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Almost-L^p-projections and L^p isomorphisms

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Synopsis

L^p-summands and L^p-projections in Banach spaces have been studied by E. Behrends, who showed that for a fixed value of p, $1 \leqq p \leqq \infty$, $p \neq 2$, any two L^p-projections on a given Banach space E commute. Here we introduce the notion of almost-L^p-projections, and we establish a result which generalises Behrends' theorem, while also simplifying its proof. Almost-L^p-projections are then applied to the study of small-bound isomorphisms of Bochner L^p-spaces. It is shown that if the Banach space E satisfies a geometric condition which, in the finite-dimensional case, reduces to the absence of non-trivial L^p-summands, then for separable measure spaces, the existence of a small-bound isomorphism between $L^p(\mu_1, E)$ and $L^p(\mu_2, E)$ implies that these Bochner spaces are, in fact, isometric.

0. Introduction

Throughout this article, the letter E will denote a Banach space, while $\mathscr{B}(E)$ denotes the space of continuous linear operators on E. For $1 \leqq p \leqq \infty$, an L^p-projection on E is a projection $P \in \mathscr{B}(E)$ satisfying

$$\|e\| = \begin{cases} (\|Pe\|^p + \|(I-P)e\|^p)^{1/p}, & 1 \leqq p < \infty \\ \max\{\|Pe\|, \quad \|(I-P)e\|\}, & p = \infty \end{cases}$$

for all $e \in E$, where I denotes the identity in $\mathscr{B}(E)$. L^p-projections were thoroughly studied by E. Behrends [3], who showed that for a fixed value of p, $1 \leqq p \leqq \infty$, $p \neq 2$, any two L^p-projections on E commute. If, moreover, E is not isometric to the space $(\mathbb{R}^2, \|\cdot\|_\infty)$, then there exist nontrivial L^p-projections on E for, at most, one value of $p \in [1, \infty]$. The proof in [3] uses only elementary methods, but is nevertheless very involved.

In Section 2 of this article, we consider almost-L^p-projections which are projections $Q \in \mathscr{B}(E)$ satisfying

$$\left.\begin{array}{ll} (1+\delta)^{-1}\|e\| \leqq (\|Qe\|^p + \|(I-Q)e\|^p)^{1/p} \leqq (1+\delta)\|e\|, & 1 \leqq p < \infty \\ (1+\delta)^{-1}\|e\| \leqq \max\{\|Qe\|, \quad \|(I-Q)e\|\} \leqq (1+\delta)\|e\|, & p = \infty \end{array}\right\} \quad (0.1)$$

for some $\delta \geqq 0$ and all $e \in E$. If (0.1) holds for the particular value $\delta = \varepsilon$, we may refer to Q as an $\varepsilon - L^p$-projection. We show in Section 2 that if $P, Q \in \mathscr{B}(E)$, with P an $\varepsilon - L^p$-projection and q an $\varepsilon - L^q$-projection for $p \in (1, \infty)$ and $q \in [1, \infty]$, $q \neq 2$, or for $p = q = 1$ or $p = q = \infty$, then P and Q almost commute in a manner which we make quite precise. This theorem, together with passage to the limit, then gives a simplified proof of the results of Behrends cited above.

In Section 3, our results concerning almost-L^p-projections are applied to isomorphisms of Bochner L^p spaces. The isometries of scalar L^p spaces were investigated by Banach [**2**] and Lamperti [**9**]. Generalisations of these results for L^p spaces of vector-valued functions were considered in [**6–8, 12**]. Y. Benyamini [**5**] obtained, for the case of scalar L^p functions, results which applied to isomorphisms with small bound, rather than merely to isometries. He showed that if $1 \leqq p < \infty, p \neq 2$, then there exists a positive number $\varepsilon(p)$ such that if (X_i, Σ_i, μ_i), $i = 1, 2$ are separable measure spaces with $L^p(\mu_1)$ isomorphic to $L^p(\mu_2)$ under a map T satisfying $\|T\| \, \|T^{-1}\| < 1 + \varepsilon(p)$, then the two measure spaces are isomorphic. Hence $L^p(\mu_1)$ and $L^p(\mu_2)$ are, in fact, isometric. Our theorem of Section 3 provides such a result for Bochner spaces $L^p(\mu, E)$.

