{-1}$ which proves that T is surjective. ∎

The last Theorem shows that the algebras $C(X)$ are stable with respect to the most natural set of seminorms - maximum on compact subsets of X. As the next example shows, the same algebra with the same topology may not be stable however with respect to some other less obvious, set of seminorms.

Example 2. *Put $A = C(\mathbb{R})$ with the usual pointwise multiplication, and*

$$p_n(f) = n \sup_{-n \leq t \leq n} |f(t)|, \quad \text{for } f \in C(\mathbb{R}) \text{ and } n \in \mathbb{N}.$$

Fix $\varepsilon > 0$ and define a new multiplication $\times$ on A by

$$(f \times g)(t) = (1 + \varepsilon t) f(t) g(t), \; t \in \mathbb{R}.$$

It is easy to check that the new multiplication is an ε-deformation of the original one, however the algebras A and $(A, \times)$ are not isomorphic, as the first one has a unit while the second one does not.

In the remaining portion of this section we will deal mostly with perturbations of topological algebras of analytic functions. The next two lemmas will provide us with technical tools, the first one gives a detailed analysis of small deformations of the disc algebra. We denote by $A(\mathbb{D}_R)$ the Banach algebra of all functions that are continuous on $\mathbb{D}_R \overset{df}{=} \{z \in \mathbb{C} : |z| \leq R\}$ and analytic on $\operatorname{int}(\mathbb{D}_R)$, we equip it with the standard sup norm: $\|f\| = \sup\{|f(z)| : |z| \leq R\}$ and pointwise multiplication.

Lemma 1. *There is an $\varepsilon_0 > 0$ such that for any positive $\varepsilon < \varepsilon_0$, any $R > 0$, and any multiplication $\times$ on $A(\mathbb{D}_R)$ with*

$$\|f \cdot g - f \times g\| \leq \varepsilon \|f\| \|g\|, \quad f \in A(\mathbb{D}_R),$$

there is an algebra isomorphism T_R from $(A(\mathbb{D}_R), \times)$ onto $A(\mathbb{D}_R)$ and a homeomorphism ψ_R from $\mathbb{D}_R$ onto itself such that

$$\|T_R\| \leq 1 + \varepsilon, \quad \left\|T_R^{-1}\right\| \leq \frac{1}{1 - \varepsilon}, \tag{5.8}$$

$$|T(f)(\psi_R(z)) - f(z)| \leq 2\varepsilon, \text{ for } z \in \mathbb{D}_R, \quad \text{and} \tag{5.9}$$

$$(1 - \varepsilon)|z| \leq |T_R(Z)(z)| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R.$$

Moreover, if the functional $A(\mathbb{D}_R) \ni f \longmapsto f(0)$ is $\times$-multiplicative, we also have

$$(1 - \varepsilon)|z| \leq \left|T_R^{-1}(Z)(z)\right| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R,$$

and if $f_0 \in A(\mathbb{D}_R)$ satisfies

$$(1 - \varepsilon)|z| \leq |f_0(z)| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R,$$

then

$$(1 - \varepsilon)^2 |z| \leq |T_R(f_0)(z)| \leq (1 + \varepsilon)^2 |z|, \quad z \in \mathbb{D}_R, \tag{5.10}$$

where Z is the identity function: $Z(z) = z$.

Proof. Since the algebra $A(\mathbb{D}_R)$ is isometrically isomorphic with the stable disc algebra [**27**], there is an algebra isomorphism T from $(A(\mathbb{D}_R), \times)$ onto $A(\mathbb{D}_R)$. We show that any such a map satisfies the remaining properties listed in the Lemma, or may be modified to satisfy them. The property (5.9) follows directly from the proof of stability of the disc algebra: Lemma 3.3 of [**27**]. Based on Remark 1 we can assume that both multiplications have the same unit $\mathbf{1}$.

For any $f \in A(\mathbb{D}_R)$ we have

$$(1-\varepsilon)\|f\|^2 \leq \|f \times f\| \leq (1+\varepsilon)\|f\|^2,$$

so for any $k \in \mathbb{N}$

$$(1-\varepsilon)^{2^k-1}\|f\|^{2^k} \leq \Bigg\|\underbrace{f \times \ldots \times f}_{2^k \text{ times}}\Bigg\| \leq (1+\varepsilon)^{2^k-1}\|f\|^{2^k}$$

and consequently the spectral radius $\rho_\times(f)$ of f in $(A(\mathbb{D}_R), \times)$ satisfies

$$(1-\varepsilon)\|f\| \leq \rho_\times(f) \leq (1+\varepsilon)\|f\|.$$

Since $\rho_\times(f) = \|Tf\|$, we get

$$\|T\| \leq 1+\varepsilon, \qquad \left\|T^{-1}\right\| \leq \frac{1}{1-\varepsilon}.$$

