Perturbations of function algebras

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INTRODUCTION

In this note we discuss the small metric and the small algebraic perturbations of the function algebras. It is not our aim to present a self contained theory with all details and proofs. This is intended as an informal introduction to a theory that is starting to develop. The emphasis on general themes has perhaps been at some cost of precision, we usually only sketch or just omit the proofs. To minimize technical issues we assume throughout that all algebras considered are function algebras which have their Choquet boundary equal to their Shilov boundary. This note is based on the author's monograph [1], where one can find all the technical details, more general farms of the theorems and references, as well as a list of open problems. We also present here some very recent, unpublished results ([2-3]).

In the paper by A we always denote a function algebra, that is a complex Banach algebra with a unit such that $\|f^2\| = \|f\|^2$ for all $f \in A$. By $\mathfrak{M}(A)$, ∂A, ChA we denote the maximal ideal space, the Shilov boundary and the Choquet boundary of A, respectively. We usually consider a function algebra A, via the Gelfand transforms, as a subalgebra of $C(\mathfrak{M}(A))$ or $C(\partial A)$.

A fundamental question in Banach algebra theory is the relation between the algebraic and the metric structures of a Banach algebra A and the structure of $\mathfrak{M}(A)$. The focus of perturbation theory is on understanding the relationship between small changes in these structures as well as on characterizing the stable properties of Banach algebras, that is the properties which are invariant under small perturbations.

There are two basic concepts of a small perturbation of a function algebra: algebraic and metric.

DEFINITION 1. By an ϵ-algebraic perturbation of A we mean any new multiplication $\times$ defined on the same Banach space A such that

$$\|f\times g - f\cdot g\| \le \epsilon\|f\|\|g\| \text{ for all } f,g \text{ in } A. \tag{1}$$

Notice please that we do not assume here that the second multiplication is commutative, continuous nor has a unit, it is by the definition just a bilinear, associative map such that (1) is satisfied. But we will prove later on that it has all these additional properties automatically.

DEFINITION 2. By an ϵ-metric perturbation of A we mean any Banach algebra B, with a unit, such that there is a linear bijection $T: A \to B$ such that

$$\|T\|\|T^{-1}\| \le 1+\epsilon \tag{2}$$

We do not assume here, of course, that T preserves the algebraic structure, it preserves only the linear

M. Hazewinkel and M. Gerstenhaber (eds.),
Deformation Theory of Algebras and Structures and Applications, 557–563.

structure and "almost preserves" the metric structure. We do not also assume that B is a function algebra, but we will show that it is a function algebra automatically.

In addition to the above two definitions there are several other possible ways to make precise the notion of a small change in a uniform algebra. For example suppose that we have two algebras A and B defined on the same compact set X and that there is a linear map T from A onto B such that $\|Id - T\|$ is small. If A and B are close in this sense then clearly B is an $\frac{2\epsilon}{1-\epsilon}$-metric perturbation of B. By setting

$$f \times g = T^{-1}(Tf \cdot Tg) \text{ for all } f, g \text{ in } A \tag{3}$$

one can also check that $\times$ is a 5ϵ-algebraic perturbation of A.

One of the basic general result is that all the above definitions are, in some sense, equivalent. First let us consider some classical results and examples which bear on this equation and will be of interest to us.

THEOREM. (Nagasawa, 1959). Let A and B be function algebras and let $T: A \to B$ be a linear and surjective isometry between them. Then

a) there is a linear isometry T from A onto B such that $T1_A = 1_B$, and

b) if $T1_A = 1_B$ then T is an algebra isomorphism.

The theorem suggests two questions concerning our situation:

Q1. Let B be an ϵ-metric perturbation of A, with ϵ sufficiently small. Does there exist a linear isomorphism $T: A \to B$ with $T1_A = 1_B$ and such that

$$\|T\|\|T^{-1}\| \leq 1 + \epsilon?$$

Q2. Let T be as in Q1. Is then T an algebra isomorphism?

The answer to the first question is positive.

PROPOSITION 1. For any ϵ-metric perturbation B of A, with $\epsilon < 0.1$ there is a linear isomorphism T from A onto B such that $T1_A = 1_B$ and $\|T\|\|T^{-1}\| \leq 10\epsilon$.

