

THE UNIQUENESS OF MULTIPLICATION IN FUNCTION ALGEBRAS

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ABSTRACT. Let A be a function algebra. We prove that the original multiplication of A is a unique multiplication on the underlying Banach space of A which produces a Banach algebra with the same unit as the original one.

Let A be a complex Banach algebra with unit. We denote by $\mathbf{1}$ the unit of A , the norm by $\|\cdot\|$ and the product of f and g by fg or $f \cdot g$. By the definition of a Banach algebra, for any elements f, g in A we have

$$(1) \quad \|f \cdot g\| \leq \|f\| \|g\|$$

and

$$(2) \quad \mathbf{1} \cdot f = f.$$

Suppose now that A is uniform algebra, that is, A is a commutative Banach algebra with unit and $\|f^2\| = \|f\|^2$ for all f in A . Our goal is to prove that there exists exactly one (associative) multiplication on the Banach space A which satisfies (1) and (2). This result follows upon considering a more general situation.

By an ε -deformation of A we mean an associative multiplication \times on the Banach space A such that

$$(3) \quad \|f \times g - f \cdot g\| \leq \varepsilon \|f\| \|g\| \quad \text{for all } f, g \text{ in } A.$$

This definition was formulated by Johnson [2] (see also [3, 4]). He investigates whether all multiplications on a Banach algebra A near the given multiplication share some of the properties of the original one. Small deformations of function algebras were studied deeply by R. Rochberg [5].

If \times is an ε -deformation of the multiplication of a Banach algebra A then for all f, g in A

$$(4) \quad \|f \times g\| \leq (1 + \varepsilon) \|f\| \|g\|$$

and

$$(5) \quad \|\mathbf{1} \times f - f\| \leq \varepsilon \|f\|.$$

Our main theorem shows that for uniform algebras the converse implication also holds.

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THEOREM 1. *Suppose (A, \cdot) is a complex uniform algebra. There are positive constants ε_0, c which do not depend on A such that for any $0 \leq \varepsilon \leq \varepsilon_0$ and any multiplication with unit on A satisfying the conditions*

- (i) $\|f \times g\| \leq (1 + \varepsilon)\|f\|\|g\|$ and
- (ii) $\|\mathbf{1} \times f - f\| \leq \varepsilon\|f\|$ for all f, g in A , we have
- (iii) $\|f \times g - f \cdot g\| \leq c\sqrt{\varepsilon}\|f\|\|g\|$ for all f, g in A . Moreover the new multiplication \times is commutative.

PROOF. If $\varepsilon_0 < 1$ then the condition (ii) implies that the operator $T: A \rightarrow A: f \mapsto \mathbf{1} \times f$ is an isomorphism so there exists an element e of A such that $\mathbf{1} \times e = \mathbf{1}$. It is easy to check that e is the unit of the algebra (A, \times) and that the element $\mathbf{1}$ is invertible in this algebra. A simple computation using (i) and (ii) proves that

$$\|f \times g - \mathbf{1}^{-1} \times f \times g\| \leq \frac{2(1 + \varepsilon)^2 \varepsilon}{1 - \varepsilon} \|f\|\|g\|$$

for all f, g in A . Hence the multiplication $\hat{\times}$ defined by $f \hat{\times} g = \mathbf{1}^{-1} \times f \times g$ has the same unit as the original multiplication of the function algebra A and the multiplication denoted by $\hat{\times}$ is a $k\varepsilon$ -deformation of the multiplication \times . This proves that without loss of generality we may assume that the element $\mathbf{1}$ is a common unit of both multiplications \times and $\hat{\times}$.

Let us now introduce some notation.

By ∂A and $\text{Ch } A$ we denote the Shilov and the Choquet boundaries of A , respectively. Let

$$\begin{aligned} \Omega &= \left\{ x + iy \in \mathbf{C}: \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 < \frac{1}{2} \right\} \\ &\cap \left\{ x + iy \in \mathbf{C}: \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 < \frac{1}{2} \right\} \end{aligned}$$

and for $r > 0$

$$D(r) = \{x + iy \in \mathbf{C}: x^2 + y^2 < r^2\}.$$

Notice that without loss of generality we may assume that A is an algebra of continuous functions on ∂A .

Fix $\delta > 0$, and let $\kappa: \overline{D(1)} \rightarrow \overline{\Omega}$ be a continuous map of $\overline{D(1)}$ onto $\overline{\Omega}$ such that κ is analytic on $\overline{D(1)}$ and

$$\kappa(1) = 1 \quad \text{and} \quad \kappa(0) = \delta/2.$$

Let $V \subset \mathbf{C}$ be a neighborhood of 0 such that

$$\kappa(V) \subset \overline{\Omega} \cap D(\delta).$$

Now fix any point $s_0 \in \text{Ch } A$ and any of its neighborhoods $U \subset \partial A$, and let $f \in A$ be such that

$$\|f\| = f(s_0) = 1 \quad \text{and} \quad f(\partial A - U) \subset V.$$

The function $\kappa \circ f \in A$ has the following properties:

- (a) $\kappa \circ f(\partial A) \subset \overline{\Omega}$;
- (b) $\|\kappa \circ f\| = \kappa \circ f(s_0) = 1$;
- (c) $\kappa \circ f(\partial A - U) \subset \overline{\Omega} \cap D(\delta)$.

