Noncommutative function algebras

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Introduction

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- **Commutative, real** uniform algebras - subalgebras of

$$C_{\mathbb{C}}(X,\tau) \overset{df}{=} \left\{f \in C_{\mathbb{C}}(X) : f(\tau(x)) = \overline{f(x)},\ x \in X\right\}$$

where $\tau^2 = id_X$. Often $X = X_1 \cup X_2 \cup X_3$ where $A_{|X_1} =$ complex uniform algebra,
$A_{|X_2} =$ complex conjugates of $A_{|X_1}$, and
$A_{|X_3} = C_{\mathbb{R}}(X_3)$.

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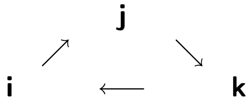
- **Noncommutative real uniform algebras.**

Quaternions: $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$



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$$\begin{array}{ccccc} & & \mathbf{j} & & \\ & \nearrow & & \searrow & \\ \mathbf{i} & & \longleftarrow & & \mathbf{k} \end{array}$$

There is no theory of analytic functions of quaternion variables - the standard definition

$$f'(a) = \lim_{q \to 0, q \in \mathbb{H}} \left(f(a+q) - f(a) \right) q^{-1}$$

is not useful since only linear functions are differentiable. Hence there is no obvious example of a nontrivial noncommutative real uniform algebra.

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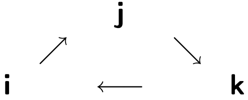

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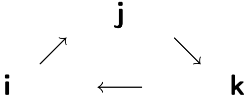

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Test question: Is $\mathbb{H}$ a complex Banach algebra?

Answer: NO!

$$\mathbf{j} \cdot i\mathbf{k} \neq i\mathbf{j} \cdot \mathbf{k}$$

PROPOSITION. **A surjective map $T : \mathbb{H} \to \mathbb{H}$ is linear (over the field of real numbers) and multiplicative if and only if**

$$T(a, b, c, d) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} a \\ \begin{bmatrix} b \\ c \\ d \end{bmatrix} \end{bmatrix}, \text{ for } a + bi + cj + dk \in \mathbb{H} \quad (4)$$

where M is an isometry of 3 dimensional Euclidean space $\mathbb{R}^3$ preserving the orientation of that space. The space of all such maps forms a 5 dimensional compact connected group $\mathcal{M}$.
If $\mathbb{K}$ is a 2 dimensional subalgebra of $\mathbb{H}$ (which must be isomorphic with $\mathbb{C}$) then the set of linear and multiplicative functionals from $\mathbb{K}$ into $\mathbb{H}$ is homeomorphic with the unit sphere in $\mathbb{R}^3$.

DEFINITIONS

Df. $A \subset C(X)$ **separates** the points of X if

$$\forall \mathbf{x_1}, \mathbf{x_2} \in \mathbf{X} \quad \exists \mathbf{f} \in \mathbf{A} \text{ such that } \mathbf{f}(\mathbf{x_1}) \neq \mathbf{f}(\mathbf{x_2}).$$

$A \subset C(X)$ **strongly separates** the points of X if

$$\forall \mathbf{x_1}, \mathbf{x_2} \in \mathbf{X} \quad \exists \mathbf{f} \in \mathbf{A} \text{ such that } \mathbf{f}(\mathbf{x_1}) \neq \mathbf{f}(\mathbf{x_2}) = \mathbf{0}.$$

Df. A real algebra A is **fully noncommutative** if any nonzero linear and multiplicative functional $F : A \to \mathbb{H}$ is surjective.

A is not fully noncommutative iff $\{F(a) : a \in A\}$ is isomorphic with $\mathbb{R}$ or $\mathbb{C}$ for some $\mathbb{H}$-valued multiplicative functional F on A.