It has been shown by D. Alspach [**1**] that given such a map T between $L^p(\mu_1)$ and $L^p(\mu_2)$, then, if $\|T\| \, \|T^{-1}\|$ is sufficiently small, T is already close to an isometry. It is obvious that the Alspach result cannot be expected to hold in the vector case for arbitrary Banach spaces E, since if the measure spaces reduce to single atoms of mass one, then $L^p(\mu_i, E) \cong E$ for $i = 1, 2$ and there are Banach spaces E, (e.g. $A(D)$, [**10**, p. 93]), which admit isomorphisms T with $\|T\| \, \|T^{-1}\| < 1 + \varepsilon$ for arbitrarily small $\varepsilon > 0$, whereas $\|T - S\| \geqq 2 - \varepsilon$ for *any* isometry S of E. Even our generalisation of Benyamini's result cannot possibly hold without some geometric condition on the Banach spaces involved, since if, for example, $E = L^p([0, 1])$, then $L^p(\mu_1, E) \cong L^p(\mu_2, E) \cong E$ for arbitrary separable measure spaces (X_i, Σ_i, μ_i), $i = 1, 2$. The condition we thus impose in order to obtain the results of Section 3 is, essentially, that E fails to have much L^p structure. (Examples show that, even in the presence of our condition, the vector analogue of Alspach's result need not hold. It can, in fact, be established that one such example is provided by $A(D)$.)

1. Linear maps which are close to projections

In this section, we establish a theorem which will be essential for our results in Section 3.

THEOREM 1.1. *Let E be a Banach space and let $Q \in \mathscr{B}(E)$ be such that $\|Q^2 - Q\| \leqq \varepsilon < \frac{1}{4}$. Then there is a projection $\hat{Q} \in \mathscr{B}(E)$ such that $\|Q - \hat{Q}\| \leqq \dfrac{\varepsilon}{1 - 4\varepsilon}(2\|Q\| + 1)$. Moreover, $\hat{Q}$ commutes with any element of $\mathscr{B}(E)$ which commutes with Q.*

Proof. Set $P = 2Q - I$. We then have

$$\|P^2 - I\| = \|4Q^2 - 4Q\| \leqq 4\varepsilon < 1. \tag{1.1}$$

Hence in the commutative algebra generated by Q and I in $\mathscr{B}(E)$, we can define a square root A of P^2 by

$$A=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(P^2-I)^n,$$

and we set

$$\hat{Q}=\frac{AP^{-1}+I}{2}.$$

Note that P^{-1} exists, since by (1.1) P^2 is invertible, and hence so is P. Also by the definition of A, we have $A^2=P^2$ (although A need not be equal to P) and direct computation shows that $\hat{Q}^2=\hat{Q}$. We have

$$\|A-I\|\leqq\sum_{n=1}^{\infty}\left|\binom{\frac{1}{2}}{n}\right|\|P^2-I\|^n\leqq\sum_{n=1}^{\infty}\left|\binom{\frac{1}{2}}{n}\right|(4\varepsilon)^n=$$
$$-\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-4\varepsilon)^n=1-\sqrt{1-4\varepsilon}=:2\varepsilon'$$

and

$$\hat{Q}-Q=\frac{AP^{-1}+I-2Q}{2}=\frac{AP^{-1}-P}{2}=AP^{-1}\left(\frac{I-P^2A^{-1}}{2}\right)=AP^{-1}\left(\frac{I-A}{2}\right).$$

Hence

$$\|\hat{Q}\|-\|Q\|\leqq\|\hat{Q}-Q\|\leqq\varepsilon'\,\|AP^{-1}\|=\varepsilon'\,\|2\hat{Q}-I\|\leqq\varepsilon'(2\,\|\hat{Q}\|+1).\quad(1.2)$$

Thus

$$\|\hat{Q}\|\leqq\frac{\|Q\|+\varepsilon'}{1-2\varepsilon'}$$

and, finally, from (1.2) we obtain

$$\|Q-\hat{Q}\|\leqq\frac{\varepsilon'}{1-2\varepsilon'}(2\,\|Q\|+1)=\frac{1-\sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}}(2\,\|Q\|+1)\leqq\frac{\varepsilon}{1-4\varepsilon}(2\,\|Q\|+1).$$

The fact that $\hat{Q}$ commutes with any element of $\mathscr{B}(E)$ with which Q commutes follows from the form of the power series representation of A.

