TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 2, February 1985

FINITE CODIMENSIONAL IDEALS IN FUNCTION ALGEBRAS

BY
KRZYSZTOF JAROSZ

ABSTRACT. Assume S is a compact, metric space and let M be a finite codimensional closed subspace of a complex space $C(S)$. In this paper we prove that if each element from M has at least k zeros in S, then for some $s_1, \ldots, s_k \in S$, $M \subseteq \{f \in C(S)\colon f(s_1) = \cdots = f(s_k) = 0\}$.

0. Let M be a subspace of codimension 1 in a commutative, complex Banach algebra with unit. A. M. Gleason [**4**] and independently J. P. Kahane and W. Żelazko [**5**] proved that M is an ideal if and only if M consists of noninvertible elements or equivalently if and only if each element x in M belongs to an ideal I_x which may depend on x.

In [**8** and **9**] C. R. Warner and R. Whitley considered the following more general problem:

Problem 1. Let A be a commutative, complex Banach algebra with unit and let M be a closed n codimensional subspace of A. Assume that every element from M belongs to at least n different maximal ideals. Is M then an ideal?

They showed that the answer to the above problem is positive if A is any of $L^1(\mathbf{R})$ and $C(S)$ where S is a compact subset of $\mathbf{R}$, and they also gave simple examples showing that this does not hold in general (e.g.: S a compact Hausdorff space such that there is an $s_0 \in S$ with $\{s_0\}$ not a G_δ set and M any subspace of $\{f \in C(S)\colon f(s_0) = 0\}$).

Chang-Pao Chen [**2**] generalized the above results to some other commutative Banach algebras.

The original Gleason-Kahane-Żelazko work has also been extended in another direction by B. Aupetit [**1**], S. Kowalski and Z. Słodkowski [**6**], and by M. Roitman and Y. Sternfeld [**7**].

C. R. Warner and R. Whitley asked whether the answer to the above problem is positive for $A = C(D)$ where D is a disc.

In this paper we give a positive answer to this problem for $A = C(S)$, S being an arbitrary metric space (Theorem 1). We also consider the following problem:

Problem 2. Let A be a commutative complex Banach algebra with unit and let M be a finite codimensional subspace of A consisting of noninvertible elements. Is M contained in some maximal ideal of A?

We prove that the answer to the second problem is positive for $A = C(S)$, for S any compact, Hausdorff space (Theorem 2). The author does not know any function algebra A which fails to possess the property described in Problem 2.

In the paper we consider only complex algebras. Simple examples prove that in the real case the original Gleason-Kahane-Żelazko theorem does not work [**3**].

Received by the editors March 2, 1984.
1980 *Mathematics Subject Classification*. Primary 46J10.

1. THEOREM 1. *Let S be a compact, Hausdorff space such that each point of S is a G_δ set, let M be a closed finite codimensional subspace of $C(S)$ and let k be a positive integer. Suppose that each element from M has at least k different zeros in S. Then there are k different elements $s_1, \dots, s_k$ in S such that*

$$M \subset \{f \in C(S) \colon f(s_1) = \cdots = f(s_k) = 0\}.$$

The theorem answers Problem 1 for $A = C(S)$ with S a metric, compact space:

COROLLARY. *Let S be a compact, Hausdorff space such that each point of S is a G_δ set, let M be a closed subspace of $C(S)$ with* $\operatorname{codim}(M) = k < +\infty$. *Suppose that each element from M has at least k different zeros in S. Then M is an ideal of the algebra $C(S)$.*

PROOF. By Theorem 1 there are k different elements $s_1, \dots, s_k$ in S such that

$$M \subset M' = \{f \in C(S) \colon f(s_1) = \cdots = f(s_k) = 0\}.$$

We have $\operatorname{codim}(M) = k = \operatorname{codim}(M')$ and hence $M = M'$.

As the main step in proving Theorem 1 we prove the following theorem, valid for any compact, Hausdorff space S.