However the following example shows that the answer to the second question is negative.

EXAMPLE. For any $\epsilon < 0$ put $P_\epsilon = \{z \in \mathbb{C}: \frac{1}{2} < |z| < 1 + \epsilon\}$, $A_\epsilon = \{f \in C(\bar{P}_\epsilon): f|_{P_\epsilon} \text{ is analytic}\}$ and let us define $T_\epsilon: A_0 \to A$ by

$$A_0 \ni f = \sum_{n=-\infty}^{+\infty} a_n z^n \xrightarrow{T_\epsilon} \sum_{n=-\infty}^{-1} a_n z^n + \sum_{n=0}^{\infty} a_n (1+\epsilon)^{-n} z^n \in A.$$

It can be check that $T_\epsilon 1 = 1$ and that

$$\|T_\epsilon\|$$

It can be check that $T_\epsilon 1 = 1$ and that

$$\|T_\epsilon\| \, \|T_\epsilon^{-1}\| \to 1 \text{ as } \epsilon \to 0.$$

On the other hand any algebra isomorphism between A_0 and A_ϵ would be defined by an analytic homeomorphism between P_ϵ and P_0 but it is well known that such a homeomorphism does not exist if $\epsilon > 0$.

So we have an example of a non-trivial small metric perturbation of a function algebra. In this example the algebras A_0 and A_ϵ are not isomorphic but they look very similar and are in some sense *almost isomorphic*. We will show that this is a general phenomenon.

In all the paper we will merely assume that ϵ in the definition of a small perturbation is close to zero, so by Proposition 1 we can always assume that an ϵ-metric perturbation is defined by a map T which preserves the unit. It can also be easily proved that any ϵ-algebraic perturbation $\times$ of ($\epsilon < 1$) has a unit, and that without loss of generality we can assume that $\times$ has the same unit as the original multiplication of A.

1. General Results

One of the fundamental general results is that the definitions of small perturbations we have are equivalent.

THEOREM 1. Let A be a function algebra with $\partial A = ChA$ and let $\times$ be a new multiplication defined on the same Banach space A such that the units of A and $(A, \times)$ coincide. Then the following five conditions are equivalent in the sense that all ϵ_i, $i = 1, \cdots, 5$ tend to zero simultaneously:

(i) $\|f\times g - f\cdot g\| \leq \epsilon_1\|f\|\|g\|$ for all f, g in A;

(ii) $|\|f\times g\| - \|f\cdot g\|| \leq \epsilon_2\|f\|\|g\|$ for all f, g in A;

(iii) $\|f\times g\| \leq (1+\epsilon_3)\|f\|\|g\|$ for all f, g in A;

(iv) there is a Banach algebra B and a linear isomorphism $T\colon A \to B$ with $T1_A = 1_B$ and $\|T\|\|T^{-1}\| \leq 1+\epsilon_4$ and such that multiplication $\times$ is defined by

$$f\times g = T^{-1}(Tf\cdot Tg) \quad \text{for all } f, g \text{ in } A; \tag{*}$$

(v) there is a function algebra B contained in $C(\partial A)$ and a linear map $T\colon A \to B$ such that the multiplication $\times$ is given by (*) and such that

$$|f(s) - Tf(s)| \leq \epsilon_5\|f\| \quad \text{for all } s \in \partial A, f \in A.$$

Notice please that we do not claim that conditions (i) - (v) are equivalent in the sense that $\epsilon_1 = \cdots = \epsilon_5$, we mean only that there is a function $\alpha\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $\lim_{t\to 0^+}\alpha(t) = 0$ and such that $\epsilon_j \leq \alpha(\epsilon_i)$ for $i, j = 1, \cdots, 5$.

A moment of reflection shows that the following implications are trivial

$$(i) \to (ii) \to (iii) \text{ and } (v) \to (iv) \to (iii).$$

The proof of the implications (iii) $\to$ (i) $\to$ (v) is the hard part of the theorem. In fact the condition (iii) seems to be much weaker than (i) and the implication (iii) $\to$ (i) is a little surprising. For $\epsilon = 0$ it means that there is only one multiplication $\times$ on A (the original one) such that $(A, \times)$ is a Banach algebra with the same unit, this means that the algebraic structure of a function algebra A is totally described by its norm!