We now show that $Z \times A(\mathbb{D}_R)$ is a closed maximal ideal in $A(\mathbb{D}_R)$. Let $S : A(\mathbb{D}_R) \to A(\mathbb{D}_R)$ be defined by

$$S(f) = T\left(\frac{f-f(0)}{Z}\right)T(Z) + f(0).$$

Notice that since the function $f - f(0)$ vanishes at zero, $\frac{f-f(0)}{Z}$ is a well defined element of $A(\mathbb{D}_R)$; furthemore since $\|f - f(0)\| \leq 2\|f\|$, and $|Z| = R$ on the boundary of $\mathbb{D}_R$, we have $\left|\frac{f-f(0)}{Z}(z)\right| \leq \frac{2\|f\|}{R}$, for $z \in \partial\mathbb{D}_R$, and consequently for all z in $\mathbb{D}_R$, so$\left\|\frac{f-f(0)}{Z}\right\| \leq \frac{2\|f\|}{R}$. As both multiplication have the same unit we have $T(\mathbf{1}) = \mathbf{1}$ and

$$\begin{aligned}
\left\|\left(T^{-1} \circ S\right)(f) - f\right\| &= \left\|T^{-1}\left(T\left(\frac{f-f(0)}{Z}\right)T(Z)\right) - \frac{f-f(0)}{Z}Z\right\| \\
&= \left\|\frac{f-f(0)}{Z} \times Z - \frac{f-f(0)}{Z}Z\right\| \\
&\leq \varepsilon\frac{2\|f\|}{R}R = 2\varepsilon\|f\|.
\end{aligned}$$

Hence, provided $2\varepsilon < 1$, the map $T^{-1} \circ S$, and consequently S, are invertible. Since $ZA(\mathbb{D}_R)$ is a codimension one closed subspace of $A(\mathbb{D}_R)$, its image under S is also a codimension one closed subspace. So $T(Z \times A(\mathbb{D}_R)) = T(Z)A(\mathbb{D}_R) = S(ZA(\mathbb{D}_R))$ is a maximal ideal in $A(\mathbb{D}_R)$. Consequently, there is a $z_0 \in \overline{\mathbb{D}}_R$ such that $T(Z)A(\mathbb{D}_R) = \{f \in A(\mathbb{D}_R) : f(z_0) = 0\}$, and the point z_0 can not be in the boundary of $\mathbb{D}_R$, since the corresponding ideal is principal. Put

$$B_{z_0}(z) = \frac{Rz_0 - z}{R - z\overline{z_0}}$$

and define $T_R : A(\mathbb{D}_R) \to A(\mathbb{D}_R)$ by

$$T_R(f) = T(f) \circ B_{z_0}, \quad f \in A(\mathbb{D}_R).$$

The new map T_R still has all the properties of T, that is satisfies (5.8) and is an algebra isomorphism from $(A(\mathbb{D}_R), \times)$ onto $A(\mathbb{D}_R)$, in addition

$$T_R(Z \times A(\mathbb{D}_R)) = T_R(Z) A(\mathbb{D}_R) = \{f \in A(\mathbb{D}_R) : f(0) = 0\}.$$

Hence $T_R(Z)$ has only one single zero so $g_0 \overset{df}{=} T_R(Z)/Z$ is an invertible element of $A(\mathbb{D}_R)$. Since

$$\|g_0\| = \left\|\frac{T_R(Z)}{Z}\right\| = \frac{\|T_R(Z)\|}{R} \leq \frac{\|T_R\| \|Z\|}{R} = \|T_R\| \leq 1 + \varepsilon$$

we get

$$|T_R(Z)(z)| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R.$$

Since $T_R(Z)/Z$ is invertible it attains minimum of its absolute value on the boundary of $\mathbb{D}_R$. Assume there is a point $w_0 \in \partial\mathbb{D}_R$ such that $\left|\frac{T_R(Z)}{R}\right| = |(T_R(Z)/Z)(w_0)| < 1 - \varepsilon$ and let $h_0 \in A(\mathbb{D}_R)$ be a norm one function such that $\|h_0 T_R(Z)/R\| < 1 - \varepsilon$. We have

$$\begin{aligned} 1 - \varepsilon &> \left\|h_0 \frac{T_R(Z)}{R}\right\| \geq (1 - \varepsilon)\left\|T_R^{-1}\left(h_0 \frac{T_R(Z)}{R}\right)\right\| \\ &= \left\|h_0 \times \frac{Z}{R}\right\| \geq \left\|h_0 \frac{Z}{R}\right\| - \left\|h_0 \times \frac{Z}{R} - h_0 \frac{Z}{R}\right\| \geq 1 - \varepsilon. \end{aligned}$$

The contradiction shows that

$$\|g_0^{-1}\| = \left\|\frac{Z}{T_R(Z)}\right\| \leq \frac{1}{1 - \varepsilon},$$

so $(1 - \varepsilon)|z| \leq |T_R(Z)(z)|$ for $z \in \mathbb{D}_R$.