Df. Let $\Phi : M \to Hom(X) : \Phi(T) = \Phi_T$ be an homomorphism of the group $\mathcal{M}$ onto a group of homeomorphisms of a compact set X. We define

$$C_{\mathbb{H}}(X, \Phi) \stackrel{df}{=} \left\{ \begin{array}{c} f \in C_{\mathbb{H}}(X) : f \circ \Phi_T(x) = T \circ f(x) \\ \text{for } x \in X, T \in \mathcal{M} \end{array} \right\}.$$

Gelfand Transformation

For a commutative real or complex uniform algebra A :

$$A \ni a \longmapsto \hat{a} \in C_{\mathbb{C}}(X) : \hat{a}(x) \stackrel{df}{=} x(a), \tag{5}$$

$$A \cong \hat{A} \subseteq C_{\mathbb{C}}(X),$$

where $X = \mathfrak{M}(A)$ .For a real algebra A and for any $x \in X$ also the complex conjugate $\bar{x} = \tau(x)$ of x is an element of X hence the representation of A as a subalgebra of

$$C_{\mathbb{C}}(X, \tau) \stackrel{df}{=} \left\{ f \in C_{\mathbb{C}}(X) : f(\tau(x)) = \overline{f(x)} \text{ for } x \in X \right\}.$$

An important feature of such representation is that $^\wedge$ maps preserves the spectrum, that it maps the set of invertible elements of A exactly onto the subset of $\hat{A}$ consisting of functions that do not vanish on X. We will construct a similar representation for a noncommutative uniform algebra.

Theorem. If A is a real uniform algebra then there is a compact set X and an isomorphism $\Phi : \mathcal{M} \to Hom(X)$ such that A is isometrically isomorphic with a subalgebra $\hat{A}$ of $C_{\mathbb{H}}(X, \Phi)$.
Furthermore $a \in A$ is invertible iff the corresponding element $\hat{a} \in \hat{A}$ does not vanish on X.
If A is fully noncommutative then $\hat{A} = C_{\mathbb{H}}(X, \Phi)$.

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We have obvious candidates for X, Φ, $^\wedge$:

$$\begin{aligned} & X \overset{df}{=} \mathfrak{M}_{\mathbb{H}}(A) \\ = \; & \mathbb{H}\text{-valued real-linear \& multiplicative functionals,} \\ & \Phi_T(x) \overset{df}{=} T \circ x, \qquad \hat{a}(x) \overset{df}{=} x(a). \end{aligned} \tag{6}$$

For $x \in X$ we may encounter three distinct cases - the set $\{T \circ x : T \in \mathcal{M}\}$ may be

- equal to the singleton $\{x\}$ if $x(A) = \mathbb{R}$, or
- homeomorphic with the unit sphere in $\mathbb{R}^3$ if $x(A)$ is a 2 dimensional commutative subalgebra of $\mathbb{H}$ isomorphic with $\mathbb{C}$, or
- homeomorphic with $\mathcal{M}$ if $x(A) = \mathbb{H}$.

Consequently the set $\mathfrak{M}_{\mathbb{H}}(A)$ can be divided into three parts X_1, X_2, X_3:

- $$\begin{aligned} X_1 &= \{x \in X : T \circ x = x, \text{ for all } T \in \mathcal{M}\} \\ &= \{x \in X : x(A) = \mathbb{R}\} \end{aligned}$$

  - that set is equal to the subset of X consisting of the points where all the functions from $\hat{A}$ are real valued;
- $X_2 = \{x \in X : \dim x(A) = 2\}$ = union of disjoint copies of the unit sphere in $\mathbb{R}^3$; the algebra $\hat{A}$ restricted to $X_1 \cup X_2$ is commutative;
- $X_3 = \{x \in X : x(A) = \mathbb{H}\}$ = union of disjoint copies of $\mathcal{M}$ and the algebra $\hat{A}$ restricted to X_3 is fully noncommutative.

To prove the Theorem we need to show that

1. $\mathfrak{M}_{\mathbb{H}}(A)$ is compact and the map $^{\wedge}$ defined by (6) preserves the norm, and
2. any noninvertible element of A is contained in a kernel of a functional from $\mathfrak{M}_{\mathbb{H}}(A)$, and
3. $\hat{A} = C_{\mathbb{H}}(X, \Phi)$ if A is fully noncommutative.

Theorem [Aupetit-Zemanek]. Let A be a real unital Banach algebra. If $\lim \sqrt[n]{\|a^n\|} = \|a\|$, $a \in A$ *then for every irreducible representation* $\pi : A \to L(E)$*, the algebra* $\pi(A)$ *is isomorphic with its commutant* C_π *in the algebra* $L(E)$ *of all linear transformation on* E.