2. Almost-L^p-projections

We begin this section with an example. Of course, any L^p-projection on a space E is an ε-L^p-projection for all $\varepsilon>0$. But for a non-trivial example of such a map, fix p with $1\leqq p\leqq\infty$ and let $E=L^p(\sigma)$, where σ denotes normalised Lebesgue measure on the unit circle $\{e^{it}: t\in[0,2\pi]\}$ – i.e. $d\sigma=dt/2\pi$. We denote by u the identity function on the circle $u(e^{ix})=e^{ix}$. Then if $n\in\mathbb{N}$, $n\geqq 2$, and if B is any measurable subset of the circle, define, for $f\in E$,

$$Pf=\chi_B\left[f+\frac{u}{n}\int f\,d\sigma\right]-\frac{u}{n}\left[\int\chi_B f\,d\sigma+\frac{1}{n}\left(\int\chi_B u\,d\sigma\right)\left(\int f\,d\sigma\right)\right].$$

The fact that P is a $((2n+1)/n^2)$-L^p-projection on E, if not already obvious, will be transparent as a result of the proofs given in Section 3.

The example we have given is simply a small perturbation of an L^p-projection on the space $E = L^p(\sigma)$. There exist, however, ε-L^p-projections on spaces E which admit no L^p-projections whatsoever (see, e.g. [**11**, Theorem 2]).

The principal result of this section is the following theorem:

THEOREM 2.1. *Let* $1 < p < \infty$ *and* $1 \leqq q \leqq \infty$, $q \neq 2$, *or let* $p = q = 1$, *or* $p = q = \infty$. *Then there is a function* $\eta: \mathbb{R}^+ \to \mathbb{R}^+$ *(depending on p and q) such that* $\eta(\varepsilon)$ *tends to* 0 *if* $\varepsilon \to 0$ *and* $\|PQ - QP\| \leqq \eta(\varepsilon)$ *for all* $\varepsilon - L^p$*-projections P and all* $\varepsilon - L^q$*-projections Q.*

Proof. We first assume that $1 < p < \infty$, $1 \leqq q \leqq \infty$, $q \neq 2$, and define for $e \in E$, $\|e\|_0 := \sqrt[p]{\|Pe\|^p + \|e - Pe\|^p}$. Obviously $\|\cdot\|_0$ is a norm on E equivalent to the original one: $\|e\|/(1+\varepsilon) \leqq \|e\|_0 \leqq \|e\|\,(1+\varepsilon)$. Moreover, in $(E, \|\cdot\|_0)$, P is an L^p-projection and Q a δ–L^q-projection, where $\delta := \varepsilon^3 + 3e^2 + 3\varepsilon$. We claim that there exists a function $\tilde{\eta}$ with $\tilde{\eta}(\delta) \to 0$ if $\delta \to 0$ and $\|Qe - PQe\|_0 \leqq \tilde{\eta}(\delta)$ for every $e \in PE$ with $\|e\|_0 = 1$, L^p-projection P and δ–L^q-projection Q on $(E, \|\cdot\|_0)$. Once this claim is established, we obtain $\|QP - PQP\| \leqq (1+\varepsilon)^2\tilde{\eta}(\delta)$ for every L^p-projection. Then, if we replace P by the L^p-projection I–P, this yields $\|PQP - PQ\| \leqq (1+\varepsilon)^2\tilde{n}(\delta)$, whence $\|QP - PQ\| \leqq 2(1+\varepsilon)^2\tilde{\eta}(\delta)$, so that we obtain the desired result if we set $\eta(\varepsilon) := 2(1+\varepsilon)^2\tilde{\eta}(\varepsilon^3 + 3\varepsilon^2 + 3\varepsilon)$.