Assume now that the functional $A(\mathbb{D}_R) \ni f \longmapsto f(0)$ is $\times$-multiplicative and put $h_0 = T_R^{-1}(g_0^{-1})$. We have

$$T_R^{-1}(Z)(0) = T_R^{-1}\left(g_0^{-1} T_R(Z)\right)(0) = h_0 \times Z(0) = 0,$$

and

$$\left\|\frac{T_R^{-1}(Z)}{Z}\right\| \leq \frac{\left\|T_R^{-1}\right\| \|Z\|}{R} \leq 1 + \varepsilon$$

hence

$$\left|T_R^{-1}(Z)(z)\right| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R.$$

Using very similar arguments as before one can now show that $T_R^{-1}(Z) A(\mathbb{D}_R)$ is a closed maximal ideal in $A(\mathbb{D}_R)$, so $\frac{T_R^{-1}(Z)}{Z}$ is invertible with $\left\|\frac{Z}{T_R^{-1}(Z)}\right\| \leq \frac{1}{1-\varepsilon}$, consequently

$$(1 - \varepsilon)|z| \leq \left|T_R^{-1}(Z)(z)\right|, \quad z \in \mathbb{D}_R.$$

Assume $f_0 \in A(\mathbb{D}_R)$ satisfies

$$(1 - \varepsilon)|z| \leq |f_0(z)| \leq (1 + \varepsilon)|z|, \quad z \in \mathbb{D}_R.$$

If we replace Z with f_0 in the first part of our proof we will get

$$\left\| \frac{T_R(f_0)}{Z} \right\| \leq (1+\varepsilon)^2 \text{ and } \left\| \frac{Z}{T_R(f_0)} \right\| \leq \frac{1}{(1-\varepsilon)^2}$$

so (5.10) follows. ∎

LEMMA 2. *Let $\mathcal{F}$ be a free ultrafilter on the set of natural numbers $\mathbb{N}$, let $r < R$ be positive real numbers, and let f_n be a sequence of holomorphic functions on $\mathbb{D}_R$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_R < \infty$. Then $\lim_{\mathcal{F}} f_n(z)$ is holomorphic on $\mathbb{D}_R$ and $\lim_{\mathcal{F}} \|f_n\|_r = \|\lim_{\mathcal{F}} f_n\|_r$.*

Notice that if we only assumed that f_n were continuous then the limit $\lim_{\mathcal{F}} f_n(z)$ could be discontinuous and $\|\lim_{\mathcal{F}} f_n\|_r$ could be strictly smaller than $\lim_{\mathcal{F}} \|f_n\|_r$. We refer to [**9**] for a review of basic applications of ultrafilters in the Banach space theory (see also [**7**]).

PROOF. Let z_0 be a point in $\mathbb{D}_R$, we show that $f \stackrel{df}{=} \lim_{\mathcal{F}} f_n(z)$ is holomorphic at z_0 and $\lim_{\mathcal{F}} \|f_n\|_r = \|\lim_{\mathcal{F}} f_n\|_r$. To simplify the notation we can assume without loss of generality that

- $z_0 = 0$ (compose all f_n with a suitable holomorphic transformation of $\mathbb{D}_R$ onto itself mapping 0 onto z_0),
- $f_n(0) = 0$, for $n \in \mathbb{N}$,
- $f_n'(0) = 0$, for $n \in \mathbb{N}$ (subtract the bounded sequence $f_n'(0) z$)
- $R > 1 > r$, and $\sup_{n\in\mathbb{N}} \|f_n\|_R \leq 1$.

For any $z \in \mathbb{D}_r$ we have

$$|f_n'(z)| \leq \left| \frac{1}{2\pi} \oint_{|\xi|=1} \frac{f_n(\xi)}{(\xi - z)^2} d\xi \right| \leq \frac{4}{(1-r)^2},$$

so $|f_n(z)| \leq \frac{4}{(1-r)^2} |z|$, hence $|\lim_{\mathcal{F}} f_n(z)| \leq \frac{4}{(1-r)^2} |z|$ and $(\lim_{\mathcal{F}} f_n)'(0) = 0$.

Assume s is such that $\|\lim_{\mathcal{F}} f_n\|_r < s < \lim_{\mathcal{F}} \|f_n\|_r$ and let $U = \{U_1, ..., U_p\}$ be a finite cover of $\mathbb{D}_r$ consisting of nonempty sets with the diameter less than $\frac{(1-r)^2}{8} (s - \|\lim_{\mathcal{F}} f_n\|_r)$. For any j fix $w_j \in U_j$. For arbitrary $n \in \mathbb{N}$ and $w \in U_j$ we have

$$\begin{aligned} |f_n(w_j) - f_n(w)| &\leq \sup_{z\in U_j} |f_n'(z)| \, |w_j - w| \\ &\leq \frac{4}{(1-r)^2} \cdot \frac{(1-r)^2}{8} \left(s - \left\| \lim_{\mathcal{F}} f_n \right\|_r \right) \\ &= \frac{s - \|\lim_{\mathcal{F}} f_n\|_r}{2}. \end{aligned}$$

Put $\mathbb{N}_j = \left\{ k \in \mathbb{N} : \sup_{z\in U_j} |f_k(z)| > s \right\}$. Since $\bigcup_{j=1}^{p} \mathbb{N}_j$ contains all but finitely many natural numbers and $\mathcal{F}$ is a free ultrafilter there is a j_0 such that $\mathbb{N}_{j_0} \in \mathcal{F}$.

