Proceedings of the Edinburgh Mathematical Society (1985) 28, 121–131 ©

SMALL ISOMORPHISMS BETWEEN OPERATOR ALGEBRAS

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(Received 23rd January 1984)

0. Introduction

Let A and B be function algebras. The well-known Nagasawa theorem [**5**] states that A and B are isometric if and only if they are isomorphic in the category of Banach algebras. In [**2**] it was shown that this theorem is stable in the sense that if the Banach Mazur distance between the underlying Banach spaces of A and B is close to one then these algebras are almost isomorphic, that is there exists a linear map T from A onto B such that $\|T^{-1}(Tf\,Tg)-fg\|\leqq\varepsilon\|f\|\|g\|$. On the other hand one can get from Theorems 1 and 3 of [**3**] that the Nagasawa theorem can be extended to some operator algebras as follows:

Theorem. *Let X,Y be real Banach spaces with the approximation property and such that X^*, X^{**}, Y^*, Y^{**} are all strictly convex. Assume that T is a linear isometry from $K(X)=X^*\check{\otimes}X$ onto $K(Y)=Y^*\check{\otimes}Y$ then one of the following two possibilities holds*

(*a*) $T=T_1\otimes T_2$ *where* $T_1:X^*\to Y^*$, $T_2:X\to Y$ *are onto isometries.*
(*b*) $T=T_1\otimes T_2$ *where* $T_1:X\to Y^*$, $T_2:X^*\to Y$ *are onto isometries.*

Consequently $K(X)$ and $K(Y)$ are isomorphic or anti-isomorphic in the category of Banach algebras.

If $X=Y=$ Hilbert space then this result is a consequence of Kadison's result on isometries in C^*-algebras.

In this paper we combine the method of [**3**], [**1**] and [**2**] to prove that, in the case of uniformly convex spaces, the above theorem is also stable.

1. Definitions and notation

For Banach spaces U and V

$B(U)$ denotes the closed unit ball in U,

$E(U)$ denotes the set of extreme point of $B(U)$,

$U\check{\otimes}V$ denotes the injective tensor product of U and V,

$L(U,V)$ ($K(U,V)$) denotes the Banach space of all continuous (compact) linear operators from U into V. If $U=V$ we write $L(U)$ ($K(U)$) in place of $L(U,U)$ ($K(U,U)$), the Banach–Mazur distance between U and V is defined by

$$d_{B-M}(U,V)=\inf\{\|T\|\|T^{-1}\|: T \text{ is a linear isomorphism from } U \text{ onto } V\},$$

and we put $d_{B-M}(U,V)=\infty$ if the spaces U and V are nor isomorphic.

For a Hausdorff space S we denote by $C(S)$ the Banach space of all continuous, bounded scalar-valued functions on S with the sup-norm.

In this paper we often consider a Banach space V as a closed subspace of $C(E(V^*))$ where $E(V^*)$ is equipped with the weak *topology. The space $V\check{\otimes}U$ is regarded as a subspace of $C(E(V^*)\times E(U^*))$.

For a Banach space V, δ_V denotes the modulus of convexity of V i.e. the function $\delta_V:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\delta_V(\varepsilon)=1-\sup\{\tfrac{1}{2}\|v+v'\|: v,v'\in V, \|v\|=\|v'\|=1, \|v-v'\|\geqq\varepsilon\}$$

Also we define $\delta_V^*:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\delta_V^*(\delta)=\sup\{\varepsilon\in\mathbb{R}^+:\delta_V(\varepsilon)\leqq\delta\}.$$

Notice that V is uniformly convex if and only if $\lim_{\delta\to 0^+}\delta_V^*(\delta)=0$.

Let A and B be Banach algebras and let T be a continuous map from A onto B. We say that T is a linear isomorphism or isomorphism in the category of Banach spaces if T is an isomorphism of underlying Banach spaces of A and B. If, in addition, T preserves the algebra multiplication we call it an algebra isomorphism or isomorphism in the category of Banach algebras.

