

Uniqueness of Translation Invariant Norms

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Let A be a Banach function space and let \mathcal{M} be a family of multipliers on A . We provide conditions on \mathcal{M} so that the original topology of A is the only complete norm topology on A making all of the maps from \mathcal{M} continuous. As a corollary we show that for a compact abelian group G , and a circle group \mathbb{T}

- for $A = L^p(\mathbb{T})$, $1 < p < \infty$, the L^p -norm is the only one that makes all translations continuous, while
- for $A = C(G)$, $A = L^\infty(G)$, or $A = L^1(G)$ there are other norms with that property.

For noncompact groups the situation is different—on the space $L^1(\mathbb{R})$ the L^1 -norm is the only one that makes a single nontrivial translation continuous. © 2000

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1. INTRODUCTION

Let A be a Banach space, and let \mathcal{M} be a family of linear maps on A . Unlike in most automatic continuity problems (see, for example, [2, 5, 10–12]) we do not ask for the algebraic conditions that *automatically* force the continuity of maps from \mathcal{M} . We assume that the maps are continuous and would like to know when this condition uniquely determines the complete norm topology of A . Similar problem has been recently considered for multiplication maps on semisimple commutative Banach algebras by A. R. Villena [13], and by the author [4].

First we provide conditions for a family of multipliers \mathcal{M} to determine the complete norm topology. As a direct corollary we show that an operator of multiplication by a single nonconstant bounded analytic function determines the complete norm topology on Hardy spaces, Bloch spaces, and most other classical Banach spaces of analytic function of one

or many variables. Then we consider various Banach spaces A of functions defined on a unit circle \mathbb{T} , or on a general locally compact group G , and the family of translations $T_g, g \in G$. While translations are not multipliers on such a space A directly, they correspond to multipliers on the space \hat{A} of Fourier transforms of functions from A . Using this representation and our characterization of multipliers determining the norm topology we prove that the family of all translations determines the complete norm topology on $L^p(\mathbb{T})$ ($1 < p < \infty$) but does *not* determine the complete norm topology on spaces $L^1(G), L^\infty(G)$, or $C(G)$ for a compact group G . For noncompact groups the situation is different—even a single nontrivial translation determines the complete norm topology on $L^1(\mathbb{R})$. The results are related to the classical problem of characterizing function spaces on a topological group G so that all translation invariant functionals are continuous.

2. NOTATION

For a locally compact Hausdorff space X we denote by $C_0(X)$ the vector space of all continuous scalar-valued functions on X vanishing at infinity. We say that A is a Banach function space on X if A is a Banach space contained in $C_0(X)$ in such a way that the functionals $x \in X$ are linearly independent and continuous; here we use the same notation for an element of X and for the corresponding functional on A —the evaluation at the point x . We do *not* assume that the norm on A is equivalent to the natural sup-norm of $C_0(X)$. If M is a continuous function on X such that $f \mapsto Mf$ is a well defined (necessarily continuous and linear) map from A into itself then we call M a multiplier. We usually use the same symbol to denote the function M and the corresponding linear map on A of multiplication by M . For a family \mathcal{M} of multipliers and an element x of X , we put $\mathcal{M}_x = \{M - \lambda : M \in \mathcal{M}, M(x) = \lambda\}$. As we do not assume \mathcal{M} to be a linear space \mathcal{M}_x may not be contained in \mathcal{M} , however \mathcal{M} determines the complete norm topology of A if and only if $\text{span}(\mathcal{M})$ does, and if and only if \mathcal{M}_x does.

For a Banach space $A = (A, \|\cdot\|)$ we denote by $B(A) = B((A, \|\cdot\|))$ the space of all continuous linear maps from A into itself. We say that $\mathcal{B} \subset B((A, \|\cdot\|))$ determines the complete norm topology of A if, for any complete norm $|\cdot|$ on A such that $\mathcal{B} \subset B((A, |\cdot|))$, the topologies defined by the norms $|\cdot|$ and $\|\cdot\|$ are the same. In general, the requirement that the topologies are identical is stronger than the requirement that the Banach spaces $(A, \|\cdot\|)$ and $(A, |\cdot|)$ are isomorphic.

We denote by \mathbb{T} the unit circle, by $H^p(\mathbb{D}), L^p = L^p(\mathbb{T}), 1 \leq p \leq \infty$ the Hardy spaces and the L^p -spaces on the unit circle, respectively. For a

locally compact group G we denote by \hat{G} the dual group; for a function f on G and an element g in G we put $T_g(f)(t) = f(t + g)$.