THEOREM 2. *Let S be a compact, Hausdorff space and let M be a finite codimensional subspace of $C(S)$. Suppose that each element from M has at least one zero in S. Then there is an s_0 in S such that*

$$M \subset \{f \in C(S) \colon f(s_0) = 0\}.$$

PROOF OF THEOREM 2. Without loss of generality we assume that M is a closed subspace of $C(S)$. For a compact subset K of the complex plane we denote by $\mathcal{P}(K)$ the linear space of all restrictions to K of polynomials; if K is an infinite set we define the norm on $\mathcal{P}(K)$ by $|p| = \max_{z\in K} |p(z)|$, and by $P(K)$ we denote the completion of $\mathcal{P}(K)$ in this norm. We regard $P(K)$ as a closed subalgebra of $C(K)$.

LEMMA 1. *Let K be a compact subset of the real line in the complex plane and let B be a linear subspace of $\mathcal{P}(K)$. Assume that there is a p_0 in B with $p_0'(z) \neq 0$ for $z \in K$ and that any element from B has at least one zero in K. Then there is a $z_0 \in K$ such that*

$$B \subset \{p \in \mathcal{P}(K) \colon p(z_0) = 0\}.$$

PROOF. It is sufficient to prove that there is a common zero for any finite subset of B, so without loss of generality we can assume that B is finite dimensional and we let $\{p_0, p_1, \dots, p_m\}$ be a linear basis of B. Next we can assume that there is an open, connected, bounded neighbourhood G of K in $\mathbf{C}$ such that $\inf_{z\in G} |p_0'(z)| \geq 1$. We define an entire function $F \colon \mathbf{C} \times \mathbf{C}^m \to \mathbf{C}$ by

$$F(z, w) = p_0(z) + \sum_{j=1}^{m} w_j p_j(z) \quad \text{where } w = (w_1, \dots, w_m).$$

We assume that there is no common zero for B and we define, by induction two sequences: $\{U_j\}_{j=0}^{\infty}$ of open, connected subsets of $\mathbf{C}^m$, and $\{g_j\}_{j=0}^{\infty}$ of analytic functions defined on U_j, respectively, such that

$$\varnothing \neq \overline{U}_{j+1} \subset U_j \quad \text{and} \quad g_j(U_{j+1}) \cap K = \varnothing \quad \text{for } j = 0, 1, \dots,$$

and

$$F(g_j(w), w) = 0 \quad \text{for } j = 0, 1, \dots \text{ and } w \in U_j$$

and

$$g_j(U_j) \cap g_i(U_j) = \varnothing \quad \text{for any } i < j.$$

Let $V_0 = \{w \in \mathbf{C}^m \colon \sum_{j=1}^m |w_j| \sup_{z\in G} |p_j'(z)| < 1\}$. By our assumption for any $w \in \mathbf{C}^m$ there is a z in K such that $F(z, w) = 0$ and for any $(z, w) \in G \times V_0$ we have $\partial F(z, w)/\partial z \neq 0$, so by the rank theorem there is an open, nonvoid, connected subset U_0 of V_0 and an analytic function $g_0 \colon U_0 \to G$ such that $F(g_0(w), w) = 0$ for $w \in U_0$.

Suppose we have defined U_j and g_j for $j = 0, 1, \dots, p$. If g_p were constant then $g_p(w)$ would be a common zero for B, so g_p is a nonconstant analytic function defined on a connected set; so $g_p(U_p)$ is open. Hence there is an open, nonvoid set V_{p+1} such that $\overline{V}_{p+1} \subset U_p$ and $g_p(\overline{V}_{p+1}) \cap K = \varnothing$. In the same manner as before we prove that there is an open, nonvoid, connected subset U_{p+1} of V_{p+1} and an analytic function $g_{p+1} \colon U_{p+1} \to G$ such that $F(g_{p+1}(w), w) = 0$ for $w \in U_{p+1}$ and $g_{p+1}(w) \in K$ for some $w \in U_{p+1}$. Taking U_{p+1} smaller if necessary we can assume that

$$g_{p+1}(U_{p+1}) \cap \bigcup_{j=0}^{p} g_j(U_{p+1}) = \varnothing\,.$$

Let $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_m) \in \bigcap_{j=0}^\infty U_j$. The function $z \mapsto F(z, \tilde{w})$ is a polynomial which has infinitely many different zeros: $g_0(\tilde{w}), g_1(\tilde{w}), \dots$ in G, so it is the zero function, which proves that p_0 is a linear combination of $p_1, \dots, p_m$ and contradicts the fact that $\{p_0, p_1, \dots, p_m\}$ is a linear basis of B.