Proof - Part 1. Since C_π is a normed real division algebra it is isomorphic with $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. The map $^\wedge$ is an isomorphisms of A into the algebra of $\mathbb{H}$-valued functions on $X = \mathfrak{M}_{\mathbb{H}}(A) =$ the set of all irreducible representations of A. Let σ be the strongest topology on X such that all the functions $\hat{a}$, $a \in A$, are continuous. Assume (X, σ) is not compact and let x_0 be a point from $\beta X \backslash X$, the operator $A \ni a \longrightarrow \hat{a}(x_0) \in \mathbb{H}$ is an irreducible representation on A so $x_0 \in X$, contradiction.

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To show that $^\wedge$ is injective we need to show that A is semisimple. Fix $0 \neq a \in A$ and let A_a be the closed subalgebra of A generated by all the elements of the form $q(a)$, where q is a rational function with real coefficients and with poles outside the spectrum of a. Notice that A_a is a commutative uniform algebra such that so $radA_a = \{0\}$. If $b \in A^{-1} \cap A_a$ then b^{-1} as given by a rational function is in A_a, that means that $A^{-1} \cap A_a = A_a^{-1}$. Hence $A_a \cap radA \subset radA_a = \{0\}$, so $a \notin radA$ and, since a was arbitrary, A is semisimple.

Part 2. Put

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$\forall w_p = a_p + b_p\mathbf{i} + c_p\mathbf{j} + d_p\mathbf{k}, p = 1, 2$

$$\begin{aligned} & (w_1 w_2 - w_2 w_1)^2 \\ = & \left(\begin{array}{c} 2(b_1 c_2 - b_2 c_1)\,\mathbf{ij} + 2(b_1 d_2 - b_2 d_1)\,\mathbf{ik} \\ +2(c_1 d_2 - c_2 d_1)\,\mathbf{jk} \end{array} \right)^2 \\ = & \; 4\left((b_1 c_2 - b_2 c_1)\,\mathbf{k} - (b_1 d_2 - b_2 d_1)\,\mathbf{j} + (c_1 d_2 - c_2 d_1)\,\mathbf{i}\right)^2 \\ = & -4\left((b_1 c_2 - b_2 c_1)^2 + (b_1 d_2 - b_2 d_1)^2 + (c_1 d_2 - c_2 d_1)^2\right) \\ \in & \; C_{\mathbb{R}}(X). \end{aligned}$$

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Let $\sim$ be an equivalence relation on X defined by

$$x_1 \sim x_2 \text{ iff } h(x_1) = h(x_2) \text{ for all } h \in \mathcal{R},$$

let $X_0 = X_{/\sim}$ be the quotient space, and let $\pi : X \to X_0$ be the natural projection.

For $x \in X$, $\exists f \in \mathcal{R} \quad f(x) \neq 0 \Longleftrightarrow x \in X_3$

$$x_1 \sim x_2 \Longleftrightarrow \ker x_1 = \ker x_2$$

$$x_1 \sim x_2 \Longleftrightarrow \{T \circ x_1 : T \in \mathcal{M}\} = \{T \circ x_2 : T \in \mathcal{M}\}.$$

Let $A_{\mathcal{R}}$ be the closed unital subalgebra of $\hat{A}$ generated by $\mathcal{R}$. By the Stone-Weierstrass Theorem

$$\begin{aligned} A_{\mathcal{R}} &= \{g \in C_{\mathbb{R}}(X) : g = const \text{ on } \{T \circ x : T \in \mathcal{M}\}\} \\ &\simeq C_{\mathbb{R}}(X_0). \end{aligned}$$

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Suppose that $\hat{f} \in \hat{A}$ does not vanish on X and put $h_1 = \left|\hat{f}\right| \in C_{\mathbb{R}}(X)$. Since $\mathcal{M}$ consists of isometries h_1 is constant on the sets $\{T \circ x : T \in \mathcal{M}\}$ so $h_1^{-1} \in A_{\mathcal{R}} \subset \hat{A}$. Put $h_2 = \mathrm{Re}\left(fh_1^{-1}\right)$; again h_2 is constant on the sets $\{T \circ x : T \in \mathcal{M}\}$ so $h_2^{-1} \in A_{\mathcal{R}} \subset \hat{A}$. For any quaternion $w = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ of absolute value one the reciprocal w^{-1} of w is equal to its conjugate $\bar{w} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. Since fh_1^{-1} is unimodular function in A with its real part also in A the inverse of fh_1^{-1} and consequently the inverse of f are in A as well.

