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ISOMETRIES OF BLOCH SPACES

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ABSTRACT. Let Ω_1 and Ω_2 be domains in $\mathbf{C}^n$ and let $\mathcal{A}$ and $\mathcal{B}$ be associated Bloch spaces. In this note we discuss the question whether any surjective isometry $T : \mathcal{A} \to \mathcal{B}$ is of the form

$$Tf = \lambda(f \circ \phi - f(\phi(0))),$$

where $\phi : \Omega_2 \to \Omega_1$ is a biholomorphic map, and $|\lambda| = 1$. We also introduce a concept of an abstract Bloch space associated with an arbitrary function algebra, and investigate isometries between such spaces.

1. INTRODUCTION

Isometries of various spaces of analytic functions have been studied by a number of authors. At first great deal of work has been devoted to Banach spaces of analytic functions equipped with the sup norm or with an L^p-norm.

The investigation of the sup norm Banach algebras of analytic functions has been facilitated by the general theory of uniform algebras and it is now very well known that in all 'resonable cases' any surjective isometry is given by

$$(*) \qquad Tf = \lambda f \circ \phi.$$

This can be extended to isometries of certain spaces of vector-valued analytic functions with sup norm [2,9]. The isometries of Hardy spaces were studied by de Leeuw-Rudin-Wermer [8], and lately, the vector-valued case has been investigated in [2]. In more recent years the interest switched to Banach spaces of analytic functions equipped with norms more closely related to the analytic structure of the underlying domains in $\mathbf{C}^n$, like the Bloch spaces. For a unit disc $\mathbf{D}$ we define the big Bloch space $\mathcal{B}(\mathbf{D})$ by

$$\mathcal{B}(\mathbf{D}) = \{f \in \mathrm{Hol}(\mathbf{D}) : \sup_{|z|<1} (1 - |z|^2)|f'(z)| = \|f\| < \infty,\ f(0) = 0\},$$

and we define the small Bloch space $\mathcal{B}_0(\mathbf{D})$ by

$$\mathcal{B}_0(\mathbf{D}) = \{f \in \mathcal{B}(\mathbf{D}) : \lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0\}.$$

The latter space is equal to the closure, in $\mathcal{B}(\mathbf{D})$, of the algebra of all polynomials. It is also well known that $\mathcal{B}(\mathbf{D})$ can be identified with the bidual of $\mathcal{B}_0(\mathbf{D})$. In 1980 Cima and Wogen [4] proved that any surjective isometry of a small, as well as of a big Bloch space is canonical, that is of the form

$$(**) \qquad Tf = \lambda(f \circ \phi - f(\phi(0))).$$

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In general the Bloch space $\mathcal{B}(\Omega)$ on a domain Ω in $\mathbf{C}^n$ containing the origin is defined by

$$\mathcal{B}(\Omega) = \{f \in \operatorname{Hol}(\Omega) : \sup \frac{|D_{\xi,z}f|}{H(z,\xi)} = \|f\| < \infty,\ f(0) = 0\},$$

where

$$D_{\xi,z}f = \sum_{j=1}^{n} \xi_j \frac{\partial f(z)}{\partial z_j}, \tag{1}$$

and $H(z,\xi)$ is the length of the vector $\xi \in T_z(\Omega)$ in the Kobayashi metric. The small Bloch space $\mathcal{B}_0(\Omega)$ is defined by

$$\mathcal{B}_0(\Omega) = \{f \in \mathcal{B}(\Omega) : \lim_{z \to \partial\Omega} \sup \frac{|D_{\xi,z}f|}{H(z,\xi)} = 0\}.$$

As in the classical, one dimensional case, $\mathcal{B}_0(\Omega)$ is the closed span of the algebra of all polynomials. In [7] Krantz and Ma proved that any linear surjective isometry of $\mathcal{B}_0(\mathbf{B}^n)$, where $\mathbf{B}^n$ is a unit ball in $\mathbf{C}^n$, is canoni. For general properties of the Bloch spaces see [1,6,10].

