Contemporary Mathematics

Function Spaces - selected open problems

Krzysztof Jarosz

ABSTRACT. We discuss briefly selected open problems concerning various function spaces.

1. Introduction

We discuss several open problems concerning various function spaces and algebras. All these problems can be phrased using rather elementary language without introducing a lot of technical definitions and notation, they also seem to be very natural yet remain open for many years.

2. Almost Corona?

Assume A is a Banach algebra, G is a linear multiplicative functional on A, and Δ is a linear functional with a very small norm. Then $F = G + \Delta$ is obviously almost multiplicative:

$$|F(fg) - F(f)F(g)| \leq \varepsilon \|f\| \|g\| .$$

Is this the only way to construct an almost multiplicative functional? In other words, is any almost multiplicative functional near a multiplicative one? The answer is an easy *yes* for the algebras $C(K)$ of all continuos functions on a compact set K and easy yet still somewhat surprising *no* in general, e.g., for the convolution radical Banach algebra $L^1[0,1]$ [**23**]. The answer is *yes* for the disc algebra and some similar separable algebras in $\mathbb{C}^n$, but in this case the proof is far from trivial [**18**].

For most classical algebras the question remains open. The most interesting open case is the H^∞ algebra. The celebrated Carleson Theorem states that H^∞ does not have a corona: any multiplicative linear functional on H^∞ is close to the disc. We do not know whether it has an almost corona, i.e., whether there are almost multiplicative functionals far from the disc. We know more about the quotient algebra H^∞/BH^∞ which may or may not have such property depending on the distribution of zeros of Blaschke product B [**18, 19**].

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3. Finite codimensional ideals in function algebras

Assume A is a complex unital Banach algebra and G is a nonzero linear multiplicative functional on A. Obviously $G(a) \neq 0$ for any invertible $a \in A$ since $G(a)\,G\left(a^{-1}\right) = G(\mathbf{1}) = 1$. The classical Gleason-Kahane-Żelazko Theorem [**12**] states that the opposite is true as well: any linear functional whose kernel does not contain invertible elements is proportional to a multiplicative functional. Since there is a one-to-one correspondence between kernels of linear functionals and subspaces of codimension one, we can rephrase this theorem as follows; to concentrate on a specific case let us assume that $A = A(\mathbb{D})$ is the disc algebra and that $M \subset A$ is a subspace of codimension one. We have:

$$\left(\forall f \in M \quad \exists z \in \overline{\mathbb{D}} \qquad f(z) = 0\right) \Longrightarrow \left(\exists z \in \overline{\mathbb{D}} \quad \forall f \in M \qquad f(z) = 0\right).$$

It is very natural to ask whether the above implication holds true for more general subspaces, for example for all subspaces of finite codimension. It is indeed the case for $C(K)$ algebras [**14**], but surprisingly the question is open in general, even for the disc algebra and for subspaces of codimension two. No counterexample is known either. There are only few very partial results known [**14, 18, 19**].

4. Riemann Mapping Theorem in $\mathbb{C}^n$?

The classical Riemann Mapping Theorem states that a nontrivial simply connected domain Ω in $\mathbb{C}$ is holomorphically homeomorphic with the open unit disc $\mathbb{D}$. Furthermore, if the boundary of Ω is homeomorphic with the unit circle, then that homeomorphism from $\mathbb{D}$ onto Ω can be extended to the boundary. It is very well known that simply connected domains in C^n, for $n > 1$, are generally not holomorphically equivalent.

Are "similar" domains "almost" holomorphically equivalent? That may of course depend on the meaning of these two words. For Banach spaces A, B the closeness is normally measured by the Banach-Mazur distance: $d_{B-M}(A, B) = \inf\left\{\|T\|\left\|T^{-1}\right\| : T : A \to B\right\}$. For domains Ω_1, Ω_2 in $\mathbb{C}$ the quasiconformal distance $d_q(\Omega_1, \Omega_2)$ is the most natural measure of closeness. In a series of papers on deformations of Banach algebras, culminating in his 1985 paper [**25**], R. Rochberg proved that these two concepts coincide:

THEOREM 4.1. *Let $S_i, i = 1, 2$ be bordered one dimensional Riemann surfaces and $A(S_i)$ be the algebras of functions continuous on S_i and analytic on $intS_i$. Then $d_q(S_1, S_2) < 1 + \varepsilon$ iff $d_{B-M}(A(S_1), A(S_1)) < 1 + \varepsilon'$, where ε and ε' tend to zero simultaneously.*

For example for $S_\varepsilon = \{z \in \mathbb{C} : 1 < |z| < 2 + \varepsilon\}$ all these domains are not holomorphically equivalent, but $d_q(S_0, S_\varepsilon) \simeq d_{B-M}(A(S_0), A(S_\varepsilon))$ as $\varepsilon \to 0$.