Finally for a metric space S we put

$$\operatorname{diam} S=\sup\{d(s_1,s_2): s_1,s_2\in S\}.$$

2. The results

Theorem 1. *Let $X,\tilde{X},Y,\tilde{Y}$ be Banach spaces with uniformly convex duals. Then there is an $\varepsilon_0>0$ such that for any $\varepsilon\leqq\varepsilon_0$ and any linear isomorphism T from $X\check{\otimes}\tilde{X}$ onto $Y\check{\otimes}\tilde{Y}$ with $\|T\|\|T^{-1}\|\leqq 1+\varepsilon$ there are linear isomorphisms $\Phi:X\to Y$ and $\Psi:\tilde{X}\to\tilde{Y}$ or $\Phi:X\to\tilde{Y}$ and $\Psi:\tilde{X}\to Y$ with $\|\Phi\|\|\Phi^{-1}\|\leqq 1+c(\varepsilon)$ and $\|\Psi\|\|\Psi^{-1}\|\leqq 1+c(\varepsilon)$ such that*

$$\|T-\Phi\otimes\Psi\|\leqq c(\varepsilon).$$

The constant ε_0 and the function c depend only on the modulus of convexity of the considered Banach spaces and $\lim_{\varepsilon\to 0^+}c(\varepsilon)=0$.

Corollary 1. *Let X,Y be Banach spaces with the approximation property and such that X,X^*,Y and Y^* are uniformly convex. Then there is an $\varepsilon_0>0$ such that if the Banach Mazur distance between $K(X)$ and $K(Y)$ is less than $1+\varepsilon_0$ then $K(X)$ and $K(Y)$ are isomorphic in the category of Banach algebras. The constant ε_0 depends only on the modulus of convexity of Banach spaces X,X^*,Y,Y^*.*

Proof. It is an immediate consequence of Theorem 1, of the fact that any uniformly convex space is reflective and that $K(X)=X^*\otimes X$ whenever X has the approximation property.

Corollary 2. *Let X, Y, be finite dimensional Banach spaces such that X, X^*, Y, Y^* are strictly convex. Then there is an $\varepsilon_0>0$ such that for any $\varepsilon \leqq \varepsilon_0$ and any linear map T from $L(X)$ onto $L(Y)$ with $\|T\|\|T^{-1}\| \leq 1+\varepsilon$ and $T(\mathrm{Id}_X)=\mathrm{Id}_Y$ there is an algebra isomorphism $\tilde{T}$ from $L(X)$ onto $L(Y)$ such that*

$$\|T-\tilde{T}\| \leq c'(\varepsilon).$$

where $\lim_{\varepsilon\to 0^+} c'(\varepsilon)=0$.

Proof of Theorem. We assume, without loss of generality, that $\|T\| \leqq 1+\varepsilon$ and $\|T^{-1}\| \leqq 1+\varepsilon$.

At various points of the proof we shall use the inequalities involving ε which are valid only is ε is sufficiently small, in those cases we will merely assume that ε is near 0 and this assumption gives rise to the constant ε_0.

Lemma 1. *Let U and V be normed, linear spaces, let δ be a positive number and assume that*

$$\|u_1 \otimes v_1 + u_2 \otimes v_2 + u_3 \otimes v_3\| \leq \delta 1 \tag{1}$$

where

$$u_1, u_2, u_3 \in U, v_1, v_2, v_3 \in V$$

and

$$\|u_1\| = \|u_2\| = 1 = \|v_1\| = \|v_2\| = \|v_3\|.$$

Then there is a number λ of modulus one such that

$$\|u_1-\lambda u_2\| \leq 3\sqrt{\delta} \text{ or } \|v_1-\lambda v_2\| \leq 3\sqrt{\delta}.$$

Proof. If $\inf_{|\lambda|=1}\|\lambda v_i - v_3\| \leqq \frac{3}{2}\sqrt{\delta}$ for both $i=1$ and 2, then we get $\|v_1-\lambda v_2\| \leqq 3\sqrt{\delta}$ for some λ of modulus one, so we can assume that

$$\inf_{|\lambda|=1} \|\lambda v_1 - v_3\| > \tfrac{3}{2}\sqrt{\delta}. \tag{2}$$

Assume there is an $\alpha \in C$ with $\|v_1-\alpha v_3\| \leqq \frac{3}{4}\sqrt{\delta}$. We get

$$1+\tfrac{3}{4}\sqrt{\delta} \geqq |\alpha| \geq 1-\tfrac{3}{4}\sqrt{\delta} > 0$$

and we have

$$\left\|\frac{\bar{\alpha}}{|\alpha|}v_1 - v_3\right\| = \left\|v_1 - \frac{\alpha}{|\alpha|}v_3\right\| \leqq \left\|\frac{\alpha}{|\alpha|} - \alpha\right| + \|v_1 - \alpha v_3\| \leqq = \left|\frac{\alpha(1-|\alpha|)}{|\alpha|}\right| + \tfrac{3}{4}\sqrt{\delta} \leqq \tfrac{3}{2}\sqrt{\delta}.$$