Thus to prove our claim, fix $e \in PE$ with $\|e\|_0 = 1$. Decompose e with respect to the projection Q: $e = \lambda w_1 + \mu w_2$, $\|w_1\|_0 = 1 = \|w_2\|_0$, $w_1 \in QE$, $w_2 \in (I - Q)E$. Next, decompose w_i, $i = 1, 2$ with respect to the projection P:

$$w_1 = v_1 + v_2, \quad v_1 = Pw_1, \quad v_2 = (I - P)w_1,$$

$$w_2 = v_3 + v_4, \quad v_3 = Pw_2, \quad v_4 = (I - P)w_2.$$

We now have $e = (\lambda v_1 + \mu v_3) + (\lambda v_2 + \mu v_4)$, where the left-hand side and the first term on the right-hand side are contained in PE, while the second term on the right-hand side is contained in $(I - P)E$. Hence $\lambda v_2 + \mu v_4 = 0$ and $Qe - PQe = \lambda v_2 = -\mu v_4$. If one of the quantities λ, $\|v_2\|_0$, μ, $\|v_4\|_0$ is zero, then there is nothing further to show. Thus we assume that all are distinct from zero, and set $\alpha := \min\{\|v_2\|_0, \|v_4\|_0\}$.

Now define $\rho := |\lambda|/|\mu| = \|v_4\|_0/\|v_2\|_0$ and choose σ such that $\rho v_2 = \sigma v_4$. Note that $|\sigma| = 1$. Let $0 < t < \min\{1, 1/\rho\}$. We now establish some equalities and inequalities which will be applied later. Since P is an L^p-projection, we have

$$1 = \|v_1\|_0^p + \|v_2\|_0^p. \tag{2.1}$$

Next, if one notes that because of (2.1) we have $\alpha \leqq 1$, it follows from the definition of ρ that

$$\alpha^{p+2} \leqq \|v_2\|_0^p \rho^2. \tag{2.2}$$

Then because the function defined on $[0, \infty)$ by $t \to t^p$ is convex, we obtain that, for every $u \in E$,

$$\tfrac{1}{2}\|v_1 + u\|_0^p + \tfrac{1}{2}\|v_1 - u\|_0^p \geqq (\tfrac{1}{2}\|v_1 + u\|_0 + \tfrac{1}{2}\|v_1 - u\|_0)^p \geqq \|v_1\|_0^p,$$

and thus it follows that

$$2\|v_1\|_0^p \leqq \|v_1 + \sigma t v_3\|_0^p + \|v_1 - o t v_3\|_0^p. \tag{2.3}$$

Next we assert that

$$2(1+\gamma_p\rho^2t^2)\leqq(1+\rho t)^p+(1-\rho t)^p, \tag{2.4}$$

where

$$\gamma_p := \begin{cases} \frac{1}{2}p(p-1) & \text{if } \ p\leqq 2 \\ 1 & \text{if } \ p\geqq 2 \end{cases}.$$

First we treat the case $p\leqq 2$. By using the binomial expansion of $(1\pm z)^p$ we obtain that, for $z\in[0,1)$, $\frac{1}{2}((1+z)^p+(1-z)^p)=\sum_{n=0}^{\infty}\binom{p}{2n}z^{2n}\geqq 1+\frac{1}{2}p(p-1)z^2$. In the case $2\leqq p$, we obtain that for all $z\in[0,1]$, $\sqrt[p]{1+z^2}\leqq\sqrt[2]{1+z^2}=\sqrt[2]{\frac{1}{2}(1+z)^2+\frac{1}{2}(1-z)^2}\leqq\sqrt[p]{\frac{1}{2}(1+z)^p+\frac{1}{2}(1-z)^p}$. Finally, we note that

$$\|v_2\|_0(1\pm\rho t)=\|v_2\pm\sigma t v_4\|_0. \tag{2.5}$$

One readily checks that Q is an $\varepsilon-L^q$-projection on E if and only if Q^* is an ε–$L^{q'}$-projection on E^*, where q and q' are conjugate exponents (see, e.g. [**4**, p. 8]). We also have $\|QP-PQ\|=\|Q^*P^*-P^*Q^*\|$. Hence, without loss of generality, we may suppose that $q>2$. Moreover, we assume that $q\neq\infty$, since that case can be treated in the same manner as the case $2<q<\infty$ with only obvious modifications. We have