LEMMA 2. *Let K be a compact subset of the real line in the complex plane and let M be a finite codimensional subspace of $C(K)$. Suppose that each element from M has at least one zero in K. Then there is a $z_0 \in K$ such that*

$$M \subset \{f \in C(K) \colon f(z_0) = 0\}.$$

PROOF. Without loss of generality we can assume that M is a closed subspace of $C(K)$ and, since the lemma is trivial for K being a finite set, we can assume that K is an infinite subset of the complex plane. Put $M_0 = \mathcal{P}(K) \cap M$. Since $P(K) = C(K)$ and M is a finite codimensional closed subpsace of $C(K)$, then M_0 is a finite codimensional closed subspace of $\mathcal{P}(K)$ and M_0 is dense in M. To end the proof it is sufficient to show that M_0 has a common zero in K. Put

$$n = \max\{j \in \mathbf{N} \colon \forall p \in M_0 \ \exists z \in K \ p^{(j)}(z) = 0\}.$$

Since M_0 consists of polynomials evidently $n < \infty$. By Lemma 1 there is a $z_0 \in K$ such that

$$\{p^{(n)} \colon p \in M_0\} \subset \{p \in \mathcal{P}(K) \colon p(z_0) = 0\}.$$

Assume $n \geq 1$. M_0 is a finite codimensional closed subspace of the normed space $\mathcal{P}(K)$ which is contained in the kernel of a discontinuous functional $\mathcal{P}(K) \ni p \mapsto p^{(n)}(z_0)$, which is impossible. Therefore $n = 0$ and this proves

$$M_0 \subset \{p \in \mathcal{P}(K) \colon p(z_0) = 0\}.$$

LEMMA 3. *Let μ be a finite, positive, Borel measure on a compact subset of the Euclidean space $\mathbf{R}^n$. For any subspace E of $\mathbf{R}^n$ we define a measure μ_E on E*

by $\mu_E(B) = \mu(\pi_E^{-1}(B))$ *for any Borel subset* B *of* E *where* π_E *is the orthogonal projection from* $\mathbf{R}^n$ *onto* E. *Assume that for every one dimensional subspace* E *of* $\mathbf{R}^n$ *the measure* μ_E *has at least one atom. Then* μ *has at least one atom.*

PROOF. We prove the lemma by induction. It is trivial for $n = 1$. Assume that it is true for $n-1$ but not for n and let μ be as in the lemma. Fix an $n-1$ dimensional subspace E_0 of $\mathbf{R}^n$ and denote $\mu_0 = \mu_{E_0}$. For any one dimensional subspace E of E_0 the measure $(\mu_0)_E = \mu_E$ has at least one atom so by the induction hypothesis μ_0 has an atom. Thus we have shown that for any $n-1$ dimensional subspace E_0 of $\mathbf{R}^n$ the measure μ_{E_0} has an atom. By the definition of μ_E this means that for any nonzero vector v in $\mathbf{R}^n$ there is a segment l in $\mathbf{R}^n$ parallel to v and such that $\mu(l) > 0$. Hence there is an uncountable family $\mathcal{A}$ of pairwise nonparallel segments in $\mathbf{R}^n$, each of positive μ-measure. There is an infinite subfamily $\mathcal{A}_0$ of $\mathcal{A}$ such that each segment in $\mathcal{A}_0$ has μ-measure greater than some positive constant c. Assuming that μ has no atoms, we get that the measure of the common part of any two elements of $\mathcal{A}_0$ is zero so $\bigcup \mathcal{A}_0$ contains a subset of infinite μ-measure which contradicts the assumption μ is finite.

LEMMA 4. *Let* S *be a compact subset of the Euclidean space* $\mathbf{R}^n$ *and let* M *be a finite codimensional closed subspace of* $C(S)$. *Assume that any* f *in* M *has at least one zero in* S. *Then there is an* $s_0 \in S$ *such that*

$$M \subset \{f \in C(S) \colon f(s_0) = 0\}.$$

PROOF. Let measures $\mu_1, \dots, \mu_m$ on S constitute a linear basis of $(C(S)/M)^*$. Put $\mu_0 = \sum_{i=1}^m |\mu_i|$. Let E be a one dimensional subspace of $\mathbf{R}^n$; by an appropriate choice of coordinates we can assume E is the first coordinate axis. Put