We have

$$\begin{aligned}\left\|\lim_{\mathcal{F}} f_n\right\|_r &\geq \left|\lim_{\mathcal{F}} f_n\left(w_{j_0}\right)\right| = \lim_{\mathcal{F}} \left|f_n\left(w_{j_0}\right)\right| \\ &\geq \lim_{\mathcal{F}} \left(\sup_{w\in U_{j_0}} |f(w)| - \sup_{w\in U_{j_0}} \left|f_n\left(w_{j_0}\right) - f_n(w)\right|\right) \\ &\geq s - \frac{s - \left\|\lim_{\mathcal{F}} f_n\right\|_r}{2} > \left\|\lim_{\mathcal{F}} f_n\right\|_r .\end{aligned}$$

The contradiction shows that $\lim_{\mathcal{F}} \|f_n\|_r = \|\lim_{\mathcal{F}} f_n\|_r$. ∎

THEOREM 2. *Let $A = Hol(\mathbb{D})$ be the algebra of all holomorphic functions on an open unit disc $\mathbb{D}$ with the usual pointwise multiplication, let r_n be an increasing sequence of positive numbers with $\lim r_n = 1$, and let*

$$\|f\|_n = \sup\{|f(z)| : |z| \leq r_n\}.$$

Then the topological algebra $(A, \{\|\cdot\|_n : n = 1, 2, 3, ...\})$ is stable.

PROOF. Fix an $\varepsilon > 0$ and assume $\times$ is another multiplication on A such that for all $n \in \mathbb{N}$

$$\|f \cdot g - f \times g\|_n \leq \varepsilon \|f\|_n \|g\|_n, \quad f, g \in A. \tag{5.11}$$

Put $A_n \overset{df}{=} A(\mathbb{D}_{r_n})$, $n \in \mathbb{N}$, $A_\infty \overset{df}{=} A(\mathbb{D})$, and let $\|\cdot\| \overset{df}{=} \sup \|\cdot\|_n$ be the usual sup norm on $\mathbb{D}_\infty \overset{df}{=} \mathbb{D}$. Since A is dense in A_n and the multiplication $\times$ is jointly continuous, it can be uniquely extended to a multiplication on A_n, $n \in \mathbb{N}$; the extension still satisfies (5.11). We also have

$$\|f \cdot g - f \times g\| \leq \varepsilon \|f\| \|g\|, \quad f, g \in A_\infty.$$

Let T_n, $n \in \mathbb{N} \cup \{\infty\}$ be an isomorphism from $(A_n, \times)$ onto $(A_n, \cdot)$, given by Lemma 1. We shall show that T_∞ can be extended to an isomorphism from $(A, \times)$ onto $(A, \cdot)$.

Let $n_1 < n_2 \in \mathbb{N} \cup \{\infty\}$. The composition map $T_{n_1} \circ T_{n_2}^{-1} : A_{n_2} \to A_{n_1}$ is a homeomorphism between two uniform Banach algebras so it must be given by a continuous map $\varphi_{n_1,n_2} : \mathbb{D}_{r_{n_1}} \to \mathbb{D}_{n_2}$ between the maximal ideal spaces of these algebras:

$$\left(T_{n_1} \circ T_{n_2}^{-1}\right)(f) = f \circ \varphi_{n_1,n_2}, \qquad f \in A_{n_2}.$$

Hence

$$T_\infty(f)\left(\varphi_{n_1,\infty}(z)\right) = T_{n_1}(f)(z), \qquad f \in A_\infty, z \in \mathbb{D}_{r_{n_1}}. \tag{5.12}$$

Since A_{n_1} is dense in A_{n_2}, $\left(T_{n_1} \circ T_{n_2}^{-1}\right)(A_{n_2})$ is dense in A_{n_1} and must separate points of $\mathbb{D}_{r_{n_1}}$ - the maximal ideal space of A_{n_1}, so φ_{n_1,n_2} is injective, moreover $\varphi_{n_1,n_2} = \left(T_{n_2} \circ T_{n_1}^{-1}\right)(Z)$. Notice also that

$$\varphi_{n_1,\infty}(z) = \left(\varphi_{n_2,\infty} \circ \varphi_{n_2,n_1}\right)(z), \text{ for } z \in \mathbb{D}_{r_{n_1}}.$$

We show that $\text{int}\mathbb{D} = \bigcup_{n=1}^{\infty} \varphi_{n,\infty}(\mathbb{D}_{r_n})$.

Assume again that $n_1 < n_2$. Since $\mathbb{D}_{r_{n_1}}$ is a compact subset of $\text{int}\mathbb{D}_{r_{n_2}}$ and φ_{n_2,n_1} is injective it follows that $\varphi_{n_2,n_1}\left(\mathbb{D}_{r_{n_1}}\right)$ is compact and contained in $\text{int}\mathbb{D}_{r_{n_2}}$ so $\varphi_{n_1,\infty}\left(\mathbb{D}_{r_{n_1}}\right) = \varphi_{n_2,\infty}\left(\varphi_{n_2,n_1}\left(\mathbb{D}_{r_{n_1}}\right)\right) \subset \text{int}\mathbb{D}$. Hence $\bigcup_{n=1}^{\infty} \varphi_{n,\infty}(\mathbb{D}_{r_n}) \subset \text{int}\mathbb{D}$.

