

FINITE CODIMENSIONAL IDEALS IN BANACH ALGEBRAS WITH ONE GENERATOR

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Let A be a complex Banach algebra and let M be a closed n -codimensional subspace of A with $n < \infty$. A.M. Gleason [1] and, independently, J.P. Kahane and W. Zelazko [4] proved that for $n = 1$, M is an ideal if and only if M consists exclusively of noninvertible elements. Equivalently, a linear functional F is multiplicative if and only if $F(f)$ is contained in the spectrum of f for any f in A . In the following years this result has been extended in various directions. In [3] one can find the history of this subject and a list of open problems. The main question here, still to large extent open, is when and how we can extend the Gleason-Kahane-Zelazko theorem for $n > 1$. In [2] we get a positive result for $A = C(S)$, but the examples given by C.R. Warner and R. Whitley [5,6] show that it fails in general.

In this note we give a partial solution to this question.

Theorem. *Let A be a Banach algebra with one generator and let M be a closed n -codimensional subspace of A with $n < \infty$. Assume that each element of M is contained in at least n distinct maximal ideals of A . Then M is an ideal, namely an intersection of n maximal ideals of A .*

Proof. A is a Banach algebra with one generator so there is an f in A such that the polynomials of f are dense in A . Considering $f + \lambda e$ in place of f , we can assume without loss of generality that $K = \sigma(f)$ is contained in $\{x + iy : x > 0\}$. We can also assume that K is infinite since otherwise $A = C(K)$ and the result (trivial in this case) follows from [2]. For any complex number λ the functions $f^\lambda = \exp(\lambda \ln f)$, $f^{\lambda+1}, \dots, f^{\lambda+n}$ are well defined and linearly independent so there is a nontrivial linear combination of these $n + 1$ functions contained in M . Assume there are two nonproportional such combinations. Then there

Received by the editors on February 11, 1989, and in revised form on October 6, 1989.

is a nontrivial linear combination of functions $f^\lambda, f^{\lambda+1}, \dots, f^{\lambda+n-1}$ in M . We have

$$f^\lambda \sum_{j=0}^{n-1} b_j(\lambda) f^j = \sum_{j=0}^{n-1} b_j(\lambda) f^{\lambda+j} \in M.$$

Our assumption that any element of M is contained in at least n maximal ideals means that the Gelfand transform

$$z^\lambda \sum_{j=0}^{n-1} b_j(\lambda) z^j$$

of

$$f^\lambda \sum_{j=0}^{n-1} b_j(\lambda) f^j$$

has at least n distinct zeros in K . This is not possible since $0 \notin K$ and $\sum_{j=0}^{n-1} b_j(\lambda) z^j$ is a polynomial of degree at most $n-1$. Hence, for any $\lambda \in \mathbf{C}$ there is exactly one n -tuple $a_1(\lambda), \dots, a_n(\lambda)$ of complex numbers, with $a_n(\lambda) \neq 0$ and such that

$$(1) \quad \sum_{j=0}^{n-1} a_j(\lambda) f^{\lambda+j} + f^{\lambda+n} \in M$$

(we assume here, without loss of generality, that $a_n = 1$). We show now that $a_j(\cdot)$ are entire functions. To this end, let F_1, \dots, F_n be continuous functionals on A such that $M = \bigcap_{j=1}^n \ker F_j$. The numbers a_j from (1) are the solutions of a system of n linear equations in n unknowns:

$$\sum_{j=1}^{n-1} a_j(\lambda) F_s(f^{\lambda+j}) + F_s(f^{\lambda+n}) = 0, \quad s = 1, \dots, n.$$

Since this system has exactly one solution, its determinant is never zero and, from Cramer's formula, the solution is a well-defined fraction of combinations of the analytic functions $\lambda \mapsto F_s(f^{\lambda+j})$.

From (1) and the definition of M , for any $\lambda \in \mathbf{C}$ there are distinct complex numbers $c_1(\lambda), \dots, c_n(\lambda)$, all contained in K such that

$$\sum_{j=1}^{n-1} a_j(\lambda) f^{\lambda+j} + f^{\lambda+n} = f^\lambda \prod_{j=1}^n (f - c_j(\lambda)) \in M.$$

Hence, all the functions $a_j(\cdot)$ are bounded and entire, and thus constant, so the functions $c_j(\cdot)$ are also constant.

We proved that there are distinct complex numbers c_1, \dots, c_n in K such that for any nonnegative integer k the element $f^k \prod_{j=1}^n (f - c_j)$ is in M . Since M is closed and linear combinations of f^k are dense in A , this shows that $M_1 = A \prod_{j=1}^n (f - c_j) \subseteq M$. We also have $M_1 \subseteq J = \{f \in A : f(c_j) = 0, j = 1, \dots, n\}$. Polynomials of f are dense in A so M_1 is dense in J , hence $J \subseteq M$ but the codimensions of these subspaces are the same so they are actually equal. \square

Remarks. We assumed in our theorem that there is an f in A such that the linear combinations of nonnegative powers of f are dense in A . This assumption can be weakened. It is enough to assume that there is an f in A such that the linear combinations of powers of f (negative or positive) are dense in A . This can be easily deduced from our theorem by considering $B =$ the closure, in A , of the algebra of all polynomials of f and $M_k = \{gf^{-k} \in B : g \in M\}$, where k is an arbitrary integer.

The following is the main open problem related to our result.

Conjecture. *Let A be a Banach algebra (a Banach algebra with one generator), let M be a closed subspace of A of codimension $2 \leq n < \infty$, consisting of noninvertible elements only. Then M is contained in a nontrivial ideal of A .*

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