$$\begin{aligned}
2(1+\alpha^{p+2}\gamma_p t^2) &\overset{(2.2)}{\leqq} 2(1+\|v_2\|_0^p\gamma_p\rho^2t^2)\\
&\overset{(2.1)}{=} 2(\|v_1\|_0^p+\|v_2\|_0^p+\|v_2\|_0^p\gamma_p\rho^2t^2)\\
&= 2(\|v_1\|_0^p+\|v_2\|_0^p(1+\gamma_p\rho^2t^2))\\
&\overset{(2.3),(2.4)}{\leqq} \|v_1+\sigma t v_3\|_0^p+\|v_1-\sigma t v_3\|_0^p+\|v_2\|_0^p((1+\rho t)^p+(1-\rho t)^p)\\
&\overset{(2.5)}{=} \|v_1+\sigma t v_3\|_0^p+\|v_1-\sigma t v_3\|_0^p+\|v_2+\sigma t v_4\|_0^p+\|v_2-\sigma t v_4\|_0^p\\
&= \|w_1+\sigma t w_2\|_0^p+\|w_1-\sigma t w_2\|_0^p\\
&\leqq 2(1+\delta)^p(1+t^q)^{p/q}
\end{aligned}$$

(the latter inequality following since Q is a $\delta-L^q$-projection in $(E,\|\cdot\|_0)$). Hence

$$\begin{aligned}
\alpha^{p+2} &\leqq \gamma_p^{-1}t^{-2}((1+\delta)^p((1+t^q)^{p/q}-1)+((1+\delta)^p-1))\\
&\leqq \gamma_p^{-1}(1+\delta)^pM_{p,q}t^{q-2}+\gamma_p^{-1}p(1+\delta)^{p-1}\delta t^{-2},
\end{aligned}$$

where

$$M_{p,q} := \begin{cases} p & \text{if } \ p\leqq q \\ p2^{(p/q)-1} & \text{if } \ q\leqq p \end{cases}.$$

The last inequality was obtained by the mean value theorem and is valid for all $t\in[0,1]$. Now if $\delta^{\frac{1}{4}}<\min\{1,1/\rho\}$ then we can substitute $\delta^{\frac{1}{4}}$ for t in the last inequality, to obtain

$$\alpha^{p+2}\leqq\gamma_p^{-1}(1+\delta)^pM_{p,q}\delta^{(q-2)/4}+\gamma_p^{-1}p(1+\delta)^{p-1}\sqrt{\delta}=:\eta_0(\delta).$$

If $\delta^{\frac{1}{4}} \geqq \min\{1, 1/\rho\}$, then it is easy to see that $\alpha \leqq \delta^{\frac{1}{4}}$. (Obviously we have $\|v_2\|_0 \leqq 1$ and $\|v_2\|_0 \leqq \|v_2\|_0/\|v_4\|_0 = 1/\rho$. Hence $\alpha \leqq \|v_2\|_0 \leqq \min\{1, 1/\rho\}$.) Thus noting that $|\lambda|$ and $|\mu|$ are each $\leqq 1+\delta$, so that $\|Qe - PQe\|_0 \leqq (1+\delta)\alpha$, we conclude the proof of the theorem in the case $1 < p < \infty$, $1 \leqq q \leqq \infty$, $p \neq 2$, if we define

$$\tilde{\eta}(\delta) := (1+\delta)\max\{\delta^{\frac{1}{4}}, (\eta_0(\delta))^{1/(p+2)}\}.$$

For the remaining cases, it suffices to assume $p = q = \infty$. Here we define, for $e \in E$, $\|e\|_0 = \max\{\|Pe\|, \|e - Pe\|\}$. Again $\|\cdot\|_0$ is a norm on E equivalent to the original one, $\|e\|/(1+\varepsilon) \leqq \|e\|_0 \leqq \|e\|(1+\varepsilon)$, and in $(E, \|\cdot\|_0)$ P is an L^∞-projection and Q a $\delta - L^\infty$-projection, where $\delta := \varepsilon^2 + 3\varepsilon^2 + 3\varepsilon$. Thus for any $v \in E$ and any scalar $t \geqq 0$, we have

$$\|tv + Qv\|_0 \geqq \frac{1}{\|Q\|_0}\|Q(tv+v)\|_0 \geqq \frac{1+t}{1+\delta}\|Qv\|_0,$$

$$\|tv + (I-Q)v\|_0 \geqq \frac{1+t}{1+\delta}\|(I-Q)v\|_0,$$

and

$$\max\{\|Qv\|_0, \quad \|(I-Q)v\|_0\} \geqq \frac{\|v\|_0}{1+\delta}.$$

Hence for any $v \in E$ with $\|v\|_0 = 1$ and any $t \geqq 0$, we have

$$\|tv + Qv\|_0 \geqq \frac{1+t}{(1+\delta)^2}, \quad \text{or} \quad \|tv + (I-Q)v\|_0 \geqq \frac{1+t}{(1+\delta)^2}. \tag{2.6}$$