The explicit form of $H(z,\xi)$ is known only for some simple domains, and is fairly complicated even for the unit ball [6]. The standard definition of $H(z,\xi)$ is formulated in the language of differential geometry and it is difficult to use it in the context of isometries of Bloch spaces. The following is an equivalent definition which seems to be much better suited for our purpose:

$$H(z,\xi) = \|D_{\xi,z}\|,$$

where the differential map $D_{\xi,z} : H^\infty(\Omega) \to \mathbf{C}$ is defined by (1) and $H^\infty(\Omega)$ is the Banach algebra of all bounded analytic functions on Ω, equipped with the sup norm. It follows directly from the definition that $H(\cdot,\cdot)$ is biholomorphically invariant, that is

$$H(\phi(z), \frac{\phi_*(z)(\xi)}{|\phi_*(z)(\xi)|}) = H(z,\xi) \tag{2}$$

for any automorphism $\phi : \Omega \to \Omega$. Hence for any automorphism $\phi : \Omega \to \Omega$ the canonical map $f \to f \circ \phi - f(\phi(0))$ is an isometry of the Bloch space. The question is whether all the isometries are of this form. We will consider this question in more general setting i.e. for abstract Bloch spaces; this approach apparently simplifies som proofs as we concentrate on the essential ideas rather than on the specific form of the Kobayashi length $H(z,\xi)$. Some ideas used in the proofs are adopted from [4] and [7].

2. ABSTRACT BLOCH SPACES

Let X be a compact Hausdorff space and let A be a uniform algebra on X, that is an algebra of continuous functions defined on X, such that A contains the constant functions, separates the points of X, and is equipped with the sup norm.

For an x in X, a linear functional D on A is called a derivation at x if

$$D(fg) = f(x)D(g) + D(f)g(x),$$

for $f, g \in A$. Let $\mathcal{D}_x$ be the set of all norm one derivations on A at x, and put $\mathcal{D} = \bigcup \mathcal{D}_x$. We define a seminorm $\|\cdot\|$ on A by $\|f\| = \sup\{|\mathcal{D}(f)| : D \in \mathcal{D}\}$, we define the small Bloch space $\mathcal{B}_0(A)$ as the completion of the quotient space $A/ker\|\cdot\|$. We note that for any $x \in ChA$, the Choquet boundary of A, we have $\lim D_\alpha(f) = 0$ whenever $D_\alpha \in \mathcal{D}_{x_\alpha}$ and $x_\alpha \to x$. Hence, if we put $A = A(\Omega)$, the above definition reduces to the classical one, given in the previous section. The only difference is that now an element of the Bloch space is an equivalence class of functions wheras in the classical case we imposed the condition $f(0) = 0$ to select a representative of the class. Hence now $(**)$ has even simpler form $(*)$. We define the big Bloch space $\mathcal{B}(A)$ as the bidual of $\mathcal{B}_0(A)$.

We notice that for some function algebras, e.g. the algebra $C(X)$ of all continuous functions on X, there is no non-trivial point derivation so the associated Bloch spaces are trivial; indeed, the existence of a non-trivial point derivation is related to the existence of an analytic structure on the spectrum of the algebra [5]. It is clear from the definition that for any function algebras A and B, any algebra isomorphism $T : A \to B$, and any complex number μ with $|\mu| = 1$, the map μT is an isometry of A onto B as well as of $\mathcal{B}(A)$ onto $\mathcal{B}(B)$, and of $\mathcal{B}_0(A)$ onto $\mathcal{B}_0(B)$. We would like to know for what classes of function algebras any isometry of the corresponding Bloch spaces must be of this form.

We first consider a simple example of an abstract Bloch space. Let $A(\mathbf{D})$ be the disc algebra, that is the algebra of all continuous functions defined on the closed unit disc $\overline{\mathbf{D}}$ and analytic on $\mathbf{D}$. Let K be a compact Hausdorff space, and let A be equal to $C(K)\check{\otimes}A(\mathbf{D})$ the injective tensor product of $A(\mathbf{D})$ and $C(K)$. We have

$$A = \{f \in C(K \times \mathbf{D}) : \forall k \in K, f(k, \cdot) \in A(\mathbf{D})\}.$$

The algebra A has quite simple analytic structure: at any point of $K \times \mathbf{D}$ there is only one point derivation, up to a multiplicative constant, and there is no point derivation at any point of $K \times \partial\mathbf{D}$. On the other hand the structure of $K \times \overline{\mathbf{D}}$, the maximal ideal sp of A, may be more involved as K is an arbitrary compact set. In spite of this we will prove in the next section that an isometry of $\mathcal{B}_0(A)$ onto itself is given by an algebra automorphism of A.