$$M_E = \{f \in M \colon f \text{ depends only on the first coordinate}\}.$$

M_E can be regarded as a finite codimensional subspace of $C(\pi_E(S))$. Since each element of M_E has at least one zero in $\pi_E(S) \subset \mathbf{R}$ by Lemma 2, a linear combination of the measures $(\mu_i)_E$ for $i = 1, 2, \dots, m$ gives a Dirac measure; so at least one of the measures $(\mu_i)_E$ has an atom and consequently $(\mu_0)_E$ has an atom. Since E was an arbitrary one dimensional subspace of $\mathbf{R}^n$, Lemma 3 ensures that μ_0, hence at least one of the measures $\mu_1, \dots, \mu_m$, has an atom. We can assume that $\mu_1 = \delta_0 + \nu$, $\nu(\{\mathbf{0}\}) = 0$ and $\mu_i(\{\mathbf{0}\}) = 0$ for $i = 2, \dots, m$, where δ_0 is a Dirac measure concentrated at the point $\mathbf{0} = \{0, \dots, 0\} \in S \subset \mathbf{R}^n$. Put $I = \{s = (s_1, \dots, s_n) \in S \colon s_1 = 0\}$.

For any $0 \le t < \infty$ let $\varphi_t \colon \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\varphi_t(s_1, s_2, \dots, s_n) = \left(t \sum_{i=2}^{n} |s_i| + s_1, s_2, \dots, s_n \right).$$

Any such map is an automorphism of $\mathbf{R}^n$ and $\varphi_{t_1}(I) \cap \varphi_{t_2}(I) = \{\mathbf{0}\}$ if $t_1 \neq t_2$. Hence there is $t_0 \in \mathbf{R}^+$ such that $|\mu|(I) = 0$ where $\mu = |\nu| + |\mu_1| + \cdots + |\mu_m|$. On the other hand any automorphism φ of $\mathbf{R}^n$ defines an isometric isomorphism of $M \subset C(S)$ onto $M_\varphi = \{f \circ \varphi \colon f \in M\} \subset C(\varphi^{-1}(S))$. So by applying an automorphism φ_{t_0} (it is just a curvilinear coordinate) we can assume that

$$|\nu|(I) = 0 \quad \text{and} \quad |\mu_i|(I) = 0 \quad \text{for } i = 2, \dots, m. \tag{1}$$

We end the proof of the lemma by an induction with respect to $m = \operatorname{codim}(M)$. Put

$$M' = \{f \in C(S)\colon \mu_i(f) = 0 \text{ for } i = 2, \dots, m\}.$$

M' is an $m-1$ codimensional subspace of $C(S)$ so we have two possibilities:

1. Any element from M' has at least one zero in S.
2. There is an f_0 in M' without zeros in S.

If the first possibility holds, then by the induction assumption there is an $s_0 \in S$ such that $M \subset M' \subset \{f \in C(S)\colon f(s_0) = 0\}$, so we can restrict ourselves to examining the second possibility. We assume that it prevails and deduce a contradiction. There is no loss of generality in supposing that ν is not contained in the linear span of $\mu_2, \dots, \mu_m$ (otherwise $M \subset \{f \in C(S)\colon f(\mathbf{0}) = 0\}$). Then there is a $g_0 \in M'$ with $\nu(g_0) \neq 0$ and we can assume $\|g_0\| < \inf_{s\in S} |f_0(s)|$. Moreover, in the case that $\nu(f_0) \neq 0$ we can require $\|g_0\|$ to be small enough that $|\nu(g_0)| < |\nu(f_0)|$. Therefore in any case we shall have $\nu(g_0 + f_0) \neq 0$. Put $h_0 = f_0 + g_0 \in M' \cap (C(S))^{-1}$. Notice that by putting $\{f/h_0\colon f \in M\}$, $\{f/h_0\colon f \in M'\}$ and $h_0\nu$, $h_0\mu_i$ for $i = 2, \dots, m$ in the place of M, M', ν and μ_i for $i = 2, \dots, m$, respectively, we can assume without loss of generality that

$$\mathbf{1} \in M' \quad \text{and} \quad \nu(\mathbf{1}) \neq 0.$$

Let π be an orthogonal projection on the first coordinate, and put as before $M_E = \{f \in M\colon f \text{ depends only on the first coordinate}\}$. By Lemma 2 there is a linear combination

$$\lambda_1\delta_0 + \lambda_1 \cdot (\nu)_E + \sum_{i=2}^{m} \lambda_i(\mu_i)_E = \delta_t \tag{2}$$

for some $t \in \pi(S)$.

