Perturbations of Banach Algebras and almost multiplicative functionals

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Definitions

Definition

A Banach algebra $\mathcal{B}$ is a **metric δ-deformation** of $\mathcal{A}$ if there is $T : \mathcal{A} \to \mathcal{B}$ such that $\|T\| \left\|T^{-1}\right\| \leq 1 + \delta$ (and $Te = e$).

Definition

A new multiplication $\times$ defined on the same Banach space $\mathcal{A}$ is an **algebraic δ-deformation** of $(\mathcal{A}, \cdot)$ if $\|\times - \cdot\| \leq \delta$; that is, if

$$\|a \cdot b - a \times b\| \leq \delta \|a\| \|b\|, \qquad \text{for } a, b \in \mathcal{A}$$

(and $e_\times = e$).

Examples

Example (trivial)

Put $T = S + \Delta$ where $S : \mathcal{A} \to \mathcal{B}$ is an isometric isomorphism and $\Delta : \mathcal{A} \to \mathcal{B}$ has small norm and

$$a \times b \overset{df}{=} T^{-1}\left(Ta \cdot Tb\right).$$

Example

Put $P_\varepsilon = \{z : 1 \le |z| \le 2 + \varepsilon\}$, and define $T_\varepsilon : A(P_0) \to A(P_\varepsilon)$ by

$$T_\varepsilon\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{0} a_n z^n + \sum_{n=1}^{\infty} a_n \left(\frac{2}{2+\varepsilon}\right)^n z^n.$$

We have $\|T_\varepsilon\| \left\|T_\varepsilon^{-1}\right\| \to 1$, as $\varepsilon \to 0$.

Problems

- $\mathcal{A} \approx \mathcal{B} \overset{?}{\Rightarrow} \mathcal{A} = \mathcal{B}$ ($\mathcal{A}$ is *stable*).
- Describe all small deformations of $\mathcal{A}$. Is there a continuous (analytic) structure on that family of small deformations?
- $\mathcal{A} \approx \mathcal{B} \overset{?}{\Rightarrow} \mathcal{A}$ and $\mathcal{B}$ share the same properties (e.g.: Dirichlet, logmodular, finitely generated, has analytic structure in the spectrum, etc.).
- Deformation of complex manifolds and multidimensional Riemann Mapping Theorem.
- Almost multiplicative functionals.

Early History (metric case)

Theorem (Nagasawa, 1959)

Uniform algebras are isometric iff they are isomorphic as algebras.

Definition

$\mathcal{A}$ is a uniform algebra iff $\left\|a^2\right\| = \|a\|^2$ or equivalently $\mathcal{A} \subset C(X)$.

Example

Not true for Banach algebras in general: $\left(l^1, \cdot\right)$, $\left(l^1, *\right)$.

Theorem (Cambern, 1963)

$C(X) \approx C(Y) \Longrightarrow X \approx Y$

$d_{B-M}(C(X), C(Y)) < 2 \Longrightarrow X, Y$ *are homeomorphic.*

Definition

$$d_{B-M}(\mathcal{A}, \mathcal{B}) = \inf\left\{\|T\| \left\|T^{-1}\right\| : T : \mathcal{A} \to \mathcal{B}\right\}.$$

Early History (algebraic case)

Theorem (B.E. Johnsom and I. Raeburn & J. Taylor, 1977)

A Banach algebra $\mathcal{B}$ is algebraicly stable if Hochschild cohomology groups $\mathcal{H}^2(\mathcal{B},\mathcal{B})$ and $\mathcal{H}^3(\mathcal{B},\mathcal{B})$ vanish.

Definition

$\mathcal{H}^n(\mathcal{B},\mathcal{B}) = \ker \delta_n / \operatorname{Im} \delta_{n-1}$, where

$... \overset{\delta_{n-1}}{\to} L^n(\mathcal{A},\mathcal{A}) \overset{\delta_n}{\to} L^{n+1}(\mathcal{A},\mathcal{A}) \overset{\delta_{n+1}}{\to} ...$ is defined by

$$\begin{aligned} & \delta_n(T)(a_1, ..., a_{n+1}) \\ = \; & a_1 \cdot T(a_2, ..., a_{n+1}) + \sum_{j=1}^{n} (-1)^j \, T(a_1, ..., a_j \cdot a_{j+1}, ..., a_{n+1}) \\ & + (-1)^{n+1} \, T(a_1, ..., a_n) \cdot a_{n+1}. \end{aligned}$$

Among uniform algebras only $C(X)$ is known to satisfy this condition.

More recent history - uniform algebras

Theorem (KJ, 1985)

Metric and algebraic deformations coincide for uniform algebras.
Let $\mathcal{A}$ be a complex uniform algebra then the following conditions are equivalent

1. $\|a \cdot b - a \times b\| \leq \varepsilon_1 \|a\| \|b\|$, *for* $a, b \in \mathcal{A}$
2. $|\|a \cdot b\| - \|a \times b\|| \leq \varepsilon_2 \|a\| \|b\|$, *for* $a, b \in \mathcal{A}$
3. $\|a \times b\| \leq (1 + \varepsilon_3) \|a\| \|b\|$, *for* $a, b \in \mathcal{A}$
4. $a \times b = T^{-1}(Ta \cdot Tb)$ *where* $T : \mathcal{A} \to \mathcal{B}$ *is such that* $\|T\| \|T^{-1}\| \leq 1 + \varepsilon_4$

   it follows that $\mathcal{B}$ is a uniform algebra and that the Chouquet boundaries of $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic.