The above contradicts (2) and we get

$$\inf_{\alpha \in C} \|v_1 - \alpha v_3\| > \tfrac{3}{4}\sqrt{\delta}. \tag{3}$$

We define a functional v^* on $\operatorname{span}(v_1, v_3)$ by

$$v^*(\alpha v_1 + \beta v_3) = \tfrac{3}{4}\sqrt{\delta}\alpha.$$

From (3) we have $\|v^*\| \leqq 1$. Let $\tilde{v}^*$ be a norm preserving extension of v^* from $\operatorname{span}(v_1, v_3)$ to V. From (1) we get

$$\|u_1 \tilde{v}^*(v_1) + u_2 \tilde{v}^*(v_2)\| \leqq \delta,$$

so

$$\left\|u_1 + u_2 \frac{\tilde{v}^*(v_2)}{\tilde{v}^*(v_1)}\right\| \leqq \tfrac{4}{3}\sqrt{\delta}$$

Hence, in the same manner as before we get

$$\left\|u_1 + u_2 \frac{\tilde{v}^*(v_2)}{\tilde{v}^*(v_1)} \frac{|\tilde{v}^*(v_1)|}{|\tilde{v}^*(v_2)|}\right\| \leqq 2\tfrac{4}{3}\sqrt{\delta} < 3\sqrt{\delta}.$$

For the next lemmas we need the following observations. The first one is easy to check by a direct computation.

Proposition 1. *Let V be a Banach space with uniformly convex dual and let $v \in V$, $\|v\| = 1$ then*

$$\operatorname{diam}\{v^* \in B(V^*): \operatorname{Re}(v^*(v)) \geqq 1 - \delta\} \leqq \delta^*_{V^*}(2\delta).$$

Proposition 2. *Let V, U be Banach spaces with uniformly convex duals and let $v \in V, u \in U, \|v\| = 1 = \|u\|$ then*

$$\operatorname{diam}\{v^* \otimes u^* \in B(V^*) \otimes B(U^*): \operatorname{Re}((v^* \otimes u^*)(v \otimes u)) \geqq 1 - \delta\} \leqq \delta^*_{V^*}(2\delta) + \delta^*_{U^*}(2\delta).$$

Proof. Fix $v_i^* \otimes u_i^* \in B(V^*) \otimes B(U^*)$ such that

$$\operatorname{Re}(v_i^* \otimes u_i^*)(v \otimes u) \geqq 1 - \delta \quad \text{for} \quad i = 1, 2.$$

Let $\alpha_i, i=1,2$ be complex numbers of modulus one such that $\alpha_i v_i^*(v)\in\mathbb{R}^1$. By our assumption we get

$$\alpha_1 v_i^*(v)\geq 1-\delta \quad \text{and} \quad \operatorname{Re}\frac{1}{\alpha_i}u_i^*(u)\geq 1-\delta \quad \text{for} \quad i=1,2.$$

Hence by Proposition 1 we get

$$\left\|\alpha_1 v_1^*-\alpha_2 v_2^*\right\|\leq\delta_{V^*}^*(2\delta) \quad \text{and} \quad \left\|\frac{1}{\alpha_1}u_1^*-\frac{1}{\alpha_2}u_2^*\right\|\leq\delta_{U^*}^*(2\delta)$$

so

$$\left\|v_1^*\otimes u_1^*-v_2^*\otimes u_2^*\right\|\leq\left\|\alpha_1 v_1^*\otimes\frac{1}{\alpha_1}u_1^*-\alpha_2 v_2^*\otimes\frac{1}{\alpha_1}u_1^*\right\|$$

$$+\left\|\alpha_2 v_2^*\otimes\frac{1}{\alpha_1}u_1^*-\alpha_2 v_2^*\otimes\frac{1}{\alpha_2}u_2^*\right\|\leq\delta_{V^*}^*(2\delta)+\delta_{U^*}^*2(\delta).$$

Proposition 3. *Let S be a compact Hausdorff space, let A be a closed subspace of $C(S)$ and Let F be a norm one functional on A. We denote by S_0 the subset of S consisting of all points s from S such that the norm of the functional $A\ni f\to f(s)$ is equal to one. Assume that for any $s\in S$ and any number λ of modulus one there is exactly one $s_\lambda\in S$ such that*

$$f(s)=\lambda f(s_\lambda) \quad \textit{for all} \quad f\in A.$$

Then there is a probability measure μ on S which is a norm preserving extension of F from A to $C(S)$. Furthermore for any such μ we have $\mu(S_0)=\mu(S)=1$.