For any $v \in (I-P)E$ and any $t \geqq 0$, we have

$$\left.\begin{aligned} &\|tv + Qv\|_0 = \max\{\|PQv\|_0, \quad \|tv + (I-P)Qv\|_0\},\\ &\|tv + (I-Q)v\|_0 = \max\{\|P(I-Q)v\|_0, \quad \|tv + (I-P)(I-Q)v\|_0\},\\ &\text{and} \quad (\text{since } \|P\|_0 = 1)\\ &\max\{\|PQv\|_0, \|P(I-Q)v\|_0\} \leqq (1+\delta)\,\|v\|_0. \end{aligned}\right\} \tag{2.7}$$

Hence, for any $t > (1+\delta)^3 - 1$, (2.6) and (2.7) then imply that for any $v \in (I-P)E$,

$$\left.\begin{aligned} &\|tv + (I-P)Qv\|_0 \geqq \frac{(1+t)\,\|v\|_0}{(1+\delta)^2}, \quad \text{or}\\ &\|tv + (I-P)(I-Q)v\|_0 \geqq \frac{(1+t)\,\|v\|_0}{(1+\delta)^2}. \end{aligned}\right\} \tag{2.8}$$

To end the proof of the theorem, let $e \in PE$ with $\|e\|_0 = 1$, where $e = \lambda w_1 + \mu w_2$, $w_1 = v_1 + v_2$, $w_2 = v_3 + v_4$, and $\lambda v_2 = -\mu v_4$ as before. We have to show that $\|\lambda v_2\|_0$ is small. Thus assume $\|\lambda v_2\|_0 > (1+\delta)^3 - 1$. Set $\hat{v}_2 = \lambda v_2/\|\lambda v_2\|_0$ and $e_1 = e + \hat{v}_2$, $e_2 = -e + \hat{v}_2$. Since $e \in PE$ and $v_2 \in (I-P)E$ we have $\|e_i\|_0 = 1$, $i = 1, 2$. On the other hand, $(1+\delta) \geqq \|Qe_1\|_0 = \|\lambda w_1 + Q\hat{v}_2\|_0 \geqq \|\lambda v_2 +$

$(I-P)Q\hat{v}_2\|_0$ and

$$(1+\delta) \geqq \|(I-Q)e_2\|_0 = \|-\mu w_2 + (I-Q)\hat{v}_2\|_0$$
$$\geqq \|-\mu v_4 + (I-P)(I-Q)\hat{v}_2\|_0 = \|\lambda v_2 + (I-P)(I-Q)\hat{v}_2\|_0.$$

Thus, by (2.8) we obtain

$$(1+\delta) \geqq \frac{1+\|\lambda v_2\|_0}{(1+\delta)^2},$$

which contradicts our assumption. Hence, in this case we may take $\tilde{\eta}(\delta) := \delta^3 + 3\delta^2 + 3\delta$ and $\eta(\varepsilon) := 2(1+\varepsilon)^2 \tilde{\eta}(\varepsilon^3 + 3\varepsilon^2 + 3\varepsilon)$. □

By letting $\varepsilon \to 0$ we then obtain the following result of E. Behrends [**3**, p. 76].

THEOREM 2.2. *Let $1 < p < \infty$, $1 \leqq q \leqq \infty$, $q \neq 2$, or let $p = q = 1$ or $p = q = \infty$. Then $PQ = QP$ for any L^p-projection P and any L^q-projection Q on E.*

This last theorem is also true for the case $p = 1$ and $q = \infty$, if E is not isometric to the real space l_2^∞. For the proof of this case, we refer to [**3, 4**].

Although p and q may differ in nontrivial cases of Theorem 2.1 dealing with almost-L^p-projections, we show with respect to Theorem 2.2 that (excepting the space $(\mathbb{R}^2, \|\cdot\|_\infty)$), there are nontrivial L^p-projections for, at most, one value of p. This fact was established by Behrends [**3**].