(we assume that all multiplications have the same unit).

More recent history - algebras of analytic functions

Example

Put $P_\varepsilon = \{z : 1 \le |z| \le 2+\varepsilon\}$, and define $T_\varepsilon : A(P_0) \to A(P_\varepsilon)$ by

$$T_\varepsilon\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{0} a_n z^n + \sum_{n=1}^{\infty} a_n \left(\frac{2}{2+\varepsilon}\right)^n z^n.$$

We have $\|T_\varepsilon\| \left\|T_\varepsilon^{-1}\right\| \to 1$, as $\varepsilon \to 0$.

Theorem (R. Rochberg, 1985)

If $\mathcal{A} = A(S)$, where S is a 1-dim Riemann surface, and $d_{B-M}(\mathcal{A}, \mathcal{B}) < 1+\varepsilon$ then $\mathcal{B} = A(S')$ where S, S' are ε'-quasiconformly equivalent.

Corollary

The disc algebra $A(\mathbb{D})$ is the only stable $A(S)$ algebra.

Deformations of complex manifolds

Problem

Describe small deformations of Banach algebras of analytic functions defined on n-dim manifolds.

Fact

For $n > 1$ a small deformation of an n-dim manifold Ω (e.g. a ball or a polidisc) is not holomorphically equivalent to Ω.

Definition

$$d(\Omega, \Omega') = \inf \left\{ \|T\| \left\| T^{-1} \right\| : \quad T : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega') \right\}.$$

Problem

Can the Rochberg's result be extended to domains in $\mathbb{C}^n$, $n > 1$ to provide a quantitative multidimensional Riemann Mapping Theorem?
Practically nothing is known about the multidimensional case. There are only partial results concerning $\mathcal{A}(\mathbb{D}^n)$ (KJ, 1992).

Almost multiplicative functionals

Definition

$F \in \mathcal{A}^*$ is δ-multiplicative iff $|F(ab) - F(a)F(b)| \leq \delta \|a\| \|b\|$.

Examples

- $F = G + \Delta$ where G is multiplicative and $\|\Delta\| \leq \varepsilon$,
- $F = G \circ T$ where G is multiplicative and $\|T\| \|T^{-1}\| < 1 + \varepsilon$.

Problem

Must any almost multiplicative functional be near a multiplicative one?

Definition

$\mathcal{A}$ is *functionally-stable* or *f-stable* (or *AMNM* algebra) if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall F \in \mathfrak{M}_\delta(\mathcal{A}) \; \exists G \in \mathfrak{M}(\mathcal{A}) \;\; \|F - G\| \leq \varepsilon,$$

where $\mathfrak{M}_\delta(\mathcal{A})$ is the set of δ-multiplicative functionals on $\mathcal{A}$.

Almost multiplicative functionals

Example

$C(X)$ algebras are f-stable.

Theorem (B. E. Johnson, 1986)

The disc algebra $A(\mathbb{D})$ is stable.

Example (B. E. Johnson, 1986)

The convolution algebra $L^2(0,1)$ is f-stable, while $L^1(0,1)$ is not.

Proof.

If $f_0 \in L^1(0,1)$ is "near" the Dirac measure then $f \longmapsto f * f_0$ is almost multiplicative. □

Example (S.J. Sidney, 1997)

There exists a non f-stable uniform algebra.

Almost multiplicative functionals - recent results

Theorem

- *The ball algebras $\mathcal{A}(B_n)$ are f-stable.*
- *Any uniform algebra with one generator is f-stable.*
- *If $K \subset \mathbb{C}$ is such that $\mathbb{C} \backslash K$ has finitely many components and the closures of the components are disjoint then $R(K)$ is f-stable.*

Theorem

Let $B(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$ be a Blaschke product. Then

- *if B is an interpolating Blaschke product then H^∞ / BH^∞ is f-stable;*
- *if B is not a product of finitely many interpolating Blaschke product then H^∞ / BH^∞ is not f-stable.*

Problem

Is $H^\infty(\mathbb{D})$ is f-stable? (Does $H^\infty(\mathbb{D})$ have an almost corona?)

Almost multiplicative functionals - proof

Theorem

The ball algebras $\mathcal{A}(B_n)$ are f-stable.

Proof. Let $F \in \mathfrak{M}_\delta(\mathcal{A}(B_n))$. We show that F is near a multiplicative functional.

Step 1. Show that (without loss of generality) F may be represented by a probabilistic, nonatomic measure μ_F on ∂B_n.