Proof. Let ν be a norm one extension of F from A to $C(S)$. Denote by K_r the subset of S consisting of all points $s\in S$ such that the norm of functional $A\ni f\to f(s)$ is not greater than r. For any $f\in A$ with $\|f\|=1$ we have

$$|F(f)|=\left|\int_S f\,d\nu\right|\leq\int_{K_r}|f|\,d|\nu|+\int_{S\setminus K_r}|f|\,d|\nu|\leq\sup\{|f(s)|:s\in K_r\}\cdot|\nu|(K_r)$$

$$+|\nu|(S\setminus K_r)\leq 1-|\nu|(K_r)(1-r).$$

Hence $|\nu|(K_r)=0$ for any $r<1$, because F has norm one on A. Since $S\setminus S_0$ is the union of $S\setminus K_r$ for $0<r<1$, $|\nu|(S_0)=1$.

Put $h=d\nu/d|\nu|$. We can assume $|h|\equiv 1$ on S. By our assumption there is the map $\varphi:S\to S$ such that

$$h(s)f(s)=f\circ\varphi(s) \quad \text{for} \quad f\in A, s\in S.$$

If h is continuous, then the corresponding function φ defined by the above equality is also continuous. Hence it is standard to prove that if h is a Borel function then φ is also Borel. To end the proof we define μ by $\mu(K)=|\nu|(\varphi^{-1}(K))$ for any Borel subset K of S.

Lemma 2. *Let $X, \tilde{X}, Y, \tilde{Y}$ be Banach spaces with uniformly convex duals and let T be a linear isomorphism from $X \check{\otimes} \tilde{X}$ onto $Y \check{\otimes} \tilde{Y}$ with $\|T\| \leqq 1+\varepsilon, \|T^{-1}\| \leqq 1+\varepsilon$. Then for any $y^* \in E(Y^*)$, $\tilde{y}^* \in E(\tilde{Y}^*)$ there are $x^* \in E(X^*)$, $\tilde{x}^* \in E(\tilde{X}^*)$ such that*

$$\|T^*(y^* \otimes \tilde{y}^*) - x^* \otimes \tilde{x}^*\| \leqq \alpha(\varepsilon);$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the function depends only on the modulus of convexity of $X^*, \tilde{X}^*, Y^*, \tilde{Y}^*$.

Proof. Fix $y_0^* \in E(Y^*)$, $\tilde{y}_0^* \in E(\tilde{Y}^*)$ and let μ be a measure on $B(X^*) \times B(\tilde{X}^*)$ which is a norm preserving extension of the functional $T^*(y_0^* \otimes \tilde{y}_0^*)$ from $X \otimes \tilde{X}$ to $C(B(X^*) \times B(\tilde{X}^*))$. By Proposition 3 we can assume that μ is positive and we have

$$\|\mu\| = \mu(B(X^*) \times B(\tilde{X}^*)) = \mu(E(X^*) \times E(\tilde{X}^*))$$

and

$$1-\varepsilon \leqq \|\mu\| \leqq 1+\varepsilon.$$

The spaces Y and $\tilde{Y}$ are reflective so there are $y_0 \in B(Y)$, $\tilde{y}_0 \in B(\tilde{Y})$ such that

$$y_0^*(y_0) = 1 = \tilde{y}_0^*(\tilde{y}_0).$$

Put

$$S = \{(x^*, \tilde{x}^*) \in E(X^*) \times E(\tilde{X}^*) : \operatorname{Re}(T^{-1}(y_0 \otimes \tilde{y}_0))(x^* \otimes \tilde{x}^*) \geq 1 - \sqrt{\varepsilon}\}.$$

We have

$$\|\mu\| \leq 1+\varepsilon, \ \|T^{-1}(y_0 \otimes \tilde{y}_0)\| \leqq 1+\varepsilon$$

and

$$\int T^{-1}(y_0 \otimes \tilde{y}_0)\, d\mu = 1$$

so, by a direct calculation

$$\mu(E(X^*) \times E(\tilde{X}^*) \setminus S) \leqq 2\sqrt{\varepsilon}. \tag{4}$$

We shall show that

$$\operatorname{diam}(\{(x^* \otimes \tilde{x}^*) : (x^*, \tilde{x}^*) \in S\}) \leqq \alpha'(\varepsilon) \tag{5}$$

where $\alpha'(\varepsilon)\to 0$ as $\varepsilon\to 0$, and α' depends only on the modulus of convexity of $X^*, \tilde{X}^*, Y^*, \tilde{Y}^*$.

