MULTIPLIERS AND ISOMETRIES IN H^∞_E

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ABSTRACT

Let E be a complex Banach space and H^∞_E the space of bounded analytic functions on the unit disc to E. By means of a study of the multipliers on H^∞_E it is shown that, if $\operatorname{Mult}(E) = \mathbb{C}$, then every isometry T of H^∞_E is of the form $(TF)(z) = \mathscr{T}F(\tau(z))$, where τ is a conformal map of the disc and $\mathscr{T}$ is a constant isometry of E.

1. *Introduction*

Throughout this article the letter E will denote a complex Banach space. If E is in fact a Hilbert space, it will be denoted by $\mathscr{H}$. Given E, $Z(E)$ denotes the centralizer of E and $\operatorname{Mult}(E)$ the set of multipliers on E. Recall that a bounded operator A on E is called a multiplier if, and only if, given $e_1, e_2 \in E$ and $r > 0$ then, whenever the inequality $\|e_1 - \lambda e_2\| \leqslant r$ is satisfied for all scalars λ with $|\lambda| \leqslant \|A\|$, it follows that $\|e_1 - Ae_2\| \leqslant r$. We refer the reader to [**1**] for the definition of $Z(E)$, and for equivalent definitions and the properties of multipliers. The letter D stands for the open unit disc in the complex plane, and H^∞_E is the Banach space of all bounded analytic functions on D to E provided with the supremum norm. Equivalently, H^∞_E consists of all bounded $F: D \to E$ such that $\langle F, e^* \rangle$ belongs to the Hardy class H^∞ for all $e^* \in E^*$ [**9**, p. 205].

A classical result provided by de Leeuw, Rudin and Wermer in [**5**], and quite independently by Nagasawa [**8**], characterizes the isometries of H^∞. They established that T is an isometry of H^∞ onto itself if and only if T is of the form

$$(Tf)(z) = \alpha f(\tau(z)), \quad f \in H^\infty, \; z \in D, \tag{1}$$

where α is a complex constant of modulus one and τ is a conformal map of D onto D. In [**3**] this result was extended to the space $H^\infty_{\mathscr{H}}$, for $\mathscr{H}$ a finite-dimensional (complex) Hilbert space. It was shown that T is an isometry of $H^\infty_{\mathscr{H}}$ onto itself if and only if, for $F \in H^\infty_{\mathscr{H}}$ and $z \in D$, one has

$$(TF)(z) = \mathscr{T}F(\tau(z)), \tag{2}$$

where τ is again conformal and $\mathscr{T}$ is now a constant isometry of $\mathscr{H}$. And in [**4**], necessary and sufficient conditions on a finite-dimensional Banach space E were given which allow such a result to be established for H^∞_E.

The first such result for H^∞_E with E infinite-dimensional was obtained quite recently in [**7**] by Lin, who showed that if E is a uniformly convex *and* uniformly smooth Banach space, then the description (2) holds for isometries T of H^∞_E. Either condition placed by Lin on E implies, in particular, that $\operatorname{Mult}(E) = \mathbb{C}$ [**6**, Theorem 12.7; **2**, Proposition 5.2]. Hence Lin's result is a particular case of the following theorem, which we establish in this article by means of an examination of the multipliers on H^∞_E.

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THEOREM. *Let E be a Banach space with* $\mathrm{Mult}(E) = \mathbb{C}$, *and let T be an isometry of H_E^∞ onto itself. Then T is of the form*

$$(TF)(z) = \mathscr{T}F(\tau(z)), \quad F \in H_E^\infty, \ z \in D,$$

where $\mathscr{T}$ is a constant isometry of E onto E and τ is a conformal map of the disc onto itself.

(Obviously any map of the form specified in our theorem is an isometry, so that the theorem actually characterizes the surjective isometries of H_E^∞.)

We note that the conditions imposed by Lin in [7] also imply that E is reflexive. And, for a reflexive Banach space, the requirement $\mathrm{Mult}(E) = \mathbb{C}$ is equivalent to the requirement that E possesses no nontrivial L^∞-summand [**1**, Theorem 5.9; **2**, Proposition 2.10, Corollary 5.4]. Hence the example provided on p. 119 of [**4**] shows that for reflexive spaces the condition $\mathrm{Mult}(E) = \mathbb{C}$ is both necessary and sufficient in order for our theorem to hold. The theorem thus answers, for the class of reflexive Banach spaces E, questions raised by Lin in Remarks 3 and 6 of [7]. (Outside the class of reflexive spaces, the condition $\mathrm{Mult}(E) = \mathbb{C}$, although sufficient for the conclusion of our theorem to hold, is no longer necessary. This can be seen by consideration of $E = A(D)$, the disc algebra.)

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2. *Proof of the theorem*

The proof will be established by means of a sequence of propositions. The first proposition establishes a general fact about isometries of any Banach space $E \neq \{0\}$.

PROPOSITION 1. *Let $A: E \to E$ be a surjective isometry which differs from the identity map. Then there exist $e \in E$ and $e^* \in E^*$, with* $\|e\| = \|e^*\| = 1 = \langle e, e^* \rangle$, such that $\langle A(e), e^* \rangle \neq 1$.