Proposition. *Let K be a compact Hausdorff space and $A = C(K)\check{\otimes}A(\mathbf{D})$. Then any isometry $T : \mathcal{B}_0(A) \to \mathcal{B}_0(A)$ is given by an algebra automorphism of A.*

Conjecture. *Let Ω_1,Ω_2 be bounded domains of holomorphy in $\mathbf{C}^n$. Then:*

(i) *Ω_1, Ω_2 are biholomorphic if and only if $\mathcal{B}_0(A(\Omega_1))$ and $\mathcal{B}_0(A(\Omega_2))$ are isometric;*

(ii) *Any isometry* $T : \mathcal{B}_0(A(\Omega_1)) \to \mathcal{B}_0(A(\Omega_2))$ *is of the form*

$$Tf = \lambda(f \circ \phi - f(\phi(w_0))), \tag{3}$$

for $f \in \mathcal{B}_0(A(\Omega_1))$, *where* $\phi : \Omega_2 \to \Omega_1$ *is an analytic homeomorphism,* $|\lambda| = 1$, *and* w_0 *is a fixed point in* Ω_2.

We were unable to prove the above conjecture in full generality, in the next section we will prove the following special case.

Theorem. *Let* Ω_1, Ω_2 *be totally homogeneous domains in* $\mathbf{C}^n$. *Then any isometry* T *of* $\mathcal{B}_0(A(\Omega_1))$ *onto* $\mathcal{B}_0(A(\Omega_2))$ *is of the form (3).*

We call Ω totally homogeneous if it is a bounded domain of holomorphy, and if for any $z_1, z_2 \in \Omega$, and $D_1 \in \mathcal{D}_{z_1}$, $D_2 \in \mathcal{D}_{z_2}$ there is an automorphism $\phi : \Omega \to \Omega$ such that $\phi(z_1) = z_2$ and $\phi_*(D_2) = D_1$.

3. PROOFS

We first make several general observations about isometries of arbitrary Bloch spaces, we will then use them to prove the Example and the Theorem stated in the previous section.

Let E be a Banach space and let K be a subset of the dual space E^*. Assume that for any e in E we have

$$\|e\| = \sup\{|e^*(e)| : e^* \in K\}.$$

Then ext(E^*), the set of extreme points of the unit ball in E^*, is contained in $\{\mu e^* : |\mu| = 1, e^* \in \overline{K}\}$, where the closure is taken in the weak $*$ topology. Moreover the map from E into $C(\overline{K})$, defined by $\tilde{e}(k) = k(e)$ for $k \in K$, is an isometric embedding.

Let now A and B be function algebras respectively on X and Y, $\mathcal{B}_0(A)$ and $\mathcal{B}_0(B)$ their associated Bloch spaces, $\mathcal{D}_A$ and $\mathcal{D}_B$ the sets of all point derivations with norm one, and let X_0 (Y_0), be the subset of X (Y) such that there exists a non-zero point derivation at any point of X_0 (Y_0). Since a weak $*$ limit of a net of point derivations is a point derivation, the weak $*$ closure of $\mathcal{D}_A$ is contained in the set of all point derivations of norm not greater than one. Hence the map

$$\mathcal{B}_0(A) \ni f \to \tilde{f} \in C(\overline{\mathcal{D}_A}) : \tilde{f}(D) = D(f),$$

is an isometric embedding and ext$(\mathcal{B}_0(A)^*) \subseteq \mathcal{D}_A$.

Assume that any norm one point derivation on A or B is an extreme point of the unit ball in the dual space of the corresponding Bloch space. It follows that any isometry $T : \mathcal{B}_0(A) \to \mathcal{B}_0(B)$ is of the form,

$$D(Tf) = \phi(D)(f), \tag{4}$$

for $f \in \mathcal{B}_0(A)$ and $D \in \mathcal{D}_B$, where $\phi : \mathcal{D}_B \to \mathcal{D}_A$ is a homeomorphism. Let $D_1, D_2 \in \mathcal{D}_A$ then $\frac{D_1+D_2}{\|D_1+D_2\|} \in \mathcal{D}_A$ if and only if the derivations D_1 and D_2 correspond to the same point of X. Hence ϕ maps point derivations corresponding to the same point of Y onto derivations corresponding to the same point of X. So ϕ produces a homeomorphism ϕ_0 from Y_0 onto X_0.