For this purpose let $(x_i^*, \tilde{x}_i^*)\in S$ for $i=1,2$. The spaces X and $\tilde{X}$ are reflexive so there are $x_i\in B(X)$, $\tilde{x}_i\in B(\tilde{X})$ such that $x_i^*(x_i)=1=\tilde{x}_i^*(\tilde{x}_i)$ for $i=1,2$. We have

$$\|x_i\otimes\tilde{x}_i+T^{-1}(y_0\otimes\tilde{y}_0)\|\geqq|((x_i\otimes\tilde{x}_i)+T^{-1}(y_0\otimes\tilde{y}_0))(x_i^*\otimes\tilde{x}_i^*)|\geqq 2-\sqrt{\varepsilon}.$$

hence, if $\varepsilon\leqq\frac{1}{4}$, we get

$$\|T(x_i\otimes\tilde{x}_i)+y_0\otimes\tilde{y}_0\|\geqq(2-\sqrt{\varepsilon})/(1+\varepsilon)\geqq 2-2\sqrt{\varepsilon}\quad\text{for}\quad i=1,2.$$

Let $y_i^*\otimes\tilde{y}_i^*\in E(Y^*)\otimes E(\tilde{Y}^*)$ be such that

$$\operatorname{Re}(y_i^*\otimes\tilde{y}_i^*(T(x_i\otimes\tilde{x}_i)+y_0\otimes\tilde{y}_0))\geqq 2-2\sqrt{\varepsilon}.$$

Hence

$$\operatorname{Re}y_i^*\otimes\tilde{y}_i^*(T(x_i\otimes\tilde{x}_i))\geqq 1-2\sqrt{\varepsilon},\ \operatorname{Re}y_i^*\otimes\tilde{y}_i^*(y_0\otimes\tilde{y}_0)\geqq 1-3\sqrt{\varepsilon}.$$

By Proposition 2 we get

$$\|y_0^*\otimes\tilde{y}_0^*-y_i^*\otimes\tilde{y}_i^*\|\leqq\delta_{Y^*}^*(6\sqrt{\varepsilon})+\delta_{\tilde{Y}^*}^*(6\sqrt{\varepsilon})$$

which in view of previous inequalities leads to

$$\operatorname{Re}(T(x_i\otimes\tilde{x}_i))(y_0^*\otimes\tilde{y}_0^*\geqq 1-3\sqrt{\varepsilon}-3(1+\varepsilon)[\delta_{Y^*}^*(6\sqrt{\varepsilon})+\delta_{\tilde{Y}^*}^*(6\sqrt{\varepsilon})]-\gamma(\varepsilon)\quad\text{for}\quad i=1,2$$

so

$$\|x_1\otimes\tilde{x}_1+x_2\otimes\tilde{x}_2\|\geqq 2\gamma(\varepsilon).$$

Hence there is $x^*\otimes\tilde{x}^*\in E(X^*)\otimes E(\tilde{X}^*)$ such that

$$\operatorname{Re}x_i\otimes\tilde{x}_i(x^*\otimes\tilde{x}^*)\geqq 2\gamma(\varepsilon)-1\quad\text{for both}\quad i=1\text{ and }2.$$

By Proposition 2

$$\|x_1^*\otimes\tilde{x}_1^*-x_2^*\otimes\tilde{x}_2^*\|\leqq[\delta_{X^*}^*(4-4\gamma(\varepsilon))+\delta_{\tilde{X}^*}^*(4-4\gamma(\varepsilon))]=\alpha'(\varepsilon).$$

Fix $(x_0^*,\tilde{x}_0^*)\in S$. To end the proof we observe that for any $f\in X\otimes\tilde{X}$ with $\|f\|\leqq 1$, it follows from (4) and (5) that

$$\left|f(x_0^* \otimes \tilde{x}_0^*) - T^*(y_0^* \otimes \tilde{y}_0^*)(f)\right| - \left|f(x_0^* \otimes \tilde{x}_0^*) - \int f \, d\mu\right|$$

$$\leqq 4\sqrt{\varepsilon} + \int_S \left|f - f(x_0^* \otimes \tilde{x}_0^*)\right| d\mu + \left|1 - \mu(S)\right|$$

$$\leqq 4\sqrt{\varepsilon} + \alpha'(\varepsilon)(1+\varepsilon) + 4\sqrt{\varepsilon} - \alpha(\varepsilon).$$