To show the other inclusion let $z_0 \in \operatorname{int}\mathbb{D}$. Put

$$B_{z_0}(w) = \frac{z_0 - w}{1 - \overline{z}_0 w},\ z \in \mathbb{D}.$$

Let ψ_1, ψ_{r_k} be as in Lemma 1, we have

$$\left\| \left(T_k \circ T_\infty^{-1}\right)(B_{z_0}) - T_\infty^{-1}(B_{z_0}) \circ \psi_{r_k}^{-1} \right\| \leq 2\varepsilon$$

and

$$\left\| T_\infty^{-1}(B_{z_0}) \circ \psi_{r_k}^{-1} - B_{z_0} \circ \psi_1 \circ \psi_{r_k}^{-1} \right\| \leq 2\varepsilon \frac{1+\varepsilon}{1-\varepsilon}.$$

Hence, assuming r_k is close to 1, the function $B_{z_0} \circ \psi_1 \circ \psi_{r_k}^{-1}$ is a homeomorphism from $\mathbb{D}_{r_k}$ onto a subset of the plane very close to $\mathbb{D}_{r_k}$. Such a set must contain 0, so there is $w_0 \in \mathbb{D}_{r_k}$ with $\left(T_k \circ T_\infty^{-1}\right)(B_{z_0})(w_0) = 0$.

We have

$$\begin{aligned}
\{f \in A_\infty : T_\infty(f)(z_0) = 0\} &= T_\infty^{-1}(B_{z_0} A_\infty) = T_\infty^{-1}(B_{z_0}) \times A_\infty \\
&\subset A_\infty \cap T_k^{-1}\left(T_k T_\infty^{-1}(B_{z_0})\right) \times A_k \\
&= A_\infty \cap T_k^{-1}\left(T_k T_\infty^{-1}(B_{z_0}) A_k\right) \\
&\subset \{f \in A_\infty : T_k(f)(w_0) = 0\}.
\end{aligned}$$

Since the codimension of the first and the last ideal above is the same, they must be identical and consequently the corresponding $\times$-multiplicative functionals on A_∞ must coincide, that is

$$T_\infty(f)(z_0) = T_k(f)(w_0) \text{ for } f \in A_\infty.$$

By (5.12) we get

$$\varphi_{k,\infty}(w_0) = z_0,$$

which shows that $\operatorname{int}\mathbb{D} \subset \bigcup_{n=1}^\infty \varphi_{n,\infty}(\mathbb{D}_{r_n})$.

To end the proof fix n_0 and let j_0, j_1 be such that $\mathbb{D}_{r_{n_0}} \subset \varphi_{j_0,\infty}\left(\mathbb{D}_{r_{j_0}}\right)$ and $\varphi_{j_0,\infty}\left(\mathbb{D}_{r_{n_0}}\right) \subset \mathbb{D}_{r_{j_1}}$. For any $f \in A_0$ we have

$$\begin{aligned}
\|T_\infty(f)\|_{n_0} &= \sup\left\{|T_0(f)(z)| : z \in \mathbb{D}_{r_{n_0}}\right\} \leq \sup\left\{|T_\infty(f)(z)| : z \in \varphi_{j_0,\infty}\left(\mathbb{D}_{r_{j_0}}\right)\right\} \\
&= \sup\left\{|T_{j_0}(f)(z)| : z \in \mathbb{D}_{r_{j_0}}\right\} = \|T_{j_0}(f)\|_{j_0} \leq (1+\varepsilon)\|f\|_{j_0},
\end{aligned}$$

and

$$\begin{aligned}
\|f\|_{n_0} &= \sup\left\{|f(z)| : z \in \mathbb{D}_{r_{n_0}}\right\} \leq \sup\left\{|f(z)| : z \in \varphi_{j_0,\infty}\left(\mathbb{D}_{r_{j_1}}\right)\right\} \\
&= \sup\left\{\left|T_{j_1} \circ T_\infty^{-1}(f)(z)\right| : z \in \mathbb{D}_{r_{j_1}}\right\} = \left\|T_{j_0}\left(T_\infty^{-1}(f)\right)\right\|_{j_1} \leq \frac{1}{1-\varepsilon}\left\|T_\infty^{-1}(f)\right\|_{j_1}.
\end{aligned}$$

The above shows that T_∞ and T_∞^{-1} are continuous in the topology of $(A, \{\|\cdot\|_n : n = 1,2,3,...\})$ so T_∞ can be extended to a homomorphism from the algebra $(A, \times, \{\|\cdot\|_n : n = 1,2,3,...\})$ onto $(A, \cdot, \{\|\cdot\|_n : n = 1,2,3,...\})$ as promised. ∎

Theorem 3. *Let $A = Hol(\mathbb{C})$ be the algebra of all holomorphic functions on $\mathbb{C}$ with the usual pointwise multiplication, let k_n be an increasing sequence of positive numbers with $\lim k_n = \infty$, and let*

$$\|f\|_n = \sup\{|f(z)| : |z| \leq k_n\}.$$

Then the topological algebra $(A, \{\|\cdot\|_n : n = 1, 2, 3, ...\})$ *is stable.*

Notice that unlike in Theorem 2 this time the subalgebra $\{f \in A : \sup_n \|f\|_n < \infty\}$ is trivial and can not be helpful in the proof.