For an open nonempty subset Ω of the complex space \mathbb{C}^n we denote by $H^\infty(\Omega)$ the space of all bounded analytic functions on Ω and by $A(\Omega)$ the subspace of $H^\infty(\Omega)$ consisting of the functions that can be continuously extended to the closure $\bar{\Omega}$ of Ω

3. THE RESULTS

3.1. Multipliers Determining the Complete Norm Topology

In the first theorem we consider a general case of an arbitrary Banach space A and a set of linear continuous maps \mathcal{B} on A and we provide a necessary condition for \mathcal{B} to determine the complete norm topology of A .

THEOREM 3.1. *Let \mathcal{B} be a family of continuous linear maps from a Banach space A into itself. Assume that the space $\text{span}(\bigcup_{M \in \mathcal{B}} M(A))$ is of infinite-codimension in A , and that it does not contain $\bigcap_{M \in \mathcal{B}} \ker M$. Then \mathcal{B} does not determine the complete norm topology of A .*

Notice that for any multiplier M we have $\ker M \cap M(A) = \{0\}$, so if \mathcal{B} above consists of multipliers then $\text{span}(\bigcup_{M \in \mathcal{B}} M(A))$ never contains $\bigcap_{M \in \mathcal{B}} \ker M$ unless the latter space is trivial.

THEOREM 3.2. *Let A be a Banach function space on a locally compact set X and let \mathcal{M} be a set of multipliers on A .*

(1) *If there is an $x \in X$ such that*

$$\text{span} \left(\bigcup_{M \in \mathcal{M}_x} M(A) \right) \text{ is infinite-codimensional and } \bigcap_{M \in \mathcal{M}_x} \ker M \neq \{0\}$$

then \mathcal{M} does not determine the complete norm topology of A .

(2) *If for any $x \in X$ one of the following two conditions is true:*

(i) $\bigcap_{M \in \mathcal{M}_x} \ker M = \{0\}$, or

(ii) *there is a finite number of elements M_1, \dots, M_n in \mathcal{M}_x such that $\text{span}(\bigcup_{j=1}^n M_j(A))$ is finite-codimensional,*

then \mathcal{M} determines the complete norm topology of A .

Notice that since \mathcal{M}_x determines the complete norm topology of A if and only if \mathcal{M} does, the first part of Theorem 3.2 follows immediately from Theorem 3.1.

COROLLARY 3.1. *Let A be a Banach function space on a locally compact set X and let \mathcal{M} be a finite set of multipliers on A . Then \mathcal{M} determines the complete norm topology of A if and only if for any $x \in X$*

$$\text{span} \left(\bigcup_{M \in \mathcal{M}_x} M(A) \right) \text{ is finite-codimensional or } \bigcap_{M \in \mathcal{M}_x} \ker M = \{0\}.$$

COROLLARY 3.2. *Let Ω be an open nonempty connected subset of \mathbb{C}^n , let A be a Banach space of analytic functions on Ω separating the points of Ω and such that for each \mathbf{w} the functional of evaluation at \mathbf{w} is continuous. Assume that M is a nonconstant function on Ω such that $f \mapsto Mf$ maps A into itself. Then M determines the complete norm topology of A .*

To prove the corollary let Δ be an open nonempty set such that $\bar{\Delta} \subset \Omega$. The Banach space A can be naturally embedded into $A(\Delta)$,

$$J: A \rightarrow A(\Delta) : J(f) = f|_{\Delta}.$$

So A can be seen as a subspace of $A(\Delta)$ separating the points of Δ . Of course the norm on this subspace may be different from the natural sup norm of $A(\Delta)$. The restriction of M to Δ is a multiplier on A as a subspace of $A(\Delta) \subset C(\bar{\Delta})$. As M is locally equal to a ratio of two analytic functions it can not be constant on an open nonempty set, so $\ker(M - \lambda) = \{0\}$ and Corollary 3.2 follows from Corollary 3.1.

Notice that Corollary 3.2 can be applied not only to spaces of bounded continuous analytic functions, like the disc algebra, but also to the spaces that include unbounded functions like H^p -spaces or Bloch spaces.

3.2. Uniqueness of Translation Invariant Norms

Theorem 3.2 can also be used in less direct and perhaps more interesting way to investigate uniqueness of translation invariant norms. This problem is related to the question of the existence of discontinuous, translation invariant functionals and operators and has been investigated by a number of authors, see, for example, [1, 3, 6, 8, 9, 11, 14].

Let E be a translation invariant space of functions on a locally compact abelian group G , for example E could be equal to $L^p(G)$ or to $C(G)$. A linear functional F on E is called translation invariant if

$$FT_g a = Fa, \quad \text{for } a \in E, g \in G,$$

where $T_g: E \rightarrow E$ is defined by $T_g(f)(t) = f(t + g)$. Similarly, a linear map $\Psi: E \rightarrow E$ is called translation invariant if

$$\Psi T_g a = T_g \Psi a, \quad \text{for } a \in E, g \in G.$$

Clearly a functional F is translation invariant if and only if

$$\text{span} \left(\bigcup_{g \in G} (Id - T_g)(E) \right) \subset \ker F.$$

Hence, there is a nonzero translation invariant functional if and only if $\text{span}(\bigcup_{g \in G} (Id - T_g)(E))$ is not equal to E . In most cases there is at most one (up to a multiplicative constant) continuous translation invariant functional—the integral with respect to the Haar measure. In such case there is a discontinuous translation invariant functional on E if and only if $\text{span}(\bigcup_{g \in G} (Id - T_g)(E))$ is of codimension greater than one.