Proof. Assume that the proposition is false. Let $e_o \in E$ and let e_o^* be a norm-one functional on E such that $\|e_o\| = \langle e_o, e_o^* \rangle$. By our assumption, $\langle A(e_o), e_o^* \rangle = \|e_o\|$. Hence, if A^j denotes $A \circ A \circ \ldots \circ A$ (j factors), then for any positive integer k we have

$$\|e_o + A(e_o) + \ldots + A^k(e_o)\| \geqslant |\langle e_o + A(e_o) + \ldots + A^k(e_o), e_o^* \rangle| = (k+1)\,\|e_o\|.$$

Now let e_o be such that $A(e_o) \neq e_o$ and let $e = A(e_o) - e_o$. For any positive integer k we have

$$k\|e\| \leqslant \left\| \sum_{j=1}^{k} A^j(e) \right\| = \left\| \sum_{j=1}^{k} A^j(A(e_o) - e_o) \right\| = \|A^{k+1}(e_o) - A(e_o)\| \leqslant 2\|e_o\|,$$

so that $e = 0$. This contradiction concludes the proof.

PROPOSITION 2. *Let S be a multiplier on H_E^∞. Then for any $z \in D$ there is a multiplier S_z on E such that*

$$(SF)(z) = S_z F(z), \quad F \in H_E^\infty. \tag{3}$$

Proof. Fix $z \in D$. We define a map $S_z: E \to E$ by $S_z(e) = (S\mathbf{e})(z)$. (Throughout the article, for $e \in E$, $\mathbf{e}$ will denote that function which is constantly equal to e.) Let

$e_1, e_2 \in E$ and let r be a positive real number. Assume that $\|e_1 - \lambda e_2\| \leqslant r$ for all scalars λ with $|\lambda| \leqslant \|S\|$. Then since S is a multiplier, this implies that $\|\mathbf{e}_1 - (S\mathbf{e}_2)\|_\infty \leqslant r$ so that $\|e_1 - (S\mathbf{e}_2)(z)\| \leqslant r$. But this says that $\|e_1 - S_z e_2\| \leqslant r$, so S_z is a multiplier on E and (3) holds for any constant function $\mathbf{e} \in H_E^\infty$.

We wish to show that, in fact, (3) holds for all $F \in H_E^\infty$. For this we need to show that if $F, G \in H_E^\infty$ and $F(z) = G(z)$, then $(SF)(z) = (SG)(z)$. Equivalently, if $F \in H_E^\infty$ and $F(z) = 0$ then $(SF)(z) = 0$. But if $F(z) = 0$ then there is a $G \in H_E^\infty$ such that $F(w) = (w-z)\,G(w)$. Since $\mathrm{Mult}\,(H_E^\infty)$ is a commutative operator algebra, and multiplication by $w-z$ is obviously a multiplier on H_E^∞, one has $(SF)(w) = (w-z)(SG)(w)$, and evaluating this at $w = z$ we obtain $(SF)(z) = 0$.

COROLLARY. *If* $\mathrm{Mult}\,(E) = \mathbb{C}$ *then, given* $S \in \mathrm{Mult}\,(H_E^\infty)$, *there is an* $f \in H^\infty$ *such that*

$$(SF)(z) = f(z)\,F(z), \quad z \in D, \; F \in H_E^\infty.$$

That is, $\mathrm{Mult}\,(H_E^\infty) = \{M_f : f \in H^\infty\}$, *where* M_f *is the operator on* H_E^∞ *which is multiplication by* f.

Proof. Fix $z \in D$ and let S_z be as in Proposition 2. Since $\mathrm{Mult}\,(E) = \mathbb{C}$, there is a complex number $f_o(z)$ such that $S_z = f_o(z)\,\mathrm{Id}_E$. Now fix $e \in E$ and $e^* \in E^*$, where $\|e\| = \|e^*\| = 1 = \langle e, e^* \rangle$ and set, for $w \in D$, $f(w) = \langle (S\mathbf{e})(w), e^* \rangle$. Then $f \in H^\infty$ and we have

$$f(z) = \langle S_z(e), e^* \rangle = \langle f_o(z)\,\mathrm{Id}_E\, e, e^* \rangle = f_o(z).$$

Throughout the remainder of the article we assume that E is a given Banach space with trivial multipliers, and T is a fixed isometry of H_E^∞ onto itself.

PROPOSITION 3. *For* $F \in H_E^\infty$ *and* $z \in D$ *we have* $(TF)(z) = \mathscr{T} F(\tau(z))$, *where* τ *is a conformal map of* D *onto itself and* $\mathscr{T}$ *is a constant surjective isometry of* E.