Part 3. Fix $x_0 \in X$ and let $f_1, f_2, f_3, f_4 \in A$ be such that

$$f_1(x_0) = 1,\ f_2(x_0) = \mathbf{i},\ f_3(x_0) = \mathbf{j},\ f_4(x_0) = \mathbf{k}.$$

Let U_0 be an open neighborhood of x_0 such that

$$|f_p(x) - f_p(x_0)| < 1/4 \ \text{ for } x \in U_0$$

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and $p = 1, 2, 3, 4$.
Let $g_0 \in C_{\mathbb{H}}(X, \Phi)$ and assume the support of g_0 is contained in $V_0 \stackrel{df}{=} \pi^{-1}(\pi(U_0))$. For any $x \in U_0$, the numbers $f_1(x), f_2(x), f_3(x), f_4(x)$ can be seen as linearly independent vectors in $\mathbb{H} \simeq \mathbb{R}^4$ so there are unique real valued functions h_p defined on U_0 such that

$$g_0(x) = \sum_{p=1}^{4} h_p(x) f_p(x) \text{ for } x \in U_0. \quad (7)$$

Let $x \in U_0$ and let $x' = T \circ x$ be another point of $\pi^{-1}(\pi(x))$, where $T \in \mathcal{M}$

$$g_0(x') = T \circ g_0(x) = \sum_{p=1}^{4} h_p(x)\, T \circ f_p(x) = \sum_{p=1}^{4} h_p(x)\, f_p(x').$$

Hence $h_p, p = 1, 2, 3, 4$ are constant on the sets of the form $\pi^{-1}(\pi(x))$ and consequently can be naturally extended to V_0, furthermore they can be extended to the entire set X by assigning value zero outside V_0. Since the coefficients with respect to a fixed basis in $\mathbb{R}^4$ are continuous functions of a vector in $\mathbb{R}^4$, the functions $h_p, p = 1, 2, 3, 4$ belong to $C_{\mathbb{H}}(X, \Phi) \cap C_{\mathbb{R}}(X) = A_{\mathcal{R}} \subset A$. Hence $g_0 \in A$.

Let $x \in U_0$ and let $x' = T \circ x$ be another point of $\pi^{-1}(\pi(x))$, where $T \in \mathcal{M}$

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We proved that for any point x_0 in X there is a neighborhood V_0 of $\pi^{-1}(\pi(x))$ such that $A_{|V_0} = C_{\mathbb{H}}(X, \Phi)_{|V_0}$. Let $V_s = \pi^{-1}(\pi(U_s))$, $s \in S$ be a finite open cover of X consisting of such sets. Let $\sum_{s\in S} \alpha_s = 1$ be a finite partition of unity in $C_{\mathbb{R}}(X_0)$ subordinated to the cover $\pi(U_s)$, $s \in S$ of X_0. For any $g \in C_{\mathbb{H}}(X, \Phi)$ we have $g(x) = \sum_{s\in S} \alpha_s \circ \pi(x)\, g(x)$ where $\operatorname{supp}\alpha_s \circ \pi g \subset V_s$ so $g \in A$.

Corollary. **Assume A is a fully non commutative closed subalgebra of $C_{\mathbb{H}}(Y)$. Then $A = C_{\mathbb{H}}(Y)$ if and only if A strongly separates the points of Y.**

We do not assume here that A contains all constant functions; if we assume that the constant functions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are in A then the above version of the Stone-Weierstrass Theorem easily follows from

$$\operatorname{Re} f = (f - \mathbf{i}f\mathbf{i} - \mathbf{j}f\mathbf{j} - \mathbf{k}f\mathbf{k})/4.$$

For a fully non commutative real uniform algebra A its maximal ideal space $\mathfrak{M}_{\mathbb{H}}(A)$ consists of disjoint copies of $\mathcal{M}$, and at any point is locally homeomorphic with a Cartesian product of $\mathcal{M}$ with another set. Quite often we also have a global decomposition: $\mathfrak{M}_{\mathbb{H}}(A) = Y \times \mathcal{M}$. However such global decomposition can not always be achieved neither its analog in the commutative case.

EXAMPLE.

In the commutative case when $A \subset C_{\mathbb{C}}(X, \tau)$ and we can divided X into three parts X_1, X_2, and X_3 such that $A_{|X_1}$ is a complex algebra, $A_{|X_2}$ consists of complex conjugates of the functions from $A_{|X_1}$, and $A_{|X_3}$ is equal to $C_{\mathbb{R}}(X_3)$. It is easy to select $X_3 = \{x \in X : \tau(x) = x\}$, we may also easily select one point form each equivalence class $x_1 \sim x_2$ iff $\tau(x_1) = x_2$ to get X_1 and we can form X_2 from the remaining points. However we may not be able to make the sets X_1, X_2 closed so $A_{|X_1}, A_{|X_2}$ may not be uniform algebra.