Assume E consists of integrable functions. The Fourier transform $\hat{\cdot}$ is a bijection from E onto a subspace \hat{E} of the space $C_0(\hat{G})$ of continuous functions on the dual group \hat{G} ,

$$\hat{\cdot} : E \rightarrow C_0(\hat{G}) : \hat{f}(\chi) = \int_G f(t) \chi^{-1}(t) dt,$$

where we integrate with respect to the normalized Haar measure. For any $g \in G$ we have

$$\widehat{T_g(f)}(\chi) = \int_G f(t+g) \chi^{-1}(t) dt = \int_G f(t) \chi^{-1}(t-g) dt = \chi(g) \hat{f}(\chi)$$

so T_g corresponds to the operator of multiplication by a function $\hat{g}(\chi) \stackrel{\text{df}}{=} \chi(g)$ on \hat{E} . Let $\mathcal{M} \stackrel{\text{df}}{=} \{\hat{g} : g \in G\}$ be the set of all such multipliers. For any $\chi_0 \in \hat{G}$ we have

$$\begin{aligned} \mathcal{M}_{\chi_0} &= \{M - \lambda : M \in \mathcal{M}, M(\chi_0) = \lambda\} \\ &= \{\hat{g}(\chi) - \hat{g}(\chi_0) : g \in G\}. \end{aligned}$$

The family of all translations $\mathcal{T} \stackrel{\text{df}}{=} \{T_g, g \in G\}$ determines the complete norm topology on E if and only if the family \mathcal{M} does on \hat{E} . Notice also that the set \mathcal{M}_{χ_0} corresponds to $\mathcal{T}_{\chi_0} \stackrel{\text{df}}{=} \{T_g - \chi_0(g) Id : g \in G\}$.

We will consider three cases:

- (1) $E = L^1(\mathbb{R})$,
- (2) $E = L^1(G), L^\infty(G)$, or $C(G)$, where G is a compact abelian group, and
- (3) $E = L^p(\mathbb{T})$, $(1 < p < \infty)$.

THEOREM 3.3. *Let a be a nonzero real number and T_a be an operator of translation by a on $L^1(\mathbb{R})$. Then T_a determines the complete norm topology of $L^1(\mathbb{R})$.*

To prove the result notice that for any real number x_0 in the dual group $\widehat{\mathbb{R}} = \mathbb{R}$ we have

$$((T_a - e^{2\pi i a x_0} Id)(f))^\wedge(x) = (e^{2\pi i a x} - e^{2\pi i a x_0}) \hat{f}(x), \quad f \in L^1(\mathbb{R}), x \in \widehat{\mathbb{R}}.$$

The multiplier $M_{x_0}(x) \stackrel{df}{=} e^{2\pi i a x} - e^{2\pi i a x_0}$ is injective (as a map from $\widehat{L^1(\mathbb{R})}$ into itself) and $M_{x_0}(x_0) = 0$ so the result follows from Corollary 3.2 as $\bigcap_{M \in \mathcal{M}_x} \ker M = \{0\}$.

It may be interesting to notice that for $E = L^1(\mathbb{R})$ the linear span of $\bigcup_{M \in \mathcal{M}_x} M(A)$ is infinite-codimensional so there are discontinuous translation invariant functionals on E [14].

THEOREM 3.4. *Let G be a compact abelian group. Then the family of all translations $T_g, g \in G$ does not determine the complete norm topology on $L^1(G), L^\infty(G)$, or $C(G)$.*

The theorem follows from Theorem 3.2 and a result by S. Saeki [11] (see also [7]) who proved that if E is equal to $L^1(G), L^\infty(G)$, or $C(G)$ then the linear span of $\bigcup_{M \in \mathcal{M}_x} M(\hat{E})$ is infinite-codimension. This time translations do not determine the complete norm topology and there are still discontinuous translation invariant functionals on E [11].