PROOF. Without loss of generality, discarding some of the norms $\|\cdot\|_n$ if necessary, we may assume that

$$\frac{k_n}{k_{n+1}} \leq (1-\varepsilon)^2, \quad \text{for } n \in \mathbb{N}.$$

Fix an $\varepsilon > 0$ and assume $\times$ is another multiplication on A such that for all $n \in \mathbb{N}$

$$\|f \cdot g - f \times g\|_n \leq \varepsilon \|f\|_n \|g\|_n, \quad f, g \in A. \tag{5.13}$$

Since A is dense in $A_n \overset{df}{=} A(\mathbb{D}_{k_n})$, $n \in \mathbb{N}$, the new multiplication $\times$ can be uniquely extended to a multiplication on A_n; the extension still satisfies (5.13). Let T_n be the isomorphisms from $(A_n, \times)$ onto $(A_n, \cdot)$ given by Lemma 1. By the same Lemma there is a $\times$-multiplicative functional F on A_1 defined by $F(f) \overset{df}{=} T_1(f)(\psi_{k_1}(0))$ such that $\|F - \delta_0\| \leq 2\varepsilon$. Let μ be a measure on $\mathbb{D}_{k_1}$ such that

$$var(\mu) \leq 2\varepsilon \text{ and } \int_{\mathbb{D}_{k_1}} f d\mu = F(f) - f(0), \text{ for } f \in A_1.$$

Put $\Phi : A \to A$

$$\Phi(f) = f + \int_{\mathbb{D}_{k_1}} f d\mu$$

and define another multiplication $\times'$ on A by

$$f \times' g = \Phi\left(\Phi^{-1}(f) \times \Phi^{-1}(g)\right).$$

Since $\Phi(f)(0) = F(f)$ and F is $\times$-multiplicative we get

$$\begin{aligned}(f \times' g)(0) &= F\left(\Phi^{-1}(f) \times \Phi^{-1}(g)\right) = \left(F \circ \Phi^{-1}\right)(f) F \circ \Phi^{-1}(g) \\ &= \Phi\left(\Phi^{-1}(f)\right)(0)\, \Phi\left(\Phi^{-1}(g)\right)(0) = f(0)\, g(0).\end{aligned}$$

Hence the algebras $(A, \times)$ and $(A, \times')$ are topologically isomorphic (Φ is an isomorphism), the new multiplication $\times'$ is a small deformation of $\times$, so it is also a small deformation of the original multiplication $\cdot$ on A, and the evaluation at 0 is $\times'$-multiplicative. Consequently it is enough to show that the algebras A and $(A, \times')$ are isomorphic; in order to simplify the notation we will just assume that the evaluation at 0 is already $\times$-multiplicative.

Let $\mathcal{F}$ be a free ultrafilter on the set of natural numbers $\mathbb{N}$ and define $T : A \to A$ by

$$T(f)(z) = \left(\lim_{\mathcal{F}} T_n(f)\right)(z), \quad f \in A.$$

We need to show that $\lim_{\mathcal{F}} T_n(f)$ is a well defined element of A, and that T is a bijective algebra isomorphism from $(A, \times)$ onto A.

Assume $n_2 > n_1 > n_0$. The composition map $T_{n_1} \circ T_{n_2}^{-1} : A_{n_2} \to A_{n_1}$ is a homeomorphism between two uniform Banach algebras so must be given be a continuous map $\varphi_{n_1,n_2} : \mathbb{D}_{k_{n_1}} \to \mathbb{D}_{k_{n_2}}$ between the maximal ideal spaces of these algebras:

$$\left(T_{n_1} \circ T_{n_2}^{-1}\right)(f) = f \circ \varphi_{n_1,n_2}, \quad f \in A_{n_2}.$$

Hence

$$T_{n_2}(f)\left(\varphi_{n_1,n_2}(z)\right) = T_{n_1}(f)(z), \qquad f \in A_{n_2}, z \in \mathbb{D}_{k_{n_1}}.$$

Since A_{n_2} is dense in A_{n_1}, and T_{n_i} are isomorphisms, $\left(T_{n_1} \circ T_{n_2}^{-1}\right)(A_{n_2})$ is dense in A_{n_1} and must separate points of $\mathbb{D}_{k_{n_1}}$ - the maximal ideal space of A_{n_1}, so φ_{n_1,n_2} is injective.