We have again two possibilities:

1. $\lambda_1 = 0$,
2. $\lambda_1 \neq 0$.

If the first one holds, then upon applying (2) to the function $\mathbf{1} \in M'$ we get the contradiction

$$0 = \sum_{i=2}^{m} \lambda_i\mu_i(\mathbf{1}) = \sum_{i=2}^{m} \lambda_i(\mu_i)_E(\mathbf{1}) = \delta_t(\mathbf{1}) = 1.$$

If the second possibility holds, then from evaluating (2) at $\{\mathbf{0}\}$ we get via (1) that $t = 0$ and $\lambda_1 = 1$, and so

$$\sum_{i=2}^{m} \lambda_i(\mu_i)_E = -(\nu)_E.$$

Then applying, as before, the above to the function $\mathbf{1}$, we get

$$0 = \sum_{i=2}^{m} \lambda_i(\mu_i)_E(\mathbf{1}) = -(\nu)_E(\mathbf{1}) = -\nu(\mathbf{1}) \neq 0.$$

The contradictions above end the proof of the lemma and now we are ready to end the proof of Theorem 2. We can assume S is a closed subset of some Tichonov cube $[0,1]^I$, where I is a set of indices. For any finite subset I_0 of I let

$$M(I_0) = \{f \in M\colon f \text{ depends only on the coordinates from } I_0\}$$

and

$$S(I_0) = \{s \in S : \forall f \in M(I_0)\ f(s) = 0\}.$$

Let $I_1, \ldots, I_m$ be any finite subsets of I and put $I_0 = \bigcup_{i=1}^m I_i$. We have $S(I_0) \subset \bigcap_{i=1}^m S(I_i)$. By Lemma 4 the set $S(I_0)$ is nonvoid and evidently compact. Hence there is an $s_0 \in \bigcap\{S(I_0): I_0 \subset I, \operatorname{card}(I_0) < \infty\}$. Notice that since $\tilde{C}(S) = \{f \in C(S): f$ depends on a finite number of coordinates only$\}$ is a dense subspace of $C(S)$ and M is a closed subspace of finite codimension, $\tilde{C}(S) \cap M$ is a dense subspace of M. But we have $\tilde{C}(S) \cap M \subset \{f \in C(S): f(s_0) = 0\}$ and hence $M \subset \{f \in C(S): f(s_0) = 0\}$.

PROOF OF THEOREM 1. Let S and M be as in the theorem and let $s_1, \ldots, s_l \in S$ be all of the common zeros of M. By Theorem 2, $l \geq 1$. Assume $l < k$ and put $n = \operatorname{codim}(M)$. We have $M \not\subset \{f \in C(S): f(s_1) = \cdots = f(s_l) = 0\}$ so $n > l$. Let $\mu_1 = \delta_{s_1}, \ldots, \mu_l = \delta_{s_l}$ together with $\mu_{l+1}, \ldots, \mu_n \in M^\perp$ be a basis in the n-dimensional space $[C(S)/M]^* = M^\perp$. We can replace each μ_j by

$$\mu_j - \sum_{i=1}^{l} \mu_j(\{s_i\})\delta_{s_i} \qquad (j = l+1, \ldots, n)$$

and thereby assume without loss of generality that

$$\sum_{j=l+1}^{n} |\mu_j|(\{s_1, \ldots, s_l\}) = 0. \tag{3}$$

Form

$$M' = \{f \in C(S): \mu_j(f) = 0 \text{ for } j = l+1, \ldots, n\}.$$

If s were a common zero of M', then δ_s would be in the linear span of $\{\mu_{l+1}, \ldots, \mu_n\}$ and then by (3), $s \in S\backslash\{s_1, \ldots, s_l\}$. Since $M' \supset M$, s would be an $(l+1)$st common zero of M, contrary to the definition of l. Hence no such s exists, so by Theorem 2 there is some invertible function f_0 in M'. Putting $\{f/f_0: f \in M\}$ in place of M we can assume, as in the proof of Lemma 4, that $f_0 \equiv \mathbf{1}$. The measures $\mu_{l+1}, \ldots, \mu_n$ are linearly independent elements of the dual space $A = \{f \in C(S): f(s_i) = 0$ for $i = 1, \ldots, l\}$, so there are $f_{l+1}, \ldots, f_n$ in A such that