THEOREM 3.5. *Let \mathbb{T} be the circle group. Then the family of all rotations $T_g, g \in \mathbb{T}$ determines the complete norm topology on $L^p(\mathbb{T})$, for $1 < p < \infty$.*

The proof of this theorem will require much more subtle analysis and will be given in the next section. Obviously this time $\bigcap_{M \in \mathcal{M}_n} \ker M$ is one-dimensional and consists of multiples of the character z^n . To prove the theorem we will have to analyze $\bigcup_{M \in \mathcal{M}_n} M(L^p(\mathbb{T}))$. In 1986 J. Bourgain [1] proved that the span of this union is one-codimensional, however if we take any finite number of elements M_1, \dots, M_n in \mathcal{M}_n then $\text{span}(\bigcup_{j=1}^n M_j(A))$ is infinite-codimensional. Hence no finite family of rotations determines the complete norm topology of $L^p(\mathbb{T})$. Notice that this time translations do determine the complete norm topology and all translation invariant functionals are continuous.

4. PROOFS

4.1. Preliminaries

The following lemma is a special case of what is perhaps the most basic result in the automatic continuity theory [12].

LEMMA 4.1. *Let S be a linear map between Banach spaces X and Y , and let T_n, R_n be continuous linear maps on the space X and on Y , respectively, such that $R_n S = S T_n$ for all $n \in \mathbb{N}$. Let \mathfrak{S}_n be the norm closure of $R_n \circ R_{n-1} \circ \dots \circ R_1(\mathfrak{S})$, where \mathfrak{S} is the separating space of S . Then there is an integer N such that $\mathfrak{S}_n = \mathfrak{S}_N$ for each $n \geq N$.*

Recall that the separating space \mathfrak{S} of a linear map S between Banach spaces X and Y is defined by

$$\mathfrak{S} = \{y \in Y : \text{there is a sequence } (x_n)_{n=1}^\infty \text{ in } X \text{ with } x_n \rightarrow 0 \text{ and } Sx_n \rightarrow y\}.$$

By the Closed Graph Theorem, S is continuous if and only if $\mathfrak{S} = \{0\}$.

We also need the following standard fact.

LEMMA 4.2. *Assume T is a bounded linear map from a Banach space A into a Banach space B . If the codimension of $T(A)$ is finite then $T(A)$ is closed.*

The next lemma follows from the previous one.

LEMMA 4.3. *Assume x is a linear functional on a Banach space B and \mathcal{B} is a finite set of bounded linear maps from a Banach space A into B such that $\text{span}(\bigcup_{M \in \mathcal{B}} M(A))$ is finite-codimensional in B and $M(A) \subset \ker x$ for all M in \mathcal{B} . Then $\text{span}(\bigcup_{M \in \mathcal{B}} M(A))$ is closed and the functional x is continuous.*

Proof. Let $\Phi: A^{\text{card}(\mathcal{B})} \rightarrow B$ be defined by

$$\Phi((a_M)_{M \in \mathcal{B}}) = \sum_{M \in \mathcal{B}} M(a_M).$$

Φ is a bounded linear map whose range contains the span of $\bigcup_{M \in \mathcal{B}} M(A)$, so according to our assumption it is finite-codimensional. By Lemma 4.2 the range of Φ is closed. Hence, the kernel of the functional x contains a finite-codimensional closed subspace, and consequently, the functional is continuous. ■

4.2. Proof of Theorem 3.1

Fix an $a_0 \in (\bigcap_{M \in \mathcal{B}} \ker M) \setminus (\bigcup_{M \in \mathcal{B}} M(A))$, let $A_0 = \text{span}\{a_0\}$, and let B be a linear complement of $\text{span}(A_0 \cup \bigcup_{M \in \mathcal{B}} M(A))$. Since $A_0 \cap \text{span}(\bigcup_{M \in \mathcal{B}} M(A)) = \{0\}$, the vector space A can be identified with a direct linear product $A_0 \oplus \text{span}(\bigcup_{M \in \mathcal{B}} M(A)) \oplus B$. By our assumption $\dim B = \infty$, so there is a discontinuous linear functional F on B . Let

$$P: A_0 \oplus \text{span}\left(\bigcup_{M \in \mathcal{B}} M(A)\right) \oplus B \rightarrow A_0 \oplus \text{span}\left(\bigcup_{M \in \mathcal{B}} M(A)\right) \oplus B$$

be defined by

$$P(a_1, a_2, a_3) = (a_1 + F(a_3) a_0, 0, 0).$$

The map P is a discontinuous linear projection from A onto A_0 such that $P \circ M = 0$, for $M \in \mathcal{B}$. We define a new norm $|\cdot|$ on A by

$$|a| = \|a_{/A_0}\| + \|Pa\|, \quad \text{for } a \in A,$$

where $\|a_{/A_0}\|$ is the norm of the equivalence class of a in the quotient space $A_{/A_0}$. Notice that $|a|$ is a well-defined complete norm on A . Indeed, $(A, |\cdot|)$ is isometric with the direct product of $A_{/A_0}$ and A_0 . Since P is discontinuous the new norm and the original one are nonequivalent (though the Banach spaces $(A, \|\cdot\|)$ and $(A, |\cdot|)$ may be isomorphic).