To give a feeling of the methods in our field we present a proof of (ii) $\to$ (i) and sketch briefly a proof of (i) $\to$ (v).

(ii) $\to$ (i). Replace f in (ii) by the element $\exp(\lambda f)$ where λ is a complex number and exp is the exponential function in the Banach algebra A, and replace g by $\exp(-\lambda f)\cdot g$. We get

$$\begin{aligned}|\|g\| - \|\exp(-\lambda f)\cdot g)\|| &\leq \|g\|\cdot\|\exp(-\lambda f)\|\cdot\|\exp\lambda f\| \leq \\ &\leq \epsilon\|g\|\exp(2|\lambda|\cdot\|f\|).\end{aligned}$$

This gives

$$\begin{aligned}\|g\|(1+\epsilon\exp(2|\lambda|\cdot\|f\|)) &\geq \|\exp(\lambda f)\times(\exp(-\lambda f)\cdot g)\| = \\ &= \|(1+\lambda f/1!+\lambda^2f^2/2!+\cdots)\times(g-\lambda fg/1!+\lambda^2f^2g/2!-\cdots)\| = \\ &= \|g-(f\times g-fg)\lambda/1!+(\cdots)\lambda^2/2!+\cdots\|.\end{aligned}$$

Now let F be any linear functional on the Banach space A of norm one. We have

$$\begin{aligned}|F(g) - \lambda F(f\times g - fg) + \lambda^2F(\cdots)/2! + \cdots| &\leq \\ &\leq \|g\|\cdot(\epsilon\exp(2|\lambda|\cdot\|f\|)+1).\end{aligned}$$

This shows that the modulus of the entire function

$$\phi(\lambda) = F(g) - F(f\times g - fg)\lambda/1! + F(\cdots)\lambda^2/2! + \cdots$$

on the unit disc $D = \{\lambda\in\mathbf{C}\colon |\lambda| < 1\}$ is not greater than $\|g\|\cdot(\epsilon\exp(2\|f\|)+1)$, hence the first derivative at the point zero of this function has the modulus not greater than this constant too. Because F is an arbitrary functional of norm one, it follows that

$$\|f \times g - fg\| \leq \|g\| \cdot (\epsilon\exp(2\|f\|) + 1) \quad \text{for any } f, g \text{ in } A. \tag{4}$$

Fix elements f and g in A both of norm one, and let $\alpha = -\frac{1}{2}\log\epsilon > 0$. From (4) we get

$$\|f \times g - fg\| = \|(\alpha f) \times (g / \alpha) - (\alpha f) \cdot (g / \alpha)\| \leq$$
$$= \alpha^{-1}(\epsilon\exp 2\alpha + 1) = -4 / \log\epsilon$$

for $0 < \epsilon < 1$, and since $-4 / \log\epsilon \to 0$ as $\epsilon \to 0$ we are done.

(i) $\to$ (v). By (i) for any f in A we have

$$(1-\epsilon)\|f\|^2 \leq \|f \times f\| \leq (1+\epsilon)\|f\|^2.$$

So by induction we get

$$(1-\epsilon)^{2^r-1}\|f\|^{2^r} \leq \|f \times \underset{2^r\times}{\cdots} \times f\| \leq (1+\epsilon)^{2^r-1}\|f\|^{2^r}$$

and hence we can estimate the spectral norm of f in the algebra $(A, \times)$

$$(1-\epsilon)\|f\| \leq \text{spectral radius } (f) \leq (1+\epsilon)\|f\|.$$

Hence by the theorem of Hirschfeld and Zelazko the multiplication $\times$ is commutative (if $\epsilon < 1$) and the spectral radius $\rho_x(\cdot)$ is an equivalent, uniform norm on A. Now we denote by T the identity map between the Banach space A and the Banach space $(A, \rho_x(\cdot))$ and by our estimation of the spectral radius $\rho_x(\cdot)$ we get

$$\|T\| \leq 1+\epsilon \text{ and } \|T^{-1}\| \leq (1-\epsilon)^{-1}.$$

So we have proved that (iv) holds for some function algebra $B = (A, x, \rho_x(\cdot))$ and to end the proof of (v) we have to show that there is a homeomorphism ψ from ∂B onto ∂A such that

$$|f \circ \psi(s) - Tf(s)| \leq \epsilon'\|f\| \text{ for all } f \in A,\ s \in \partial B. \tag{5}$$