Step 2. For $\mathbf{w} = (w_1, ..., w_n) \in B_n$ define $\Phi_\mathbf{w} : \bar{B}_n \to \bar{B}_n$ by

$$\Phi_\mathbf{w}(\mathbf{z}) = \frac{\mathbf{w} - P_\mathbf{z} - \sqrt{1 - \|\mathbf{w}\|^2}\,(\mathbf{z} - P_\mathbf{z})}{1 - \langle \mathbf{z}, \mathbf{w} \rangle},$$

where

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^{n} z_k \bar{w}_k \quad \text{and} \quad P_\mathbf{z} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2}\mathbf{w}.$$

Almost multiplicative functionals - proof

Step 3. Define $\varphi : B_n \longrightarrow B_n$ by

$$\varphi(\mathbf{w}) = \int \Phi_{\mathbf{w}} d\mu_F.$$

If

$$B_n \ni \mathbf{w}_j \longrightarrow \mathbf{w}_0 \in \partial B_n,$$

then

$$\Phi_{\mathbf{w}_j}(z) \longrightarrow \mathbf{w}_0 \quad \text{pointwise on } \bar{B}_n \setminus \{\mathbf{w}_0\}.$$

Since μ_F has no atoms, φ extends to a continuous function on $\bar{B}_n$ such that $\varphi(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in \partial B_n$. Hence φ is surjective and there is a $\mathbf{w}_0 \in B_n$ such that $\varphi(\mathbf{w}_0) = \mathbf{0}$. Replacing F with $f \longmapsto F(f \circ \Phi_{\mathbf{w}_0})$ we may assume without loss of generality that $\mathbf{w}_0 = \mathbf{0}$, so that

$$\varphi(\mathbf{0}) = \int \Phi_{\mathbf{0}} d\mu_F = \int Id d\mu_F = \mathbf{0}$$

hence

$$F(z_k) = 0 \quad \text{for } k = 1, ..., n.$$

Step 4. Put $\mathcal{A}_0(B_n) \stackrel{df}{=} \{f \in \mathcal{A}(B_n) : f(\mathbf{0}) = 0\}$ and define

$$T : (\mathcal{A}(B_n))^n \to \mathcal{A}_0(B_n) \qquad \text{by} \qquad T(f_1, \ldots, f_n) \stackrel{df}{=} \sum_{k=1}^{n} z_k f_k.$$

T is surjective and there are continuous linear selections S_k [E. Stout]

$$\mathcal{A}_0(B_n) \ni \sum_{k=1}^{n} z_k f_k \xmapsto{S_k} f_k \in \mathcal{A}(B_n).$$

For $f \in \mathcal{A}(B_n)$

$$\begin{aligned} F(f) &= F\left(f(\mathbf{0}) - \sum_{k=1}^{n} z_k S_k(f - f(\mathbf{0}))\right) = f(\mathbf{0}) - \sum_{k=1}^{n} F(z_k S_k(f - f(\mathbf{0})) \\ &\approx f(\mathbf{0}) - \sum_{k=1}^{n} F(z_k) \cdot F\left(S_k(f - f(\mathbf{0}))\right) = f(\mathbf{0}). \end{aligned}$$

Hence

$$F \approx \delta_0. \qquad \square$$

As a special case, for $n = 1$ we get Johnson's theorem.

Open problems - deformations of uniform algebras

Problem

For bounded pseudoconvex domains Ω in C^n describe small deformations of algebras $A(\Omega)$ and $H^\infty(\Omega)$ and characterize these domains for which the algebras are stable.

Problem

Is the injective tensor product $\mathcal{A} \bigotimes \mathcal{B}$ of uniform algebras stable iff the algebras $\mathcal{A}$ and $\mathcal{B}$ are both stable?

Problem

Does there exist a nonstable uniform algebra with only countably many small deformations?

Problem

Is the property "$\mathcal{A}$ is logmodular" stable?
Is the property "$\mathcal{A}$ has n generators" for $n \in \mathbb{N} \cup \{\infty\}$ stable?

Open problems - almost multiplicative functionals

Problem

Let Ω be a bounded pseudoconvex domain in C^n, are $A(\Omega)$ and $H^\infty(\Omega)$ f-stable? Is H^∞ f-stable?

Problem

Is any uniform algebra with two generators f-stable?

Problem

Let $\mathcal{A}$ be an f-stable uniform algebra. Is an ultrapower of $\mathcal{A}$ f-stable? Is $l^\infty(\mathcal{A})$ f-stable?

Problem

Let B be a product of finitely many interpolating Blaschke products. Is H^∞ / BH^∞ f-stable?

More open problems - separating maps

Definition

$S : \mathcal{A} \to \mathcal{B}$ is separating iff $f \cdot g = 0 \Longrightarrow S(f) \cdot S(g) = 0$.

Theorem

For compact X, Y and a separating $S : C(X) \to \mathcal{C}(Y)$

- *S may not be continuous (even if Y is a singleton),*
- *if S^{-1} exists then S is continuous and S^{-1} is separating.*

Problem

Assume $S : C(X) \to \mathcal{C}(Y)$ is a separating bijection, does it follow that S^{-1} is separating?

Theorem (J. Araujo & KJ, 1999)

Yes if $X \subset \mathbb{R}$.

THANK YOU!