For example

$$A = \left\{ f \in C_{\mathbb{C}}\left(S^1\right) : f\left(e^{it}\right) = \overline{f\left(e^{i(t+\pi)}\right)} \text{ for } 0 \leq t \leq \pi \right\}$$

where S^1 is a unit circle.

This is the reason we represent a real commutative uniform algebra as a subalgebra of $C_{\mathbb{C}}(X, \tau)$ rather then, what may seem more appealing, a direct sum of $A_{|X_1}, A_{|X_2}$, and $A_{|X_3}$.

QUESTION

We proved that the algebra $\mathbb{H}$ has the following property:

> *Any subalgebra A of $C_{\mathbb{H}}(X) = C_{\mathbb{R}}(X) \otimes \mathbb{H}$ that strongly separates the points of X and such that $\{f(x) : x \in X\} = \mathbb{H}$ for all $x \in X$ is equal to $C_{\mathbb{H}}(X)$.*

We also know (Stone-Weierstass Theorem) that the algebra $\mathbb{R}$ has the same property. It would be interesting to find other algebras with that property.

Corollary. **Assume A is a real m-convex Hausdorff algebra with the topology give by a family $\{p_\alpha : \alpha \in \Lambda\}$ of m-convex seminorms. If**

$$(p_\alpha(a))^2 = p_\alpha(a^2), \text{ for } a \in A, \alpha \in \Lambda$$

then A isometrically isomorphic with a subalgebra $\hat{A}$ of $C_{\mathbb{H}}(X, \Phi)$. Furthermore $a \in A$ is invertible iff the corresponding element $\hat{a} \in \hat{A}$ does not vanish on X.
If A is fully noncommutative then $\hat{A} = C_{\mathbb{H}}(X, \Phi)$.

Questions

- What about real semisimple subalgebras of $C_{\mathbb{H}}(X)$?
- What about real algebras A with

$$\|ab\| = \|ba\|, \text{ for } a, b \in A \text{ ?}$$

Lemma. For a complex unital Banach algebra A if

$$\left| \|ab\| - \|ba\| \leq \varepsilon \|a\| \|b\| \right|, \text{ for } a, b \in A$$

then

$$\|ab - ba\| \leq \varepsilon' \|a\| \|b\|, \text{ for } a, b \in A,$$

where $\varepsilon' = -\frac{4}{\ln \varepsilon}$.

Proof. Put

$$\varphi(\lambda) = \exp(\lambda b)\, a \exp(-\lambda b);$$

we have

$$\begin{aligned} & \left| \|a\| - \|\exp(\lambda b)\, a \exp(-\lambda b)\| \right| \\ \leq \; & \varepsilon \|\exp(\lambda b)\, a\| \, \|\exp(-\lambda b)\| \\ \leq \; & \varepsilon \|a\| \exp(2 |\lambda| \, \|b\|). \end{aligned}$$

Hence

$$\|\exp(\lambda b)\, a \exp(-\lambda b)\| \leq \|a\| \left(1 + \varepsilon \exp(2 |\lambda| \, \|b\|)\right),$$

so

$$\left\| a + (ba - ab)\lambda + (\dots)\lambda^2 + \dots \right\| \leq \|a\| \left(1 + \varepsilon \exp(2 |\lambda| \, \|b\|)\right)$$

Taking the first derivative at 0 and estimating on the unit disc $|\lambda| \leq 1$

$$\|ab - ba\| \leq \|a\| \left(1 + \varepsilon \exp\left(2 \|b\|\right)\right).$$

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$$\|ab - ba\| \leq \|a\| \left(1 + \varepsilon \exp\left(2 \|b\|\right)\right).$$

For $t = -\frac{\ln \varepsilon}{2\|b\|}$ we get

$$\begin{aligned} \|ab - ba\| &\leq \left\|\frac{1}{t} a\right\| \left(1 + \varepsilon \exp\left(2 \|tb\|\right)\right) \\ &= -\frac{4}{\ln \varepsilon} \|a\| \|b\|. \end{aligned}$$