The strategy to prove that any isometry between two given Bloch spaces is canonical, i.e. is given by an algebra isomorphism, is to show first that any point derivation is an extreme point, and then to use (4). Since both $\mathrm{ext}(\mathcal{B}_0(A)^*)$ and $\mathcal{D}_A$ are preserved by isometries and $\mathrm{ext}(\mathcal{B}_0(A)^*)$ is non-empty, in order to show that these sets are e it is enough to show that A has sufficiently many isometries.

Proof of the Proposition. Fix k in K. Let f be any norm one element of the classical Bloch space $\mathcal{B}_0(\mathbf{D})$. Let g_α be a net of continuous, norm one functions on X convergent uniformly to zero off any neighborhood of k and such that $g_\alpha(k) = 1$. A net f_α of elements of $\mathcal{B}_0(A)$ defined by $f_\alpha(k, z) = f(z)g_\alpha(k)$ is convergent to zero uniformly off any neighborhood of $\{k\} \times \overline{\mathbf{D}}$ so $\{k\} \times \mathbf{D}$ contains at least one element of $\mathrm{ext}(\mathcal{B}_0(\mathbf{D})^*)$. Now fix z and w in $\mathbf{D}$. Let ψ be an analytic automorphism of $\mathbf{D}$ such that $\psi(w) = z$. We define an automorphism φ of $K \times \overline{\mathbf{D}}$ by $\varphi(k, \omega) = \psi(\omega)$. Then $f \to f \circ \varphi$ is an isometry of A onto itself. Hence $\mathrm{ext}(\mathcal{B}_0(A)^*) = \mathcal{D}_A$ (one could prove the last equation more directly by constructing specific functions from $\mathcal{B}_0(A)$ peaking at a given point of $\mathcal{D}_A$, but we chose different arguments to show that the result can be extended to other function algebras with big group of isometries).

It follows that for any isometry $T : \mathcal{B}_0(A) \to \mathcal{B}_0(A)$ we have

$$\frac{\frac{\partial}{\partial w}(Tf)(k,w)}{H(k,w)} = \kappa(k,w)\frac{\frac{\partial}{\partial w'}f(k',w')}{H(k',w')}$$

for $k \in K, w \in \mathbf{D}$, where $|\kappa| \equiv 1$, $H(k, w)$ is the norm of the derivation at $(k, w) \in K \times \mathbf{D}$ and the map $\phi(k, w) = (k', w')$ is an automorphism of $K \times \mathbf{D}$.

Let k_1 and k_2 be distinct elements of K, and w_1, w_2 be elements of $\mathbf{D}$, then the norm distance between the derivation at (k_1, w_1) and at (k_2, w_2) is equal to 2. On the other hand for any fixed $k \in K$ the weak $*$ topology and the norm topology on $\{k\} \times \mathbf{D}$ coincide, therfore the sets of the form $\{k\} \times \mathbf{D}$, $k \in K$ are precisely the connected components of $K \times \mathbf{D}$ with respect to the norm topology, so they are preserved by T^*. Hence the map ϕ is of the following form

$$\phi(k, w) = (\phi_1(k), \phi_2(k, w)),$$

where, for any $k \in K$, $\phi_2(k, \cdot)$ is an automorphism of the unit dis So for any $k \in K$, T produces an isometry from $\mathcal{B}_0(\mathbf{D}) \equiv \mathcal{B}_0(A_{|\{\phi(k)\} \times \mathbf{D}})$ onto $\mathcal{B}_0(\mathbf{D}) \equiv \mathcal{B}_0(A_{|\{k\} \times \mathbf{D}})$. Since any isometry of the classical Bloch space is canonical [4] $\phi_2(k, \cdot)$ is analytic, $\kappa(k, \cdot)$ is constant, and

$$Tf(k, w) = \kappa(k) f(\phi_1(k), \phi_k(w)), \tag{5}$$

for all $f \in \mathcal{B}_0(A)$, where $|\kappa| \equiv 1$, ϕ_1 is an automorphism of K, and ϕ_k is an analytic automorphism of the unit disc for $k \in K$. Note that in (5) we use the notation of abstract Bloch spaces, that is we identify two functions which have the same derivatives at any point.