Proof. By the corollary we have an isometry Φ of H^∞ onto itself defined by $M_{\Phi(f)} = TM_f T^{-1}$, for $f \in H^\infty$. Moreover, Φ takes the constant function $\mathbf{1}$ to $\mathbf{1}$. Thus by the classical result of [5] and [8] characterizing the isometries of H^∞, we have $\Phi(f) = f \circ \tau$, where τ is a conformal map of the disc onto itself. Hence $TM_f T^{-1} = M_{f\circ\tau}$, or $TM_f = M_{f\circ\tau} T$. Thus if we define T_1 to be the isometry of H_E^∞ given by $(T_1 F)(z) = F(\tau^{-1}(z))$ and set $T_2 = T_1 \circ T$, then from

$$(T(fF))(z) = (TM_f F)(z) = (f \circ \tau)(z)(TF)(z)$$

it follows that

$$(T_2(fF))(z) = (T_1 \circ T(fF))(z) = f(z)(TF)(\tau^{-1}(z)) = f(z)(T_1 \circ T(F))(z) = f(z)(T_2 F)(z).$$

Thus T_2 is an isometry of H_E^∞ onto itself which commutes with multiplication by bounded analytic functions.

Define a map $\mathscr{T}_z$ taking E to E by

$$\mathscr{T}_z(F(z)) = (T_2 F)(z), \quad F \in H_E^\infty. \tag{4}$$

One must, of course, show that $\mathscr{T}_z$ is well-defined; that is, if $F, G \in H_E^\infty$ and $F(z) = G(z)$, then $(T_2 F)(z) = (T_2 G)(z)$. But since T_2 commutes with multiplication by bounded analytic functions, this follows from the same argument used in the proof of Proposition 2.

Hence $\mathscr{T}_z$ given by (4) is a well-defined linear map of E to itself which is clearly surjective. We claim, moreover, that the map is isometric. For take $e \in E$ and note that $\|\mathscr{T}_z(e)\| = \|(T_2\mathbf{e})(z)\| \leqslant \|T_2\mathbf{e}\|_\infty = \|e\|$, so that $\|\mathscr{T}_z\| \leqslant 1$. Next note that the map $\mathscr{S}_z$ defined by

$$\mathscr{S}_z(F(z)) = (T_2^{-1}F)(z), \quad F \in H_E^\infty,$$

is, like $\mathscr{T}_z$, a well-defined linear map of E to itself with $\|\mathscr{S}_z\| \leqslant 1$. Now suppose that $e_o \in E$ and $\mathscr{T}_z(e_o) = (T_2\mathbf{e}_o)(z) = e_1$. Then $e_o = (T_2^{-1}(T_2\mathbf{e}_o))(z) = \mathscr{S}_z((T_2\mathbf{e}_o)(z)) = \mathscr{S}_z(e_1)$, so that $\mathscr{S}_z = \mathscr{T}_z^{-1}$ and thus $\mathscr{T}_z$ is isometric.

We wish to show that $\mathscr{T}_z$ is, in fact, equal to a constant isometry $\mathscr{T}$ for all $z \in D$. For this assume, to the contrary, that there is an $\alpha \in D$ such that $\mathscr{T}_0$ and $\mathscr{T}_\alpha$ are not identical. Without loss of generality (by composing both maps with $\mathscr{T}_0^{-1}$), we can assume that $\mathscr{T}_0 = \mathrm{Id}_E$. Let e and e^* be given by Proposition 1 for $A = \mathscr{T}_\alpha$, and set $\chi(z) = \langle (T\mathbf{e})(z), e^* \rangle = \langle \mathscr{T}_z(e), e^* \rangle$. Then χ is an analytic function on D bounded by 1, while by our own assumptions $\chi(0) = 1$ and $\chi(\alpha) \neq 1$. This contradiction shows that indeed $\mathscr{T}_z$ is a constant isometry $\mathscr{T}$ as claimed.

We thus have

$$(T_2F)(z) = (T_1 \circ T(F))(z) = \mathscr{T}F(z), \quad F \in H_E^\infty.$$

Hence

$$(TF)(z) = (T_1^{-1} \circ T_2(F))(z) = \mathscr{T}F(\tau(z)), \quad z \in D,$$

which concludes the proof.

References

1. E. Behrends, *M-structure and the Banach–Stone Theorem*, Lecture Notes in Math. 736 (Springer, Berlin, 1979).
2. E. Behrends, 'Multiplier representations and an application to the problem whether $A \oplus_\varepsilon X$ determines A and/or X', *Math. Scand.* 52 (1983) 117–144.
3. M. Cambern, 'The isometries of $H^\infty(K)$', *Proc. Amer. Math. Soc.* 36 (1972) 173–178.
4. M. Cambern, 'On mappings of modules, with an application to spaces of analytic functions', *Bull. Inst. Math. Acad. Sinica* 5 (1977) 105–119.
5. K. de Leeuw, W. Rudin and J. Wermer, 'The isometries of some function spaces', *Proc. Amer. Math. Soc.* 11 (1960) 694–698.
6. K. Jarosz, *Perturbations of Banach algebras*, Lecture Notes in Math. 1120 (Springer, Berlin, 1985).
7. P.-K. Lin, 'The isometries of $H^\infty(E)$', *Pacific J. Math.*, to appear.
8. N. Nagasawa, 'Isomorphisms between commutative Banach algebras with an application to rings of analytic functions', *Kōdai Math. Sem. Rep.* 11 (1959) 182–188.
9. A. E. Taylor, *Introduction to functional analysis* (Wiley, New York, 1958).

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