Hence for any $s_0 \in \text{Ch } A$ there exists a net $(f_\alpha) \subset A$ such that

- (A) $f_\alpha(\partial A) \subset \bar{\Omega}$,
- (B) $\|f_\alpha\| = f_\alpha(s_0) = 1$,
- (C) (f_α) tends uniformly to zero on the compact subsets of the set $\partial A - \{s_0\}$.

Using the net (f_α) we define

$$g'_\alpha = f_\alpha + i(\mathbf{1} - f_\alpha), \quad g''_\alpha = f_\alpha - i(\mathbf{1} - f_\alpha).$$

By direct computation

$$g'_\alpha \times g'_\beta = f_\alpha + f_\beta - \mathbf{1} + i(f_\alpha + f_\beta - 2f_\alpha \times f_\beta).$$

Further observe that, by the definition of Ω , we have

$$\|g'_\alpha\| = \sup_{s \in \partial A} |f_\alpha(s) + i(1 - f_\alpha(s))| \leq \sup_{z \in \Omega} |z + i(1 - z)| = 1.$$

Hence from (i) we get

$$(6) \quad 1 + \varepsilon \geq \|g'_\alpha \times g'_\beta\| \geq |g'_\alpha \times g'_\beta(s_0)| = |1 + 2i(1 - f_\alpha \times f_\beta(s_0))|.$$

The same computations for the functions g''_α and g''_β show that

$$(7) \quad 1 + \varepsilon \geq \|g''_\alpha \times g''_\beta\| \geq |g''_\alpha \times g''_\beta(s_0)| = |1 - 2i(1 - f_\alpha \times f_\beta(s_0))|.$$

Inequalities (6) and (7) can be satisfied simultaneously only if

$$(8) \quad |1 - f_\alpha \times f_\beta(s_0)| \leq \sqrt{\varepsilon/2 + \varepsilon^2/4} \leq \sqrt{\varepsilon}.$$

Now for any $g \in A$ define two functionals $T'_g: A \rightarrow \mathbf{C}$ and $T''_g: A \rightarrow \mathbf{C}$ by

$$T'_g(f) = g \times f(s_0), \quad T''_g(f) = f \times g(s_0).$$

For each $g \in A$ fix two regular measures μ'_g and μ''_g on ∂A such that

$$\begin{aligned} \mu'_g(f) &= T'_g(f), & \text{var}(\mu'_g) &= \|T'_g\|, \\ \mu''_g(f) &= T''_g(f), & \text{var}(\mu''_g) &= \|T''_g\| \quad \text{for all } f \text{ in } A. \end{aligned}$$

Inequality (8) shows that

$$(9) \quad |\mu'_{f_\alpha}(f_\beta) - 1| \leq \sqrt{\varepsilon} \quad \text{for any } \alpha \text{ and all } \beta.$$

By the definition of (f_α) we get

$$|\mu'_{f_\alpha}(\{s_0\}) - 1| \leq \sqrt{\varepsilon}.$$

Hence, because $\text{var}(\mu'_{f_\alpha}) = \|T'_{f_\alpha}\| = 1 + \varepsilon$, the measure μ'_{f_α} is of the form

$$(10) \quad \mu'_{f_\alpha} = \delta_{s_0} + \Delta\mu'_{f_\alpha}$$

where δ_{s_0} is a Dirac measure concentrated at the point s_0 and $\text{var}(\Delta\mu'_{f_\alpha}) \leq 3\sqrt{\varepsilon}$.

Now let g_0 be any element of A such that $\|g_0\| = 1 = g_0(s_0)$. By (10) we get

$$\begin{aligned} \mu^r_{g_0}(f_\alpha) &= f_\alpha \times g_0(s_0) = \mu'_{f_\alpha}(g_0) \\ &= g_0(s_0) + \Delta\mu'_{f_\alpha}(g_0) = 1 + \Delta\mu'_{f_\alpha}(g_0). \end{aligned}$$

Hence

$$|\mu^r_{g_0}(f_\alpha) - 1| \leq 3\sqrt{\varepsilon}.$$

In the same way as previously, we get

$$(11) \quad \mu_{g_0}^r = \delta_{s_0} + \Delta\mu_{g_0}^r \quad \text{where } \text{var}(\Delta\mu_{g_0}^r) \leq 7\sqrt{\varepsilon}.$$

Using this we can estimate the norm of $g_0 \times g_0$ from below.

$$\begin{aligned} \|g_0 \times g_0\| &\geq |g_0 \times g_0(s_0)| = |\mu_{g_0}^r(g_0)| = |1 + \Delta\mu_{g_0}^r(g_0)| \\ &\geq 1 - 7\sqrt{\varepsilon}. \end{aligned}$$

Because s_0 is an arbitrary point of $\text{Ch } A$ this proves that

$$(12) \quad \|g \times g\| \geq (1 - 7\sqrt{\varepsilon})\|g\|^2 \quad \text{for any } g \text{ in } A.$$

As an immediate consequence of (12) we conclude that the spectral radius of any element g of the algebra (A, \times) is not less than $(1 - 7\sqrt{\varepsilon})\|g\|$. Hence by a theorem of Hirschfeld and Żelazko [1] one obtains the commutativity of the multiplication \times if $1 - 7\sqrt{\varepsilon} > 0$.