It would be most interesting to know if similar result is true in $\mathbb{C}^n$. For example, if a uniform algebra, which is close with respect to the Banach-Mazur distance to the algebra $A(B_n)$ or to $A(\mathbb{D}^n)$ must automatically be isomorphic with the original algebra. The above mentioned Rochberg's Theorem implies that it is true for $n = 1$, however we know very little about $n > 1$ (see [**17**] for a partial result).

5. Do multipliers determine the complete norm topology?

Let A be the disc algebra or the algebra of continuous functions defined on a compact subset of the complex plane. Assume that the operator M of multiplication by the identity function $M(f)(z) = zf(z)$ is continuous with respect to *some* complete norm $|\cdot|$ on A. It turns out that the norm $|\cdot|$ must then be automatically equivalent to the standard sup norm on A - the operator M *determines the complete norm topology* of A. This problem was first investigated by A. R. Villena [**27**] and then by the author [**20, 21**], and several other mathematicians. In particular we know the following result.

THEOREM 5.1. *Let A be a unital, semisimple, commutative Banach algebra. Then an operator M_a of multiplication by an element a of A determines the complete norm topology of A if and only if the codimension of $(a+\lambda e)A$ is finite for each scalar λ such that $(a+\lambda e)$ is a divisor of zero.*

There are similar results known for various other spaces of continuous functions and also for group translation determining the norm topology of $L^p(G)$ [**2, 3, 10, 24**]. All these proofs directly or indirectly consider evaluation functionals $A \ni f \longmapsto f(x)$, where x is a fixed point in the domain of f, and investigate the continuity of such functionals with respect to both norms. While $f(x)$ is not well defined for $f \in L^p(G)$ and $x \in G$ we consider the Fourier transformations of f and in such setting the translation operator become a multiplication by a continuous function. On the other hand, in cases when the point evaluation is not available, the question is open. For example we do not even know if multiplication by $I(t) = t$ determines the complete norm topology of $L^p([0,1])$.

6. Separating = Biseparating?

A linear map $T : A \to B$ between function spaces A, B is called separating if

$$ab = 0 \Rightarrow T(a)T(b) = 0, \qquad \text{for all } a, b \in A;$$

it is called biseparating if $T^{-1} : B \to A$ exists and is also separating.

The concept has its source in the theory of topological lattices, but is also an important generalization of multiplicative maps on Banach algebras with application to many other areas. One of such areas includes composition operators, considered in the ergodic theory and harmonic analysis since any composition operator is separating. Separating and biseparating maps have been studied intensively by many authors, an interested reader may want to start with a recent monograph by Y. Abramovich and A. K. Kitover [**1**]. In general, without extra assumptions, such maps may be discontinuous.

In spite of all this, we still do not know the answer to the following question, which seems to be the most basic one.

QUESTION 6.1. *Let $K_i, i = 1,2$ be topological spaces and let $C(K_i)$ be the vector space of all scalar valued continuous functions on K_i. Assume $T : C(K_1) \to C(K_2)$ is an invertible separating linear map. Does it follow that T is a weighted composition map?*

Notice that we do not assume that T is biseparating, or that the spaces K_i are compact, or that the functions are bounded, or that T is continuous with respect to some topology. If we made any of such, or similar additional assumptions, we would be able to prove that T is indeed a weighted composition map, see for example [**16**] or [**9**].

7. Algebras of analytic functions on planer sets

We have a number of new intriguing open questions in the area of Banach algebras. We also have surprisingly many "very simple" questions about "very simple algebras" which remain open for several decades. Let us mention just two:

Question 7.1. *Let K be a compact subset of the complex plane, let $R(K)$ be the closure with respect to the sup norm topology on K of the algebra of rational functions with poles off K, let $A(K)$ be the algebra of functions which are continuous on K and analytic on int$K$, finally let A be a function algebra such that $R(K) \subset A \subset A(K)$. Is the maximal ideal space of A equal to K?*

It has been very well known for a very long time that the maximal ideal spaces of both $R(K)$ and $A(K)$ are equal to K, however in general the maximal ideal space of an "intermediate" Banach algebra may not be equal to the maximal ideal space of the other two algebras.

Question 7.2. *Is there a uniform algebra $A \subset C(\mathbb{D})$, with $\mathbb{D}$ = unit disc = the maximal ideal space of A, and such that the Shilov boundary of A is contained in the interior of $\mathbb{D}$?.*

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Department of Mathematics and Statistics, Southern Illinois University Edwardsville, Edwardsville, IL 62026, USA
E-mail address: kjarosz@siue.edu