The idea is the following. We fix a point s_0 in $\partial A = ChA$ and a sequence (net) $(f_n)_{n=1}^{\infty} \subset A$ which peaks at the point s_0, this means such that

$$\|f_n\| = 1 = f_n(s_0) \quad \text{for all } n \text{ in } \mathbb{N} \text{ and}$$
$$f_n \to 0 \text{ uniformly on compactsubsets of } \partial A \setminus \{s_0\}.$$

By our estimation of $\rho_x(\cdot)$ there is a sequence $(F_n)_{n=1}^{\infty}$ in $\mathfrak{M}(B)$ such that

$$|F_n(f_n)| \geq 1-\epsilon \quad \text{for all } n \text{ in } \mathbb{N}.$$

Then we prove that any such sequence $(F_n)_{n=1}^{\infty}$ converges to a point F_0 in $ChB \subset \mathfrak{M}(B)$ and we conclude the proof by showing that the map $ChB \ni F_0 \to s_0 \in \partial A$ is a well defined homeomorphism and that (5) is satisfied.

2. Stability

Let $\mathscr{A}$ be a class of all function algebras modulo the equivalence relation: A is equivalent to B if the Banach-Mazur distance between A and B is equal to one (it can be shown that this relation is not trivial). We have two metrices on $\mathscr{A}$ which are the natural measures of closedness of two function algebras A and B:

$D(A,B) = \inf\{\epsilon: B \text{ is algebraically isomorphic to an algebraic } \epsilon\text{-perturbation of } A\}$,

$d(A,B) = \inf\{\log\|T\|\|T^{-1}\|: T \text{ is a linear isomorphism between Banach spaces of } A \text{ and } B\}$.

It can be proven (compare Theorem 1 (i) $\leftrightarrow$ (v)) that the topologies defined on $\mathscr{A}$ by D and d coincide. We can get even slightly more.

PROPOSITION 2. There are positive constants ϵ_0 and c such that for any function algebras A and B we have

$$\frac{1}{2}d(A,B) \leq D(A,B) \leq cd(A,B)$$

whenever $\min(d(A,B), D(A,B)) \leq \epsilon_0$.

In the previous section we studied the global properties of the complete metric space [1] $(\mathcal{A}, d)$, now we investigate the local properties of this space. We are mostly interested in properties which are called stable.

DEFINITION Let $\mathcal{P}$ be a property which is defined for every function algebra. We call $\mathcal{P}$ stable (or rigid) if for any function algebra A which has $\mathcal{P}$ there is a positive number ϵ such that for any ϵ-metric perturbation B of A the algebra B has the property $\mathcal{P}$.

Notice that a property $\mathcal{P}$ is stable if and only if the subset of $\mathcal{A}$ consisting of all function algebras which have $\mathcal{P}$ is an open subset of $\mathcal{A}$. By Theorem 1(v) for any compact Hausdorff space S the properties:

S1. "∂A is homeomorphic to S", and

S2. "$A = C(S)$ (in the category of algebras"

are stable. To prove that $S2$ is stable we have only to notice that for any $\epsilon<1$ the following implication holds:

if there is a linear isomorphism T from $C(S)$ onto a subalgebra B of $C(S)$ such that $\|Id - T\| \leq \epsilon$ then $B = C(S)$.

The stability of the property "$A = C(S)$" means that the algebra $C(S)$ is an isolated point of $\mathcal{A}$, that is any sufficiently small perturbation of $C(S)$ produces a new algebra which is isomorphic, as an algebra, to the original one. A Banach algebra with this property is called stable. It is interesting to mention that we know very few examples of stable function algebras. Namely $C(S)$, $H^\infty(D) :=$ the algebra of all bounded analytic functions on an open unit disc D([2]), $A(D) := H^\infty(D) \cap C(\bar{D})$ and the trivial, finite combinations of these three. R. Rochberg [4] proved that $A(D)$ is the only stable algebra among the algebras of the form $A(R)$, where R is a connected, finite bordered one dimensional Riemann surface. He also proved that any small perturbation of the algebra $A(R)$ is of the form $A(R')$ and that this perturbation is defined by a quasiconformal homeomorphism between R and R'. The proofs concerning small perturbations of the algebras $A(R)$ or $H^\infty(D)$ seem to be technical and too long to present here. The basic idea is to extend the algebra B, given by Theorem 1(v), from $C(\partial A)$ to a subalgebra $\tilde{B}$ of $C(\mathfrak{M}(A)) = C(R)$ and then to prove that functions from $\tilde{B}$ are "almost analytic" on R.