Proof of the Theorem. Let $A = A(\Omega_1)$, $B = A(\Omega_2)$, $\Gamma = \{z \in \mathbf{C}^n : \|z\| = 1\}$, and let $T : \mathcal{B}_0(A) \to \mathcal{B}_0(B)$ be an isometry. Assume that Ω_1 is totally homogenous, this means that for any $D_1, D_2 \in \mathcal{D}_A$ there is an isometry S of A onto itself such that $D_1 = D_2 \circ S$. Hence $\mathcal{D}_A = \operatorname{ext}(\mathcal{B}_0(A)^*) \cong \Omega_1 \times \Gamma$ (this is the only point where we use the assumption that Ω_1 is totally homogenous). Let us also assume that Ω_2 is totally homogenous. By what we said at the beginning of this section T is of the following form

$$\frac{D_\xi(Tf)(z)}{H(z,\xi)} = \frac{D_{\phi_2(z,\xi)} f(\phi_1(z))}{H(\phi_1(z), \phi_2(z,\xi))}, \tag{6}$$

for $f \in \mathcal{B}_0(A)$, where ϕ_1 is a homeomorphism of Ω_2 onto Ω_1, $D_\xi(f)(z)$ is the derivative of f at the point z in the direction ξ, and $H(z,\xi)$ is the norm of the map $D_\xi(\cdot)(z)$.

Hence for f^2 we get

$$\frac{D_\xi(Tf^2)(z)}{H(z,\xi)} = \frac{D_{\phi_2(z,\xi)} f^2(\phi_1(z))}{H(\phi_1(z), \phi_2(z,\xi))} = 2f(\phi_1(z)) \frac{D_{\phi_2(z,\xi)} f(\phi_1(z))}{H(\phi_1(z), \phi_2(z,\xi))}. \tag{7}$$

Dividing (7) by (6) we get

$$f \circ \phi_1(z) = \frac{D_\xi(Tf^2)(z)}{2D_\xi(Tf)(z)}$$

for all f, z and ξ such that the terms involved in (6) and (7) are not equal to zero. Hence $\phi_1 : \Omega_2 \to \Omega_1$ is an analytic map.

Composing T with the map $f \to f \circ \phi_1^{-1}$ we can assume, without loss of generality, that $\Omega_1 = \Omega_2 = \Omega$, and $\phi_1 = \mathrm{id}_\Omega$, so that T is of the following form

$$\frac{D_\xi(Tf)(z)}{H(z,\xi)} = \frac{D_{\phi_2(z,\xi)} f(z)}{H(z, \phi_2(z,\xi))},$$

for $f \in \mathcal{B}_0(A)$.

We now use an argument adopted from [7]. Let $f \in B_0(A)$ and put $h = T(f^2) - 2fTf$. Since $D_\xi \circ T(\cdot)(z)$ is a derivation at z we get

$$D_\xi(h)(z) = 2f(z) D_\xi \circ T(f) - 2f(z) D_\xi(Tf) - 2D_\xi(f) Tf = -2D_\xi(f) Tf(z).$$

Hence, for $f(z) = z_1$ we get

$$D_{e_1}(h) = -2Tf \text{ and } D_{e_j}(h) = 0, j = 2, ..., n.$$

So h, as well as Tf are functions of z_1 alone. By a similar argument Tf^m is a function of z_1 alone for each positive integer m.

Thus the subspace of $\mathcal{B}_0(A)$ consisting of functions depending on the first variable is mapped onto itself. Hence T^* maps the set of derivations, that anihilate this subspace, onto itself. The same is true for any $j = 1, ..., n$. Hence $\phi_2(z,\xi) = \mu_z(\xi)\xi$, where $|\mu| = 1$ and since $\phi_2(z,\cdot)$ is linear, we get

$$\phi_2(z,\xi) = \mu_z \xi,$$

for $z \in \Omega$, $\xi \in \Gamma$, where $|\mu_z| = 1$. Hence

$$D_\xi(Tf)(z) = \mu_z D_\xi(f),$$

for $f \in \mathcal{B}_0(A)$; therefore $z \to \mu_z$ is an analytic map of absolute value 1, so it is constant. We conclude that $Tf = \mu f$.

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