To show that any $M \in \mathcal{B}$ is continuous on $(A, |\cdot|)$, fix $a \in A$ and let λ be a scalar such that $\|a_{/A_0}\| = \|a + \lambda a_0\|$. By our assumption $P \circ M = 0$ and $Ma_0 = 0$, so

$$\begin{aligned} |Ma| &= \|(Ma)_{/A_0}\| + \|PMa\| = \|(Ma)_{/A_0}\| \leq \|Ma\| \\ &= \|M(a + \lambda a_0)\| \leq \|M\| \|a + \lambda a_0\| = \|M\| \|a_{/A_0}\| \leq \|M\| |a|, \end{aligned}$$

where $\|M\|$ is the norm of M in $(A, \|\cdot\|)$. The above shows that the norm of M in $(A, |\cdot|)$ is not greater than $\|M\|$.

4.3. Proof of Theorem 3.2

We already noticed that one part of the Theorem follows from Theorem 3.1. To prove the other part assume that there is another complete norm $|\cdot|$ on A such that the operators from \mathcal{M} are continuous on $(A, |\cdot|)$. Assume further that for any $x \in X$ we have: $\bigcap_{M \in \mathcal{M}_x} \ker M = \{0\}$, or there is a finite number of elements M_1, \dots, M_n of \mathcal{M}_x such that $\text{span}(\bigcup_{j=1}^n M_j(A))$ is finite-codimensional. According to Lemma 4.2 that means that for any $x \in X$ we have

$$\text{the functional } x \text{ is } |\cdot| \text{-continuous, or } \bigcap_{M \in \mathcal{M}_x} \ker M = \{0\}. \quad (1)$$

Let S be the identity map from $(A, |\cdot|)$ onto $(A, \|\cdot\|)$ and \mathfrak{S} the separating space of S .

Fix a nonzero element b of \mathfrak{S} and let $x_1 \in X$ be such that $b(x_1) \neq 0$. Since b is in the separating space, the functional $x_1 \in (A, |\cdot|)$ is discontinuous, so $\bigcap_{M \in \mathcal{M}_{x_1}} \ker M = \{0\}$, and there is an M_1 in \mathcal{M}_{x_1} such that $M_1 \cdot b \neq 0$ (here we use the same symbol M for a multiplier and for the corresponding continuous function on X). Since $M_1 \in \mathcal{M}_{x_1}$ we have $M_1 b(x_1) = 0$. Let $x_2 \in X$ be such that $M_1 b(x_2) \neq 0$. As before there is an M_2 in \mathcal{M}_{x_2} such that

$M_2 \cdot M_1 \cdot b \neq 0$. Continuing this process we get an infinite sequence of distinct points x_1, x_2, \dots in X and a corresponding sequence (M_n) in \mathcal{M} such that

$$M_n(x_n) = 0,$$

and

$$(M_1 \cdot \dots \cdot M_n \cdot b)(x_{n+1}) \neq 0, \quad \text{for } n \in \mathbb{N}. \tag{2}$$

Put $R_n = M_n$ when the map is defined on the space $(A, \|\cdot\|)$, and $T_n = M_n$ on $(A, |\cdot|)$. By Lemma 4.1, there is an N such that $\mathfrak{S}_N = \mathfrak{S}_{N+1}$. However, the space \mathfrak{S}_{N+1} is contained in the kernel of the continuous functional $x_{N+1} \in X$, while \mathfrak{S}_N contains an element $M_1 \cdot \dots \cdot M_n \cdot b$ which, by (4.7), is not in the kernel of x_{N+1} . The contradiction shows that \mathfrak{S} does not contain any nonzero element, so the norms are equivalent.

4.4. Proof of Theorem 3.5

Let $E = L^p(\mathbb{T})$ ($1 < p < \infty$) and let \hat{E} be the space of Fourier transforms $\hat{f}(n) \stackrel{df}{=} \frac{1}{2\pi} \int_{\mathbb{T}} f e^{-2\pi i n t} dt$ of the functions f from $L^p(\mathbb{T})$. Let \mathcal{T} be the family of all translations $T_g, g \in \mathbb{T}$ on $L^p(\mathbb{T})$, and \mathcal{M} the corresponding family of multipliers on \hat{E} . For $n \in \mathbb{Z}$ we denote by $L^n_p(\mathbb{T})$ (respectively \hat{E}_n) the subspace of $L^p(\mathbb{T})$ (of \hat{E} , resp.) consisting of all elements with $\hat{f}(n) = 0$ and we put $\mathcal{M}_n \stackrel{df}{=} \{M - M(n) \in C(\mathbb{Z}) : M \in \mathcal{M}\}$. For $n \in \mathbb{Z}$, and $t \in \mathbb{T}$ we put $M_{n,t}(k) = e^{2\pi i k t} - e^{2\pi i n t}$, and we use the same symbol $M_{n,t}$ for the multiplier on \hat{E} -multiplication by $M_{n,t}$. The multiplier $M_{n,t}$ corresponds to the operator on $L^p(\mathbb{T})$ of translation by t minus the multiplication by the constant $e^{2\pi i n t}$; notice that $M_{n,t} \in \mathcal{M}_n$.