Applying (12) for $g = f_\alpha$ and using the commutativity of \times we get that there exists a linear and \times -multiplicative functional F_α such that $|F_\alpha(f_\alpha)| \geq 1 - 7\sqrt{\varepsilon}$. For any f in A of norm equal one we have

$$(1 + \varepsilon)\|F_\alpha\| \geq \|F_\alpha\| \|f \times f\| \geq |F_\alpha(f \times f)| = |F_\alpha(f)|^2,$$

hence

$$(1 + \varepsilon)\|F_\alpha\| \geq \|F_\alpha\|^2, \quad \text{so } \|F_\alpha\| \leq 1 + \varepsilon.$$

Let ν_α be a regular measure on ∂A which represents the functional F_α and such that $\text{var}(\nu_\alpha) = \|F_\alpha\|$. We have

$$(13) \quad \begin{cases} |\nu_\alpha(f_\alpha)| \geq 1 - 7\sqrt{\varepsilon}, \\ \text{var}(\nu_\alpha) \leq 1 + \varepsilon, \\ \nu_\alpha(\mathbf{1}) = 1 \quad \text{for all indices } \alpha. \end{cases}$$

Taking a net finer than (f_α) and using the weak $*$ compactness of ∂A we can assume, without loss of generality, that the net (ν_α) is weak $*$ convergent to the measure ν_0 . The measure ν_0 also represents a linear and \times -multiplicative functional F_0 on A . From (13) we derive that the measure ν_0 is of the form

$$(14) \quad \nu_0 = \delta_{s_0} + \Delta\nu_{s_0} \quad \text{where } \text{var}(\Delta\nu_{s_0}) \leq c_1\sqrt{\varepsilon}.$$

From (14) for any f, g in A we find

$$\begin{aligned} f \times g(s_0) + \Delta\nu_{s_0}(f \times g) &= \nu_0(f \times g) = \nu_0(f) \cdot \nu_0(g) \\ &= (f(s_0) + \Delta\nu_{s_0}(f)) \cdot (g(s_0) + \Delta\nu_{s_0}(g)) \\ &= f(s_0) \cdot g(s_0) + \Delta\nu_{s_0}(f) \cdot g(s_0) + f(s_0) \cdot \Delta\nu_{s_0}(g) + \Delta\nu_{s_0}(f) \cdot \Delta\nu_{s_0}(g). \end{aligned}$$

Hence

$$\begin{aligned} |f \times g(s_0) - f \cdot g(s_0)| &\leq \text{var}(\Delta\nu_{s_0})(2\|f\|\|g\| + \text{var}(\Delta\nu_{s_0}) \cdot \|f\|\|g\|) \\ &\leq c_1\sqrt{\varepsilon}(2 + c_1\sqrt{\varepsilon}) \cdot \|f\|\|g\| = c\sqrt{\varepsilon}\|f\|\|g\|. \end{aligned}$$

Because s_0 is an arbitrary point in $\text{Ch } A$, which is a dense subset of ∂A , the above statement proves (iii) and ends the proof of the theorem.

COROLLARY 1. *Suppose (A, \cdot) is a complex function algebra. Let \times be any associative multiplication on the Banach space A with the same unit and such that (A, \times) is a Banach algebra (this means such that $\|f \times g\| \leq \|f\| \|g\|$ for all f, g in A). Then the new multiplication \times and the original one coincide.*

The above corollary can be also formulated in the following way, giving a generalization of Nagasawa's Theorem.

COROLLARY 2. *Let (A, \cdot) be a complex function algebra with unit $\mathbf{1}_A$ and let B be any Banach algebra with unit $\mathbf{1}_B$. Suppose T is a linear isometry from A onto B such that $T\mathbf{1}_A = \mathbf{1}_B$. Then T is an algebra isomorphism of A and B .*

Notice that we have only considered complex Banach algebras. The theorem and the corollaries are not valid for real function algebras, even in two dimensions. To prove this let $A = (\mathbf{R}^2, \cdot, \|\cdot\|_\infty)$ be the two dimensional real function algebra and let $\rho_t: \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$.

$$\rho_t((x, y), (x', y')) = (xx' - t(x - y)(x' - y'), yy' - t(x - y)(x' - y')).$$

A direct computation shows that for any $0 \leq t \leq 1/2$ the bilinear map ρ_t is a commutative, associative multiplication on \mathbf{R}^2 such that $\|\rho_t\| = 1$ and

$$\rho_t((1, 1), (x, y)) = (x, y) \quad \text{for any } (x, y) \in \mathbf{R}^2.$$

Let us end the paper with a natural problem arising from Corollary 1.

PROBLEM. Characterize those Banach spaces A and elements e which admit a unique multiplication \times on A so that (A, \times) is a Banach algebra with unit e .

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