

A CHARACTERIZATION OF WEAK PEAK SETS FOR FUNCTION ALGEBRAS

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Let $A \subset C(S)$ be a function algebra. In this paper we prove that $S_0 = \overline{S_0} \subset S$ is a weak peak set for A if and only if for any open neighbourhood U of S_0 there is an f in A such that $\|f\| \leq 2$, $|f(x) - 1| \leq 1/3$ on S_0 and $|f(x)| \leq 1/3$ on $S \setminus U$.

Let S be a compact Hausdorff space. We shall denote by $C(S)$ ($C_R(S)$) the Banach algebra of all complex (real) valued continuous functions on S , provided with a usual supremum norm. Let A be a function algebra on S , i.e. a closed subalgebra of $C(S)$ which contains the constant functions and which separates points of S . A closed subset S_0 of S is called a weak peak set for A if for any open neighbourhood U of S_0 there is an f in A such that $1 = \|f\| = f(s)$ for $s \in S_0$ and $|f(s)| < 1$ for $s \in S \setminus U$. A one-point weak peak set is called a weak peak point. A classical Bishop's the so-called "1/4 - 3/4" criterion (see e.g. [4] p. 263) gives the following characterization of weak peak points:

THEOREM [Bishop]. *Let A be a function algebra on S . A point $s_0 \in S$ is a weak peak point for A if and only if there are constants $K \geq 1$ and $c < 1$ such that for any open neighbourhood U of s_0 there is an f in A such that $\|f\| \leq K$, $f(s_0) = 1$ and $|f(s)| \leq c$ for $s \in S \setminus U$.*

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The aim of this note is to give the following generalization of the above criterion:

THEOREM. *Let A be a function algebra on a compact Hausdorff space S and let S_0 be a closed subset of S . Then S_0 is a weak peak set for A if and only if there are constants $K \geq 1$, $c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 < 1$ such that for any open neighbourhood U of S_0 there is an f in A such that $\|f\| \leq K$, $|f(s) - 1| \leq c_2$ for $s \in S_0$ and $|f(s)| \leq c_1$ for $s \in S \setminus U$.*

Please note that the definition of a weak peak set is evidently equivalent to the following one ([4]):

Let A be a function algebra on S and let S_0 be a closed subset of S then S_0 is a weak peak set for A if and only if for any $\varepsilon > 0$ and any open neighbourhood U of S_0 there is an f in A such that $1 = \|f\| = f(s)$ for $s \in S_0$ and $|f(s)| < \varepsilon$ for $s \in S \setminus U$.

Hence the "only if" part of our theorem is trivial: that is, if $S_0 \subset S$ is a weak peak set for A then we can take c_1, c_2 any positive numbers and put $K = 1$.

We divide the proof of our theorem into four steps; the proof of the second step is based on Bishop's proof of his criterion.

Assume that the assumption of Theorem is fulfilled.

STEP 1. For any $\varepsilon > 0$ there is a positive constant $K(\varepsilon)$ such that for any open neighbourhood U of S_0 there is an f in A such that $|f(s)| \leq \varepsilon$ for $s \in S \setminus U$; $|f(s)| \leq 1$ and $|f(s) - 1| \leq \varepsilon$ for $s \in S_0$ and $\|f\| \leq K(\varepsilon)$.

Proof. Let $f_1 \in A$ be such that $|f_1(s)| \leq c_1$ for $s \in S \setminus U$, $|f_1(s) - 1| \leq c_2$ for $s \in S_0$ and $\|f_1\| \leq K$. Since the discs $D_1 = \{z \in \mathbb{C} : |z| \leq c_1\}$ and $D_2 = \{z \in \mathbb{C} : |z - 1| \leq c_2\}$ are disjoint then, by Runge Theorem, there is a polynomial p such that $p(z) \leq \varepsilon$ for $z \in D_1$ and $|p(z) - (1 - \frac{\varepsilon}{2})| \leq \frac{\varepsilon}{2}$ for $z \in D_2$.

Put $K(\varepsilon) = \sup\{|p(z)| : |z| \leq K\}$ and $f = p \circ f_1 \in A$.