$$\mu_i(f_i) = 1 \quad \text{for } i = l+1, \ldots, n$$

and

$$\mu_i(f_j) = 0 \quad \text{for } i \neq j;\ i, j = l+1, \ldots, n.$$

Put $K = \max\{\|f_j\|: l+1 \leq j \leq n\}$, fix $\varepsilon > 0$ with $n\varepsilon K < 1$ and let $V \subset S$ be an open neighbourhood of the set $\{s_1, \ldots, s_l\}$ such that

$$\sum_{j=l+1}^{n} |\mu_i|(V) < \varepsilon.$$

By our assumption $\{s_1, \ldots, s_l\}$ is a G_δ subset of S so there is a real valued, continuous function g_0 on S such that

$$g_0^{-1}(1) = \{s_1, \ldots, s_l\}, \qquad g_0|_{S\backslash V} \equiv 0,$$

and

$$0 \leq g_0(s) < 1 - \varepsilon \sum_{j=l+1}^{n} |f_j(s)| \quad \text{for } s \in V\backslash\{s_1, \ldots, s_l\}.$$

Put

$$h_0 = \mathbf{1} - g_0 + \sum_{j=l+1}^{n} \mu_j(g_0) f_j.$$

We have $h_0(s_j) = 0$ for $j = 1, \dots, l$ and (recalling that $\mathbf{1} \in M'$) $\mu_j(h_0) = 0$ for $j = l+1, \dots, n$, so $h_0 \in M$. To get a contradiction it is sufficient now to show that h_0 has only l zeros in S, namely $s_, \dots, s_l$. We have

$$|h_0(s)| \geq 1 - \sum_{j=l+1}^{n} |\mu_j|(V)\|f_j\| > 0 \quad \text{for } s \in S \backslash V$$

and

$$|h_0(s)| \geq 1 - g_0(s) - \varepsilon \sum_{j=l+1}^{n} |f_j(s)| > 0 \quad \text{for } s \in V \backslash \{s_1, \dots, s_l\}.$$

2.

REMARK. The assumption in Problem 2 that M is of finite codimension is essential, as the following example, due to W. Żelazko, shows: Let A be a disc algebra, i.e., $A = P(D)$ where D is the closed unit disc in the complex plane and put $M = \text{span}\{B_1, B_2\}$ where B_1, B_2 are nonconstant Blaschke factors with disjoint zero sets. By the Rouché Theorem any element from M has a zero in D, but M is not contained in any maximal ideal. Since A can be isometrically embedded into $C([0,1])$, M can be also regarded as a two dimensional subspace of $C([0,1])$.

REFERENCES

1. B. Aupetit, *Une généralisation du théorème de Gleason-Kahane-Żelazko pour les algèbres de Banach*, Pacific J. Math. **85** (1979), 11–17.
2. Chang-Pao Chen, *A generalization of the Gleason-Kahane-Żelazko theorem*, Pacific J. Math. **107** (1983), 81–87.
3. N. Farnum and R. Whitley, *Functionals on real* $C(S)$, Canad. J. Math. **30** (1978), 490–498.
4. A. M. Gleason, *A characterization of maximal ideals*, J. Analyse Math. **19** (1967), 171–172.
5. J. P. Kahane and W. Żelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. **29** (1968), 339–343.
6. S, Kowalski and Z. Słodkowski, *A characterization of multiplicative linear functionals in Banach algebras*, Studia Math. **67** (1980), 215–223.
7. M. Roitman and Y. Sternfeld, *When is a linear functional multiplicative*?, Trans. Amer. Math. Soc. **267** (1981), 111–124.
8. C. R. Warner and R. Whitley, *A characterization of regular maximal ideals*, Pacific J. Math. **30** (1969), 277–281.
9. ——, *Ideals of finite codimension in* $C[0,1]$ *and* $L^1(R)$, Proc. Amer. Math. Soc. **76** (1979), 263–267.

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, PKIN, 00-901 WARSAW, POLAND