Since $\varphi_{n_1,n_2} = \left(T_{n_1} \circ T_{n_2}^{-1}\right)(Z)$ by Lemma 1 with $f_0 = T_{n_2}^{-1}(Z)$ we get

$$(1-\varepsilon)^2 |z| \leq \left|\varphi_{n_1,n_2}(z)\right| \leq (1+\varepsilon)^2 |z|, \quad z \in \mathbb{D}_{k_{n_1}}.$$

Let $f \in A$ and $|z| \leq k_{n_0} \leq (1-\varepsilon)^2 k_{n_1}$ so that $z \in \varphi_{n_1,n_2}\left(\mathbb{D}_{k_{n_1}}\right)$, then

$$\|T_{n_2}(f)\|_{n_0} \leq \sup_{w \in \mathbb{D}_{k_{n_1}}} \left|T_{n_2}(f)\left(\varphi_{n_1,n_2}(w)\right)\right| = \sup_{w \in \mathbb{D}_{k_{n_1}}} |T_{n_1}(f)(z)| \leq (1+\varepsilon)\|f\|_{n_1},$$

$$\|T_{n_2}(f)\|_{n_2} \geq \sup_{w \in \mathbb{D}_{k_{n_1}}} \left|T_{n_2}(f)\left(\varphi_{n_1,n_2}(w)\right)\right| = \sup_{w \in \mathbb{D}_{k_{n_1}}} |T_{n_1}(f)(z)| \geq (1-\varepsilon)\|f\|_{n_1}.$$

Hence by Lemma 2 $\lim_{\mathcal{F}} T_n(f)$ is a well defined analytic function and

$$(1-\varepsilon)\|f\|_k \leq \left\|\lim_{\mathcal{F}} T_n(f)\right\|_{k+1} \leq (1+\varepsilon)\|f\|_{k+2}.$$

So T is a topological isomorphism from $(A, \times)$ onto a closed subalgebra of A. To show that T is surjective it is enough to notice that $Z \in T(A)$. Indeed $\frac{T(Z)}{Z}$ is a bounded entire function by Lemma 1, so it is constant. ∎

Theorem 4. *Let $0 < r < R < \infty$, let $A = Hol(P)$ be the algebra of all holomorphic functions on $P = \{z \in \mathbb{C} : r < |z| < R\}$ with the usual pointwise multiplication, let $(k_n)_{n=-\infty}^{n=\infty}$ be an increasing sequence of positive numbers with $\lim_{n\to-\infty} k_n = r$ and $\lim_{n\to\infty} k_n = R$, and let*

$$\|f\|_n = \sup\{|f(z)| : k_{-n} \leq |z| \leq k_n\}.$$

*Then the topological algebra $(A, \{\|\cdot\|_n : n = 1, 2, 3, ...\})$ is **not** stable.*

Proof (sketch) . Fix an $\varepsilon > 0$, put $P_\varepsilon = \{z \in \mathbb{C} : r < |z| < (1+\varepsilon)R\}$ and define $T : A \to Hol(P_\varepsilon)$ by

$$T\left(\sum_{n=-\infty}^{\infty} a_n Z^n\right) = \sum_{n=-\infty}^{0} a_n Z^n + \sum_{n=-\infty}^{\infty} a_n \left(\frac{R}{R+\varepsilon}\right) Z^n, \text{ for } \sum_{n=-\infty}^{\infty} a_n Z^n \in Hol(P).$$

It is obvious that T is a well defined linear bijection and that the algebras $Hol(P)$ and $Hol(P_\varepsilon)$ are not isomorphic since their maximal ideal spaces are not holomorphically homeomorphic. Hence

$$f \times g \stackrel{df}{=} T^{-1}(T(f)T(g))$$

defines a new multiplication on A such that A and $(A, \times)$ are not isomorphic.

Put $P_n \stackrel{df}{=} \{z : k_{-n} \leq |z| \leq k_n\}$, $P_n' \stackrel{df}{=} \{z : k_{-n} \leq |z| \leq (1+\varepsilon)k_n\}$, and

$$A_n = \{f \in C(P_n) : f \in Hol(\text{int}P_n)\}, A_n = \{f \in C(P_n) : f \in Hol(\text{int}P_n)\}.$$

Notice that for each n, T maps $A(P_n)$ onto $A(P'_n)$. One can verify that the norms of these maps, as well as the norms of their inverses tend to one uniformly as $\varepsilon \to 0$, and hence by Theorem 3.1 of [**10**]

$$\|f \times g - fg\|_n \leq (1+\varepsilon') \|f\|_n \|g\|_n \text{, for all } f, g \in A \text{ and } n \in \mathbb{N},$$

where $\varepsilon' \to 0$ as $\varepsilon \to 0$. ∎

Theorem 5. *The property of being a Q-algebra is not stable.*

Proof. Let A be equal to the space $C(\mathbb{C})$ of all continuous functions on the complex plane with the topology defined by a family

$$\Xi = \{\|\cdot\|_n : n = 1, 2, 3, ...\},$$

where

$$\|f\|_n = \sup\{|f(z)| : |z| \leq n\},$$

and let $\cdot$ be a zero multiplication on A, that is

$$a \cdot b = 0 \text{ for all } a, b \in A.$$

It is immediate to check that A is a Q-algebra.