To present an idea of our method we give a sketch of the proofs of the stability of the following two important properties:

S3. "A is a Dirichlet algebra";

"A is antisymmetric".

Let us recall that A is called Dirichlet if $\overline{ReA} = C_R(\partial A)$, where $ReA = \{\mathrm{Re}f : f \in A \in C(\partial A)\}$ and A is called antisymmetric if there is no non constant real valued function in $A \subset C(\partial A)$.

To prove that the property S3 is stable we put

$$A^{-1} = \{f \in A : f^{-1} \in A\},\ e^A = \{f \in A : \exists g \in A \ \mathrm{s.t.}\ f = \exp g\}$$

and we first prove the following lemma.

LEMMA. Let B be a function algebra contained in $C(\partial A)$ and let T be a linear map from A onto B such that $T1 = 1$ and $\|T - Id\| < 0.1$ then we have

a) if $f \in A^{-1}$ and $\|f\| \cdot \|f^{-1}\| < 10$ then $Tf \in B^{-1}$;

b) if $f \in e^A$ and $\|f\| \cdot \|f^{-1}\| < 10$ then $Tf \in e^B$.

PROOF. By our assumption $\|T - Id\| < 0.1$ we have

$$\|T(f) \cdot T(f^{-1}) - 1\| \leq 10\|f\| \cdot \|f^{-1}\| < 1$$

and hence $T(f) \cdot T(f^{-1}) \in B^{-1}$. To prove b) recall that e^B is the connected component of B^{-1} which contains

1. Strictly speaking $\mathcal{A}$ is not a metric space since it is a class but not a set. On the other hand it becomes a set on any subclass $\mathcal{A}_0$ of $\mathcal{A}$ such that $d(A,B) < \infty$ for A, B in $\mathcal{A}_0$. Hence, from our point of view, there is no danger of ambiguity in treating $\mathcal{A}$ as a metric space.

1. Assume $f = e^h \in e^A$ and $\|f\|\|f^{-1}\| \leqslant 10$. For $0 \leqslant t \leqslant 1$ we have

$$\|e^{th}\|\|e^{-th}\| \leqslant 10.$$

Hence, by a) we have $T(e^{th}) \in B^{-1}$ and $[0,1] \ni t \to T(e^{th})$ is a continuous function so $T(e^h) \in e^B$.

Now to end the proof of the stability of the property "A is Dirichlet" it is sufficient, by the above lemma and Theorem 1(v), to prove that:

If B is a closed subalgebra of $C(\partial A)$ and there is a linear map $T: A \to B$ such that

$$T1 = 1 \text{ and } \|Id - T\| \leqslant 0.05 \tag{6}$$

then for any $\mathrm{Re} f \in Re A$ with $\|\mathrm{Re} f\| = 1$ there is an $\mathrm{Re} g \in Re B$ such that $\|\mathrm{Re} f - \mathrm{Re} g\| \leqslant \frac{1}{2}$.

To this end fix $f \in A$ with $\|\mathrm{Re} f\| = 1$. We have

$$\|e^{\pm f}\| = \sup\{e^{\pm \mathrm{Re} f(s)}: s \in \partial A\} \leqslant e$$

so

$$\|e^f\|\|e^{-f}\| \leqslant e^2 < 10.$$

By our Lemma there is a $g \in B$ such that $T(e^f) = e^g$. From (6) we get

$$0.05e \geqslant \|T(e^f) - e^f\| = \|e^g - e^f\| = \|e^f(1 - e^{g-f})\| \geqslant \frac{1}{e}\|1 - e^{g-f}\|.$$

Hence

$$\|e^{g-f}\| \leqslant 1 + -0.05e^2 \text{ and } \|e^{f-g}\| \leqslant (1 - 0.05e^2)^{-1}$$

and this gives

$$\|\mathrm{Re} f - \mathrm{Re} g\| \leqslant \max\{\ln(1 + 0.05e^2), -\ln(1 - 0.05e^2)\} < \frac{1}{2}.$$

The proof of the stability of the property "A is antisymmetric" is much harder so we only sketch it very briefly. The proof can be derived into three steps.