Lemma 3. *Let $X, \tilde{X}, Y, \tilde{Y}, T, \varepsilon, \alpha$ be as in Lemma 2. Assume $y_0^* \in E(Y^*)$, $\tilde{y}_1^*, \tilde{y}_2^*, \tilde{y}_3^* \in E(\tilde{Y}^*)$, $x_1^*, x_2^*, x_3^* \in E(X^*)$, $\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^* \in E(\tilde{X}^*)$ are such that*

$$\left\|T^*(y_0^* \otimes \tilde{y}_i^*) - x_i^* \otimes \tilde{x}_i^*\right\| \leq \alpha(\varepsilon) \quad \textit{for} \quad i = 1, 2, 3,$$

then there are numbers $\lambda_{i,j}$ for $i, j = 1, 2, 3$ of modulus one such that

$$\left\|x_i^* - \lambda_{i,j} x_j^*\right\| \leqq \beta(\varepsilon) \quad \textit{for} \quad i, j = 1, 2, 3$$

or

$$\left\|\tilde{x}_i^* - \lambda_{i,j} \tilde{x}_j^*\right\| \leqq \beta(\varepsilon) \quad \textit{for} \quad i, j = 1, 2, 3$$

where

$$\beta(\varepsilon) = 24\sqrt{\alpha(\varepsilon)}.$$

Proof. Since $\tilde{Y}^*$ is uniformly convex, by Lemma 2, there are $x_4^* \in E(X^*)$ and $\tilde{x}_4^* \in E(\tilde{X}^*)$ such that

$$\left\|T(y_0^* \otimes (\tilde{y}_1^* + \tilde{y}_2^*)) - k x_4^* \otimes \tilde{x}_4^*\right\| \leq k\alpha(\varepsilon),$$

where

$$k = \left\|\tilde{y}_1^* + \tilde{y}_2^*\right\| \leqq 2.$$

Hence

$$\left\|x_1^* \otimes \tilde{x}_1^* + x_2^* \otimes \tilde{x}_2^* - k x_4^* \otimes \tilde{x}_4^*\right\| \leqq (k\alpha(\varepsilon) + 2\alpha(\varepsilon)) \leqq 4\alpha(\varepsilon),$$

and by Lemma 1 we have

$$\left\|x_1^* - \lambda x_2^*\right\| \leq 12\sqrt{\alpha(\varepsilon)} \quad \text{or} \quad \left\|\tilde{x}_1^* - \lambda \tilde{x}_2^*\right\| \leq 12\sqrt{\alpha(\varepsilon)}$$

for some λ of modulus one.

Considering successively the pairs of indices $(1, 2), (2, 3)$ and $(1, 3)$ we obtain the assertion of the lemma.

From Lemmas 2 and 3 we deduce that for any $y_0^* \in E(Y^*)$ we have exactly two possibilities:

(a) there is an $x_0^* \in E(X^*)$ and a function $\varphi: E(\tilde{Y}^*) \to E(\tilde{X}^*)$ such that

$$\left\| T^*(y_0^* \otimes \tilde{y}^*) - x_0^* \otimes \varphi(\tilde{y}^*) \right\| \leqq \alpha(\varepsilon) + \beta(\varepsilon) = \gamma(\varepsilon) \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*)$$

or

(b) there is an $\tilde{x}_0^* \in E(\tilde{X}^*)$ and a function $\psi: E(\tilde{Y}^*) \to E(X^*)$ such that

$$\left\| T^*(y_0^* \otimes \tilde{y}^*) - \psi(\tilde{y}^*) \otimes \tilde{x}_0^* \right\| \leqq \gamma(\varepsilon) \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*). \tag{7}$$

By the same arugments applied to the map T^{-1} in place of T, we get by symmetry (replacing the space X by $\tilde{X}$ and Y by $\tilde{Y}$) and by Lemma 3 that

$$\begin{aligned} &\sup\{\inf\{\|\varphi(\tilde{y}^*) - \tilde{x}^*\| : \tilde{y}^* \in E(\tilde{Y}^*)\} : \tilde{x}^* \in E(\tilde{X}^*)\} \leqq \gamma(\varepsilon) \\ &\sup\{\inf\{\|\psi(\tilde{y}^*) - x^*\| : \tilde{y}^* \in E(\tilde{Y}^*)\} : x^* \in E(X^*)\} \leqq \gamma(\varepsilon). \end{aligned} \tag{8}$$

For any $y_0^* \in E(Y^*)$ we define, depending on which of the above possibilities takes place, a function $\Phi: \tilde{X} \to \tilde{Y}$ or $\Psi: X \to \tilde{Y}$ as follows:

(a) fix $x_0 \in B(X)$ such that $x_0^*(x_0) = 1$ and define Φ by $\tilde{y}^*(\Phi(x)) = y_0^* \otimes \tilde{y}^*(T(x_0 \otimes \tilde{x}))$ for $\tilde{y}^* \in \tilde{Y}^*$, $\tilde{x} \in \tilde{X}$;

(b) fix $\tilde{x}_0 \in B(\tilde{X})$ such that $\tilde{x}_0^*(x_0) = 1$ and define Ψ by $\tilde{y}^*(\Psi(x)) = y_0^* \otimes \tilde{y}^*(T(x \otimes \tilde{x}_0))$ for $\tilde{y}^* \in \tilde{Y}^*, x \in X$.

The above definitions may depend on the choice of $x_0(\tilde{x}_0)$ and we assume that we have fixed some $\Phi(\Psi)$ as above, for any $y_0^* \in E(Y^*)$.

We have $\|\Phi\| \leq 1 + \varepsilon$, $\|\Psi\| \leqq 1 + \varepsilon$, and

$$\left|\tilde{y}^*(\Phi(\tilde{x})) - \varphi(\tilde{y}^*)(\tilde{x})\right| \leqq \gamma(\varepsilon)\|\tilde{x}\| \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*), \tilde{x} \in \tilde{X},$$

$$\left|\tilde{y}^*(\Psi(x)) - \psi(\tilde{y}^*)(x)\right| \leq \gamma(\varepsilon)\|x\| \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*), x \in X,$$

so from (8) we infer that Φ and Ψ are one to one, onto isomorphisms with $\|\Phi^{-1}\| \leqq 1 + \gamma(\varepsilon)$, $\|\Psi^{-1}\| \leqq 1 + \gamma(\varepsilon)$ and

$$\|\Phi^*(\tilde{y}^*) - \varphi(\tilde{y}^*)\| \leq \gamma(\varepsilon) \quad \text{and} \quad \|\Psi^*(\tilde{y}^*) - \psi(\tilde{y}^*)\| \leqq \gamma(\varepsilon) \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*).$$

To end the proof we show that for all $y_0^* \in E(Y^*)$ one of the two possibilities (a) and (b) takes place and the map assigning to $y_0^* \in E(Y^*)$ a $\Phi \in L(\tilde{X}, \tilde{Y})$ ($\Psi \in L(X, \tilde{Y})$) is "ε-almost" constant.

For this end, assume that $y_1^*, y_2^* \in E(Y^*)$, $x_1^* \in E(X^*), \tilde{x}_2^* \in E(\tilde{X}^*)$, $\Phi_1 \in L(\tilde{X}, \tilde{Y})$, $\Psi_2 \in L(X, \tilde{Y})$ are such that, for all $\tilde{y}^* \in E(\tilde{Y}^*)$,

$$\left\| T^*(y_1^* \otimes \tilde{y}^*) - x_1^* \otimes \Phi_1^*(\tilde{y}^*) \right\| \leqq 2\gamma(\varepsilon) \tag{9}$$

and

$$\|T^*(y_2^* \otimes \tilde{y}^*) - \Psi_2^*(\tilde{y}^*) \otimes \tilde{x}_2^*\| \leq 2\gamma(\varepsilon). \tag{10}$$

Since $\|(\Phi_1^*)^{-1}\| \leq 1+\gamma(\varepsilon), \|(\Psi_2^*)^{-1}\| \leq 1+\gamma(\varepsilon)$ there are $\tilde{y}_1^*, \tilde{y}_2^* \in E(\tilde{Y}^*)$ such that $\|\Phi_1^*(\tilde{y}_1^*) - \tilde{x}_2^*\| \leq \gamma(\varepsilon), \|\Psi_2^*(\tilde{y}_2^*) - x_1^*\| \leq \gamma(\varepsilon)$; so we get

$$\|x_1^* \otimes \tilde{x}_2^* - T^*(y_i^* \otimes \tilde{y}_i^*)\| \leq (1+\varepsilon)\gamma(\varepsilon) + 2\gamma(\varepsilon) \quad \text{for} \quad i = 1, 2$$

and hence

$$\|y_1^* \otimes \tilde{y}_1^* - y_2^* \otimes \tilde{y}_2^*\| \leq 2(1+\varepsilon)(3+\varepsilon)\gamma(\varepsilon) \leq 7\gamma(\varepsilon)$$

leading to the inequality

$$\|y_1^* - y_2^*\| \leq 7\gamma(\varepsilon)$$

which contradicts (9) and (10).