Fix an $\varepsilon > 0$ and define a new multiplication $\times$ on A by

$$(a \times b)(z) = \varepsilon a(z) b(z), \text{ for } a, b \in A, z \in \mathbb{C}.$$

We have

$$\|a \cdot b - a \times b\|_n = \|a \times b\|_n \leq \varepsilon \|a\|_n \|b\|_n,$$

so $\times$ is an ε-perturbation of the original multiplication of A. However $(A, \times)$ is not a Q-algebra since it has a unit (the constant function $\frac{1}{\varepsilon}$), but any neighborhood

$$\left\{f \in A : \left\|f - \frac{1}{\varepsilon}\right\|_n < \delta\right\}$$

of of that unit contains functions equal zero at some point of the plane and such functions are not invertible in the algebra $(A, \times)$.

Notice that all the elements of the algebra $(A, \cdot)$ considered in the last proof are bounded however function a defined by

$$a(z) = z, \text{ for all } z \in \mathbb{C}$$

is not bounded in the algebra $(A, \times)$, hence we get the following Proposition.

Proposition 6. *Property of m-convex algebras of having* bounded elements *is not stable.*

On the other hand related uniform property is stable.

Proposition 7. *Property of m-convex algebras of having* uniformly bounded elements *is stable.*

Proof. Let $(A, \cdot, \Xi)$ be an m-convex algebra with uniformly bounded elements, where $\Xi = \{p_\lambda : \lambda \in \Lambda\}$ is a family of m-convex seminorms defining the topology of the algebra. Let $\times$ be an ε-deformation of A, and let a be a fixed element of A.

Let μ_a, M be positive real numbers such that

$$p_\lambda\left(\left(\frac{a}{\mu_a}\right)^n\right) \leq M, \text{ for all } n \in N, \text{ and } \lambda \in \Lambda.$$

Since the norms

$$q_\lambda = (1+\varepsilon)\, p_\lambda,\ \lambda \in \Lambda$$

are $\times$-convex for all $n \in N$, and $\lambda \in \Lambda$, we get

$$p_\lambda \left(\underbrace{\frac{a}{M(1+\varepsilon)\mu_a} \times ... \times \frac{a}{M(1+\varepsilon)\mu_a}}_{n \text{ times}} \right) \le (1+\varepsilon)^{n-1} \left[p_\lambda \left(\frac{a}{M(1+\varepsilon)\mu_a} \right) \right]^n$$

$$= \frac{1}{M^n (1+\varepsilon)} p_\lambda \left(\left(\frac{a}{\mu_a} \right)^n \right) \le \frac{1}{1+\varepsilon} < 1.$$

Hence elements of $(A, \times)$ are uniformly bounded. ∎

6. Almost multiplicative functionals and other problems

Small deformations of multiplication is not the only aspect of the deformation theory of Banach algebras that one may try to extend to topological algebras. We would like to mention briefly some other very natural problems.

The first one concerns almost multiplicative functionals. By an ε-multiplicative functional on a Banach algebra A we mean a linear functional F on A such that

$$|F(ab) - F(a)F(b)| \le \varepsilon \|a\| \|b\|, \quad \text{for } a, b \in A.$$

Such functionals play crucial role in the investigation of small deformations of multiplication but they are also interesting on their own right. To get an example of an almost multiplicative function one need only to take a multiplicative functional G and any linear functional $\Delta \in A^*$ with sufficiently small norm, and put $F = G + \Delta$. A Banach algebra is called functionally stable if this is the only way to obtain an almost multiplicative functional; that is, if any almost multiplicative functional is close to a multiplicative one. In 1986 B. Johnson proved [**19**] that the Banach algebras $C(X)$, the disc algebra $A(\mathbb{D})$, and some other related uniform algebras are functionally stable. He also constructed a commutative radical Banach algebra which had no multiplicative functionals but had ε-multiplicative functionals for any $\varepsilon > 0$. The first example of nonfunctionally stable uniform algebra was given by Sidney in 1997 in [**30**]. More recently the problem was investigated in [**17**], however a number of important questions remains open even for Banach algebras. For example we do not know if the algebra $H^\infty(\mathbb{D})$ is functionally stable - in view of the importance the corona theorem it would be particularly interesting to know if the algebra $H^\infty(\mathbb{D})$ has an *almost corona* consisting of almost multiplicative functionals far from the maximal ideal space of $H^\infty(\mathbb{D})$. The concept of almost multiplicative functionals can be easily extended to topological algebras along the same line as the extension of deformations of multiplication. There is an abundance of interesting natural open problems here, for example: What topological algebras are functionally stable? Is multiplicative stability of an m-convex algebra $(A, \cdot, p_\alpha)$ equivalent to functional stability of the completions of all quotient algebras $A/\ker p_\alpha$? etc.

To move even further one can ask about almost multiplicative maps between two topological algebras or about continuous/analytic structures on the family of all deformations of an algebra; partial results are again available only for very special Banach algebras ([**20**], [**10**], [**28**]).

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