We need the following theorem by J. Bourgain [1].

THEOREM 4.1 (J. Bourgain). *Assume f is in $L^p(\mathbb{T})$ ($1 < p < \infty$) with $\int_{\mathbb{T}} f dt = 0$. Then for any $\varepsilon > 0$ there is a positive number c_ε and a subset K_ε of \mathbb{T}^J with $m_J(K_\varepsilon) > 1 - \varepsilon$, such that for any $(g_1, \dots, g_J) \in K_\varepsilon$ there are functions $f_j \in L^p(\mathbb{T})$ with $\|f_j\| \leq c_\varepsilon \|f\|$ and*

$$f = \sum_{j=1}^J (f_j - T_{g_j}(f_j)), \tag{3}$$

where $J = \max\{[p], [\frac{p}{p-1}]\} + 1$.

Here we denote by m_J the normalized J -dimensional Haar measure on \mathbb{T}^J , and by $[s]$ the integer part of a real number s .

In fact, the original Bourgain Theorem is stated in [1] in a weaker form, without the estimate for the measure of K_ε . However, as Bourgain himself

mentions [1, p. 99], a more careful analysis of the same proof gives the desired estimate.

If we apply the Bourgain Theorem to the space \hat{E} rather than to $L^p(\mathbb{T})$, then for $\hat{f} \in \hat{E}_0$, (3) has the form

$$\hat{f} = \sum_{j=1}^J M_{0, g_j} \cdot \hat{f}_j, \quad \text{for some } \hat{f}_j \in \hat{E}.$$

Let $n_0 \in \mathbb{Z}$, $f \in L^p_{n_0}(\mathbb{T})$, and put $f^0(e^{2\pi it}) = f(e^{2\pi i(t+n_0)})$. Since $\hat{f}^0 \in \hat{E}_0$ we get

$$\hat{f} = \sum_{j=1}^J M_{n_0, g_j} \cdot \hat{f}_j, \quad \text{for some } \hat{f}_j \in \hat{E}. \tag{4}$$

Assume now that $n_1, \dots, n_r \in \mathbb{Z}$, and $f \in \bigcap_{j=1}^r L^p_{n_j}(\mathbb{T})$. By (4) there are functions f^1_1, \dots, f^1_J in $L^p(\mathbb{T})$ and g^1_1, \dots, g^1_J in \mathbb{T} such that $\hat{f} = \sum_{j=1}^J M_{n_1, g^1_j} \cdot \hat{f}^1_j$. Notice that if we change the functions f^1_j by subtracting such multiples of $e^{2\pi i n_k t}$, $k = 1, \dots, r$ that $\hat{f}^1_j(n_k) = 0$, for $1 \leq j \leq J$, $1 \leq k \leq r$, then we get

$$\hat{f} = \sum_{j=1}^J M_{n_1, g^1_j} \cdot \hat{f}^1_j, \quad \text{for some } \hat{f}^1_j \in \bigcap_{j=1}^r \hat{E}_{n_j}.$$

We can apply the same procedure to all of the functions f^1_1, \dots, f^1_J and write each of these functions in the form

$$\hat{f}^1_j = \sum_{i=1}^J M_{n_2, g^2_i} \cdot \hat{f}^2_i, \quad \text{for some } \hat{f}^2_i \in \bigcap_{j=1}^r \hat{E}_{n_j}.$$

Since we can continue the process, and at each stage we can arbitrarily select g^s_1, \dots, g^s_J from a subset of measure $1 - \varepsilon$, we get the following Lemma.

LEMMA 4.4. *Assume $n_1, \dots, n_r \in \mathbb{Z}$, $\varepsilon > 0$, and $f \in \bigcap_{j=1}^r L^p_{n_j}(\mathbb{T})$ ($1 < p < \infty$). Then there is a positive number C_ε and a subset L_ε of $\mathbb{T}^{J \times r}$ with $m_J(L_\varepsilon) > 1 - \varepsilon$, such that for any $(g^1_1, \dots, g^1_J, \dots, g^r_1, \dots, g^r_J) \in L_\varepsilon$ there are functions $f_{j_1, \dots, j_r} \in L^p(\mathbb{T})$ with $\|f_{j_1, \dots, j_r}\| \leq C_\varepsilon \|f\|$ and*

$$\hat{f} = \sum_{\substack{1 \leq s \leq r \\ 1 \leq j_s \leq J}} M_{n_1, g^1_{j_1}} \cdot \dots \cdot M_{n_p, g^p_{j_p}} \hat{f}_{j_1, \dots, j_r},$$

where $J = \max\{[p], [\frac{p}{p-1}]\} + 1$.