STEP 2. Assume that there are constants K_1 and $\varepsilon < 1$ such that for any open neighbourhood U of S_0 there is an f in A such that $|f(s)| \leq \varepsilon$ for $s \in S \setminus U$; $|f(s)| \leq 1$ and $|f(s) - 1| \leq \varepsilon$ for $s \in S_0$ and $\|f\| \leq K_1$, then for any open neighbourhood U of S_0 there is a g in A such that $|g(s)| \leq \varepsilon$ for $s \in S \setminus U$; $|g(s) - 1| \leq \varepsilon$ for $s \in S_0$ and $\|g\| \leq 1$.

Proof. Fix any $x < 1$ with

$$\alpha = (K_1 - 1) - x(K_1 - \varepsilon) < 0$$

and a decreasing sequence of positive numbers ε_n such that

$$\varepsilon_n(1 - x^n) + x^n \alpha < 0 \text{ for } n \geq 1.$$

We define by induction a sequence of functions $(h_n)_{n=1}^{\infty}$ from A . Let $h_1 \in A$ be any function from A such that $|h_1(s)| \leq \varepsilon$ for $s \in S \setminus U$; $|h_1(s)| \leq 1$ and $|h_1(s) - 1| \leq \varepsilon$ for $s \in S_0$ and $\|h_1\| \leq K_1$.

Assume we have defined h_1, \dots, h_n then put

$$W_n = \{s : \max_{1 \leq j \leq n} |h_j(s)| \geq 1 + \varepsilon_n\}.$$

The set W_n is a closed subset of $S \setminus S_0$ so there is an $h_{n+1} \in A$ such that

$$|h_{n+1}(s)| \leq \varepsilon \text{ for } s \in (S \setminus U) \cup W_n;$$

$$|h_{n+1}(s)| \leq 1 \text{ and } |h_{n+1}(s) - 1| \leq \varepsilon \text{ for } s \in S_0$$

and

$$\|h_{n+1}\| \leq K_1.$$

Let

$$g = (1 - x) \sum_{j=1}^{\infty} x^{j-1} \cdot h_j .$$

We have evidently $|g(s)| \leq 1$ and $|g(s) - 1| \leq \varepsilon$ for $s \in S_0$,
 $|g(s)| \leq \varepsilon$ for $s \in S \setminus U$, and $|g(s)| \leq 1$ for $s \in S \setminus \bigcup_{n=1}^{\infty} W_n$. It
 remains to show that if $s \in \bigcup_{n=1}^{\infty} W_n$ then $|g(s)| \leq 1$.

The sequence $(W_n)_{n=1}^{\infty}$ is an increasing sequence of compact sets so
 if $s \in \bigcup_{n=1}^{\infty} W_n$ then there is a positive integer m such that $s \in W_{m+1}$
 but $s \notin W_m$ for $m \geq 0$ (we put $W_0 = \emptyset$).

We have

$$|h_j(s)| \leq 1 + \varepsilon_m \quad \text{for } j \leq m ,$$

$$|h_{m+1}(s)| \leq K_1 ,$$

$$|h_j(s)| \leq \varepsilon \quad \text{for } j \geq m + 2 ,$$

hence

$$\begin{aligned} |g(s)| &\leq (1 - x) \left((1 + \varepsilon_m) \sum_{j=1}^m x^{j-1} + K_1 x^m + \varepsilon \sum_{j=m+2}^{\infty} x^{j-1} \right) \\ &= 1 + \varepsilon_m (1 - x^m) + x^m (K_1 - 1 - x(K_1 - \varepsilon)) < 1 . \end{aligned}$$

STEP 3. Let M be a closed subspace of $C(S)$, let π be a
 canonical map from M onto $M|_{S_0} = \{f|_{S_0} : f \in M\} \subset C(S_0)$ $\pi(f) = f|_{S_0}$,
 and let $\tilde{\pi} : M/\ker\pi \rightarrow M|_{S_0} : \tilde{\pi}(f + \ker\pi) = f|_{S_0}$. Then if $\tilde{\pi}$ is not an
 isometry then there are measures μ on S_0 and ν on $S \setminus S_0$ such that
 $\mu - \nu \perp M$ but the measure μ represents a non zero functional on M .

Proof. Assume $\tilde{\pi}$ is not an isometry, there is a functional F_0 on

$M|_{S_0}$ represented by a measure μ_0 on S_0 such that $\|F_0\| = \text{var}(\mu_0) = 1$ but $\|\pi^*(F_0)\| = t < 1$. Let ν_0 be any measure on S which represents the functional $\pi^*(F_0)$ with $\text{var}(\nu_0) = t$. Since $\text{var}(\nu_0|_{S_0}) \leq t < 1$ then the measure $\mu_0 - \nu_0|_{S_0}$ is not orthogonal to M and we can put $\mu = \mu_0 - \nu_0|_{S_0}$, $\nu = \nu_0|_{S \setminus S_0}$.