Thus without loss of generality we can assume that it is the first possibility that always holds.

Fix $y_0^* \in E(Y^*)$ and $\tilde{y}_0^* \in E(\tilde{Y}^*)$. There is an $x_0^* \in E(X^*)$ and $\Phi_0 \in L(\tilde{X}, \tilde{Y})$ with $\|\Phi_0\| \|\Phi_0^{-1}\| \leq (1+\varepsilon)(1+\gamma(\varepsilon))$ such that

$$\|T^*(y_0^* \otimes \tilde{y}^*) - x_0^* \otimes \Phi_0^*(\tilde{y}^*)\| \leq 2\gamma(\varepsilon), \quad \text{for all} \quad \tilde{y}^* \in E(\tilde{Y}^*). \tag{11}$$

By symmetry there is an $\tilde{x}_0^* \in E(\tilde{X}^*)$ and $\Psi_0 \in L(X, Y)$ with $\|\Psi_0\| \|\Psi_0^{-1}\| \leq (1+\varepsilon)(1+\gamma(\varepsilon))$ such that

$$\|T^*(y^* \otimes \tilde{y}_0^*) - \Psi_0^*(y^*) \otimes \tilde{x}_0^*\| \leq 2\gamma(\varepsilon) \quad \text{for all} \quad y^* \in E(Y^*). \tag{12}$$

Moreover, replacing $2\gamma(\varepsilon)$ in (11) and (12) by $4\gamma(\varepsilon)$ we can assume $\tilde{x}_0^* = \Phi_0^*(\tilde{y}_0^*)$ and $x_0^* = \Psi_0^*(y_0^*)$.

Let us compose T with $\Phi^{-1} \otimes \Psi^{-1}$. To complete the proof it is sufficient to show the following lemma:

Lemma 4. *Let $X, \tilde{X}$ be Banach spaces with uniformly convex duals, then there is an* $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ *the following implication holds:*

if T is a linear isomorphism from $X \otimes \tilde{X}$ onto itself with $\|T\| \|T^{-1}\| \leq 1+\varepsilon$ and if there exist $x_0^* \in E(X^*)$ and $\tilde{x}_0^* \in E(\tilde{X}^*)$ such that

$$T^*(x_0^* \otimes \tilde{x}^*) = x_0^* \otimes \tilde{x}^* \quad \textit{for all} \quad \tilde{x}^* \in \tilde{X}^*$$

and

$$T^*(x^* \otimes \tilde{x}_0^*) = x^* \otimes \tilde{x}_0^* \quad \textit{for all} \quad x^* \in X^*$$

then $\|T - \mathrm{Id}\| \leq 2\gamma(\varepsilon)$.

Proof. Let $x_1^* \in E(X^*)$, $\tilde{x}_1^* \in E(\tilde{X}^*)$. It follows from the assumptions and our previous considerations that there are isomorphisms $\Phi \in L(\tilde{X})$ and $\Psi \in L(X)$ such that

$$\left\|T^*(x_1^* \otimes \tilde{x}) - x_1^* \otimes \Phi^*(\tilde{x}^*)\right\| \leqq 2\gamma(\varepsilon) \quad \text{for all} \quad \tilde{x}^* \in E(\tilde{X}^*)$$

and

$$\left\|T^*(x^* \otimes \tilde{x}_1^*) - \Psi^*(x^*) \otimes \tilde{x}_1^*\right\| \leqq 2\gamma(\varepsilon) \quad \text{for all} \quad x^* \in E(X^*).$$

Substituting $\tilde{x}^* = \tilde{x}_1^*$ and $x^* = x_1^*$ we get

$$\left\|T^*(x_1^* \otimes \tilde{x}_1^*) - x_1^* \otimes \Phi^*(\tilde{x}_1^*)\right\| \leqq 2\gamma(\varepsilon)$$

and

$$\left\|T^*(x_1^* \otimes \tilde{x}_1^*) - \Psi^*(x_1^*) \otimes \tilde{x}_1^*\right\| \leqq 2\gamma(\varepsilon).$$

Hence

$$\left\|\Phi^*(\tilde{x}_1^*) - \tilde{x}_1^*\right\| \leqq 2\gamma(\varepsilon) \quad \text{and} \quad \left\|x_1^* - \Psi^*(x_1^*)\right\| \leqq 2\gamma(\varepsilon),$$

so $\left\|T^*(x_1^* \otimes \tilde{x}_1^*) - x_1^* \otimes \tilde{x}_1^*\right\| \leqq 2\gamma(\varepsilon)$ as required.

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