We are now ready to prove Theorem 3.5. Assume that there is another complete norm $|\cdot|$ on \hat{E} such that all multipliers from \mathcal{M} are $|\cdot|$ -continuous

Let S be the identity map from $(\hat{E}, |\cdot|)$ onto $(\hat{E}, \|\cdot\|)$ and \mathfrak{S} be the separating space of S . We prove that \mathfrak{S} is trivial in four steps: (i) we show that for any $b \in \mathfrak{S}$ the support of b is finite, (ii) we show that the union of the supports of all the elements from \mathfrak{S} , denoted by Ω , is finite, (iii) we show that $|\cdot|$ is equivalent to a third norm $\|\!\|\!\| \cdot \|\!\|\!\|$ which can be described using a very specific formula involving Ω and the original norm $\|\cdot\|$, (iv) finally we prove that $\|\!\|\!\| \cdot \|\!\|\!\|$ and $\|\cdot\|$ are equivalent.

Assume b is a nonzero element of \mathfrak{S} with infinite support. Let n_1, n_2, \dots be a sequence of distinct integers such that $b(n_j) \neq 0$, for $j \in \mathbb{N}$. Fix an irrational real number $0 < t < 1$ and let $g = e^{2\pi it}$ be the corresponding element of the circle group \mathbb{T} . Notice that $M_{n, g}(k) = 0$ if and only if $k = n$. We have

$$M_{n_j, g}(n_j) = 0,$$

and

$$(M_{n_1, g} \cdot \dots \cdot M_{n_j, g} \cdot b)(n_{j+1}) \neq 0, \quad \text{for } j \in \mathbb{N}. \tag{5}$$

Put $R_n = M_n$ when the map is defined on the space $(\hat{E}, \|\cdot\|)$, and $T_n = M_n$ on $(\hat{E}, |\cdot|)$. By Lemma 4.1, there is an N such that $\mathfrak{S}_N = \mathfrak{S}_{N+1}$. However, the space \mathfrak{S}_{N+1} is contained in the kernel of the continuous functional $n_{N+1} \in \mathbb{Z}$, while \mathfrak{S}_N contains an element $M_{n_1, g} \cdot \dots \cdot M_{n_j, g} \cdot b$ which, by (5), is not in the kernel of n_{N+1} . The contradiction shows that \mathfrak{S} does not contain any element with infinite support.

Assume now that $\Omega \stackrel{\text{df}}{=} \{n \in \mathbb{Z} : \delta_n \text{ is } |\cdot| \text{-discontinuous}\} = \cup \{n \in \mathbb{Z} : \exists b \in \mathfrak{S}, b(n) \neq 0\}$ is infinite and contains a sequence of distinct points n_1, n_2, \dots , here we denote by δ_n a functional on \hat{E} of evaluation at the point n . Define

$$\Phi: \mathfrak{S} \rightarrow l^\infty : \Phi(b) = (b(n_k))_{k=1}^\infty.$$

The range of Φ is infinite-dimensional and is contained in \mathfrak{K}_0 -dimensional space $\{(t_k)_{k=1}^\infty : t_k = 0 \text{ for all but finitely many } k\}$. Hence $\dim \Phi(\mathfrak{S}) = \mathfrak{K}_0$. However, the separating space is closed [12] and Φ is linear and continuous, so the range of Φ is isomorphic, as a vector space, with the quotient Banach space $\mathfrak{S}/\ker \Phi$, which can not have an infinite countable linear dimension. The contradiction proves that Ω is finite, so let

$$\Omega = \{n_1, \dots, n_r\}.$$

Let $\chi_j: \mathbb{Z} \rightarrow \mathbb{C}, j = 1, \dots, r$ be a characteristic functions of the set $\{n_j\}$. Let $P: \hat{E} \rightarrow \hat{E}$ be a $|\cdot|$ -continuous projection onto $A_0 \stackrel{\text{df}}{=} \text{span}\{\chi_1, \dots, \chi_r\}$. We define a third norm $\|\!\|\!\| \cdot \|\!\|\!\|$ on \hat{E} by

$$\|\!\|\!\| a \|\!\|\!\| = |Pa| + \|a_{/A_0}\|, \quad \text{for } a \in \hat{E}, \tag{6}$$