Step 1. We generalize the so-called Bishop $\frac{3}{4} - \frac{1}{4}$ criterion and prove that a closed subset K of ∂A is an "almost peak set" if and only if it is a peak set. We mean:

If for any open neighborhood U of K there is an f in A such that

$$\|f\| \leqslant 2, \ |f(s) - 1| \leqslant \frac{1}{3} \text{ for } s \in K \text{ and}$$

$$|f(s)| \leqslant \frac{1}{3} \text{ for } s \in \partial A \setminus U$$

then K is an intersection of peak sets of A.

Step 2. We assume that $B \subset C(\partial A)$ is a small perturbation of A. Then we prove that $K \subset \partial A$ is a peak set for A if and only if it is a peak set for B.

We assume that A is antisymmetric and that there is a non constant, real-valued function g in B. Then we notice that for any compact subset X of R the set $g^{-1}(X)$ is a peak set for B, so it is also a peak set for A. On the other hand, to get a contradiction, we prove that an antisymmetric algebra A can not have so many peak sets.

We conclude this section by the Decomposition Theorem which restricts the investigation of these small perturbations of function algebras to the investigation of the perturbations of the antisymmetric algebras. We first recall the well-known Shilov-Bishop theorem.

Theorem. Let A be a function algebra. Then ∂A is a sum of a family of disjoint, maximal sets of antisymmetry $(S_i)_{i\in I}$ of A and

$$f\in A \text{ iff } f\in C(\partial A) \text{ and } f\,|_{S_i}\in A\,|_{S_i} \text{ for all } i\in I.$$

Decomposition Theorem. Let A,B be separable function algebras defined on a compact set $S=\partial A=\partial B$ and assume that $\|Id-T\|\leq\epsilon\leq\epsilon_0$ (an absolute constant). Then the peak sets and the maximal sets of antisymmetry for A and B coincide. Moreover for any maximal set of antisymmetry $S_0\subset S$ there is a linear map $T_0\colon A\,|_{S_0}\to B\,|_{S_0}$ such that $T_0 1=1$ and $\|T_0-Id\|\leq 10\epsilon$.

3. Approximately Multiplicative Functionals

Let x be an ϵ-algebraic perturbation of a function algebra A. Then for any F in $\mathfrak{M}(A)$ we have

$$|F(f\times g)-F(f)\cdot F(g)|\leq\epsilon\|f\|\|g\| \text{ for all } f,g \text{ in } A.$$

One of our main methods in the previous sections was to show that for any such F there is a $G\in\mathfrak{M}((A,x))$ with $\|F-G\|\leq\epsilon'$.

This suggests the following problem, which seems to be of its own interest.

Problem. Let F be a linear functional defined on a Banach algebra A which is approximately multiplicative in the sense that the bilinear functional $\check{F}(f,g)=F(fg)-F(f)F(g)$ is small in norm. Is then F a sum of a multiplicative functional and a small functional? And, when it fails, whether any of the properties of multiplicative linear functionals carry over to approximately multiplicative ones.

An example of this is the following result.

Proposition 3. Let A be a Banach algebra and let F be a linear functional on A such that $\|\check{F}\|\leq\epsilon<\infty$. Then F is continuous and $\|F\|\leq 1+\epsilon$.

B.E. Johnson is his recent paper [3] concerned with algebras in which every approximately multiplicative functional is near a multiplicately one (AMNM algebras). He proved that various Banach algebras are AMNM, but that it is not so in general for some radical Banach algebras. For function algebras situation is not clear. Two most typical function algebras, namely $C(S)$, S-compact and $A(D)$ - the disc algebra, are AMNM ([3]). The first result is easy but the second one seems to be quite hard. It has been also proven that any finite injective tensor product and a finite sum of AMNM function algebras is AMNM. But it is an open problem whether all function algebras are AMNM, for example it is not known whether $H^\infty(D)$ is AMNM.

4. References

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