To end the proof of Theorem now fix an open neighbourhood U of S_0 and let $q \in C_R(S)$ be such that

$$q|_{S_0} \equiv 1, \quad q|_{S \setminus U} \equiv 0, \quad 1 \leq q \leq 1$$

and put

$$M = \{q \cdot g \in C(S) : g \in A\}.$$

We shall prove that $\tilde{\pi} : M/\ker \pi \rightarrow M|_{S_0}$ is an isometry. Assume the contrary and let μ, ν be as in Step 3. We can assume that the norm of μ on $M|_{S_0}$ is equal to 2 and let $f \in M$ be such that $\int f d\mu = 1$ and

$\|f\|_{S_0} \leq 1$. Fix $0 < \delta < \frac{1}{2}(\|f\| (1 + \text{var}(\nu)))^{-1}$. The regularity of the

measure ν provides there is an open neighbourhood V of S_0 such that

$|\nu|(V) < \delta$ and, by Steps 1 and 2, there is an f_0 in A such that $\|f_0\|_{S \setminus V} \leq \delta$, $\|f_0 - 1\|_{S_0} \leq \delta$ and $\|f_0\| \leq 1$. Since $\nu - \mu \perp M$ and

$ff_0 \in M$ we have

$$\begin{aligned} \frac{1}{2} < 1 - 2\delta &\leq \left| \int_{S_0} f d\mu \right| - \left| \int_{S_0} f(1 - f_0) d\mu \right| \\ &\leq \left| \int_{S_0} ff_0 d\mu \right| = \left| \int_{S \setminus S_0} ff_0 d\nu \right| \leq \left| \int_V ff_0 d\nu \right| + \left| \int_{S \setminus V} ff_0 d\nu \right| \\ &\leq \delta \|f\| + \text{var}(\nu) \delta \|f\| < \frac{1}{2}. \end{aligned}$$

So we have proved that $\tilde{\pi}$ is an isometry and the same time we have established that:

for any open neighbourhood U of S_0 there is an f_U in A such $\|f_U\| \leq 2$, $f_U|_{S_0} \equiv 1$ and $|f_U(s)| \leq \frac{1}{2}$ for $s \in S \setminus U$.

Now the Theorem follows from the Bishop criterion applied to the algebra $\tilde{A} = \{f \in A : f|_{S_0} = \text{const}\}$. //

REMARKS. The theorem we have just proved is applied to the theory of small perturbations of multiplication in function algebras (see the author's paper "Perturbations of Banach algebras" to appear and also [2], [3], [5]) but one can also get another application; for example, the following theorem due to Badé and Curtis is an immediate consequence of our theorem:

THEOREM. [Badé and Curtis]. *Let A be a function algebra on a compact Hausdorff space S and assume A is boundedly ϵ -normal for some $\epsilon < \frac{1}{2}$ then $A = C(S)$.*

Where A is called boundedly ϵ -normal if there is a constant K such that for any closed disjoint subsets F, G of S there is an f in A with $\|f\| \leq K$, $|f(s)| \leq \epsilon$ for $s \in F$ and $|f(s) - 1| \leq \epsilon$ for $s \in G$.

Proof. Assume $A \not\subset C(S)$ then there is a measure μ on S orthogonal to A ; let $F = \overline{F} \subset S$ be such that $\mu(F) \neq 0$, we can assume $\mu(F) = 1$, and let U be an open neighbourhood of F with $|\mu|(U \setminus F) \leq \frac{1}{4}$. By our Theorem and the assumptions F is a weak peak set so there is an f in A such that $\|f\| \leq 1$, $f|_F \equiv 1$ and $|f(s)| < \frac{1}{2} \text{var}(\mu)$ for $s \in S \setminus U$. We have

$$\begin{aligned} 0 &= \left| \int_S f d\mu \right| = \left| \int_F f d\mu + \int_{U \setminus F} f d\mu + \int_{S \setminus U} f d\mu \right| \\ &\geq 1 - \frac{1}{2} - \frac{|\mu|(S \setminus U)}{2 \text{var}(\mu)} > 0 \end{aligned}$$

and this ends the proof.

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