where $\|a_{/A_0}\|$ is the norm of the equivalence class of a in the quotient space $A_{/A_0}$. We claim that the norms $|\cdot|$ and $\|\|\cdot\|\|$ are equivalent. To show this let $n \in \mathbb{Z} \setminus \Omega$, and let $a \in \hat{E}$ be such that $\|\|a\|\| < 1$. Then $\|a_{/A_0}\| < 1$ so there is an $a' \in A_0$ such that $\|a + a'\| < 1$ and we have

$$|a(n)| = |(a + a')(n)| \leq \|a + a'\| < 1.$$

Hence δ_n is $\|\|\cdot\|\|$ -continuous (with the norm at most one), also the projection P is $\|\|\cdot\|\|$ -continuous. Let \mathcal{I} be the identity map from $(\hat{E}, \|\|\cdot\|\|)$ onto $(\hat{E}, |\cdot|)$. Since $\delta_n \circ \mathcal{I}$, $n \in \mathbb{Z} \setminus \Omega$ and $P \circ \mathcal{I}$ are all continuous, and the family $\{\delta_n : n \in \mathbb{Z} \setminus \Omega\} \cup \{P\}$ separates points of \hat{E} , it follows from the Closed Graph Theorem that \mathcal{I} is continuous, and consequently an isomorphism.

Put $M = M_{n_1, g_1} \cdot \dots \cdot M_{n_r, g_r}$ where n_j are the points of Ω and g_j are arbitrary elements of \mathbb{T} . We show that $P \circ M$ is continuous as a map from $(\hat{E}, \|\cdot\|)$ into itself; notice that as the range of $P \circ M$ is finite-dimensional it does not matter what norm we consider on the range space. Let a be an element of \hat{E} with $\|a\| < 1$. Using the facts that $M \circ P = 0$, M is $\|\|\cdot\|\|$ -continuous, and the range of P is in A_0 we get

$$\begin{aligned} |P \circ M(a)| &= |P \circ M(a - Pa)| \\ &\leq |P \circ M(a - Pa)| + \|M(a - Pa)_{/A_0}\| \\ &= \|\|M(a - Pa)\|\| \\ &\leq \|\|M\|\| (|P(a - Pa)| + \|(a - Pa)_{/A_0}\|) \\ &= \|\|M\|\| \|(a - Pa)_{/A_0}\| \\ &= \|\|M\|\| \|a_{/A_0}\| \\ &\leq \|\|M\|\| \|a\|. \end{aligned}$$

For any natural number N let

$$\mathcal{W}_N = \{(g_1, \dots, g_r) \in \mathbb{T}^r : \|P \circ M_{n_1, g_1} \cdot \dots \cdot M_{n_r, g_r}\| < N\}.$$

Since $\bigcup_{N=1}^{\infty} \mathcal{W}_N = \mathbb{T}^r$ there must be $N_0 \in \mathbb{N}$ such that \mathcal{W}_{N_0} is not of measure zero (we do not claim that \mathcal{W}_{N_0} must be measurable). Put $J = \max\{[p], [\frac{p}{p-1}]\} + 1$ and let $\varepsilon > 0$ be so small that any set $L \subset \mathbb{T}^{r \times J}$ with measure greater than $1 - \varepsilon$ has to have a nonempty intersection with $(\mathcal{W}_{N_0})^J$.

Let $\hat{f} \in \bigcap_{j=1}^r \hat{E}_{n_j}$. According to Lemma 4.4 there are functions f_{j_1, \dots, j_r} in $L^p(\mathbb{T})$ with $\|f_{j_1, \dots, j_r}\| \leq C_\varepsilon \|\hat{f}\|$ and an element $(g_1^1, \dots, g_1^J, \dots, g_r^1, \dots, g_r^J)$ of $(\mathcal{W}_{N_0})^J$ such that

$$\hat{f} = \sum_{\substack{1 \leq s \leq r \\ 1 \leq j_s \leq J}} M_{n_1, g_{j_1}^1} \cdot \dots \cdot M_{n_p, g_{j_r}^r} \hat{f}_{j_1, \dots, j_r}.$$

We have

$$\|P\hat{f}\| \leq \sum_{\substack{1 \leq s \leq r \\ 1 \leq j_s \leq J}} \|P \circ M_{n_1, g_{j_1}^1} \cdots \cdots M_{n_r, g_{j_r}^p}\| \|\hat{f}_{j_1, \dots, j_p}\| \leq rJN_0 C_\varepsilon,$$

so the restriction of P to $\bigcap_{j=1}^r \hat{E}_{n_j}$ is $\|\cdot\|$ -continuous, since $\bigcap_{j=1}^r \hat{E}_{n_j}$ is a closed finite-codimensional subspace of \hat{E} , the map P is $\|\cdot\|$ -continuous on the entire space \hat{E} . By the definition (6), it means that the norms $\|\cdot\|$ and $\|\cdot\|$ are equivalent. Since we also know that $\|\cdot\|$ and $|\cdot|$ are equivalent, this completes the proof.

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