PROPOSITION 2.3. *Let E be a Banach space not isometric to the space $(\mathbb{R}^2, \|\cdot\|_\infty)$. Let P be a nontrivial L^p-projection and Q a non-trivial L^q-projection on E, $(p, q \in [1, \infty])$. Then $p = q$.*

Proof. If $p = q = 2$, we are done. Thus we can assume $q \neq 2$. By the above we then have $PQ = QP$. We will show that there exist e_1 and e_2 in E, such that

$$\|(\lambda, \mu)\|_p = \|\lambda e_1 + \mu e_2\| = \|(\lambda, \mu)\|_q \tag{2.9}$$

for all scalars λ and μ. This will prove our proposition.

Assume first that $PQ = 0$. Choose $e_1 \in PE$ and $e_2 \in QE$ with $\|e_1\| = 1 = \|e_2\|$. Then (2.9) is true because $e_1 \in PE$ and $e_2 \in (I-P)E$, respectively $e_1 \in (I-Q)E$ and $e_2 \in QE$. The case $(I-P)(I-Q) = 0$ can be treated similarly. Thus assume that $PQ \neq 0$ and $(I-P)(I-Q) \neq 0$. Choose norm-one elements $e_1 \in (PQ)E$ and $e_2 \in (I-P)(I-Q)E$. Then again it is easy to verify (2.9).

We simply remark, without proof, that one can establish an analogous result for almost-L^p-projections: if P is a non-trivial ε-L^p-projection on E and Q a non-trivial ε-L^q-projection, then p is "close to" q.

Noting that an L^p-projection is automatically an ε-L^p-projection for any $\varepsilon > 0$, we now state the form in which Theorem 2.1 will be applied throughout Section 3.

COROLLARY 2.4. *Let P be an L^p-projection and Q an ε-L^p-projection for some p with $1 \leqq p \leqq \infty$, $p \neq 2$. Then there exists a function $\eta: \mathbb{R}^+ \to \mathbb{R}^+$ (depending on p) such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and such that $\|PQ - QP\| \leqq \eta(\varepsilon)$.*

3. Small bound isomorphisms of Bochner L^p spaces

Throughout this section, if a and b are two numerical quantities such that $a-b$ depends on the positive quantity ε and tends to 0 as $\varepsilon \to 0$, we will write $a \overset{\varepsilon}{\lessapprox} b$. If A and B are operators on a given Banach space such that $\|A-B\|$ depends on ε and tends to zero as $\varepsilon \to 0$, we will write $A \overset{\varepsilon}{\sim} B$. Note that $\overset{\varepsilon}{\lessapprox}$ and $\overset{\varepsilon}{\sim}$ are transitive relations.

LEMMA 3.1. *Let E be a Banach space, $1 \leqq p \leqq \infty$, $p \neq 2$, and let P be an L^p-projection and Q an ε-L^p-projection of E for some $\varepsilon > 0$. Then there exists a projection $\hat{Q} \in \mathcal{B}(E)$ which commutes with P such that $\hat{Q} \overset{\varepsilon}{\sim} PQP$. Moreover, $\hat{Q}$ is an ε'-L^p-projection where $\varepsilon' \to 0$ as $\varepsilon \to 0$.*

Proof. We have

$$\begin{aligned}\|(PQP)(PQP) - PQP\| &= \|P(QPQP - QP)\| \leqq \|QPQP - QP\| \\ &= \|Q(PQ - QP)P\| \leqq (1+\varepsilon)\eta(\varepsilon) \qquad (3.1)\end{aligned}$$

by Corollary 2.4, and thus the existence of $\hat{Q}$ commuting with P such that $\hat{Q} \overset{\varepsilon}{\sim} PQP$ follows from Theorem 1.1 for all sufficiently small ε. Moreover, for $e \in E$ we have

$$\begin{aligned}\|e\|^p &= \|Pe\|^p + \|(I-P)e\|^p \\ &\overset{\varepsilon}{\lessapprox} \|QPe\|^p + \|(I-Q)Pe\|^p + \|(I-P)e\|^p \\ &= \|PQPe\|^p + \|(I-P)QPe\|^p + \|P(I-Q)Pe\|^p \\ &\quad + \|(I-P)(I-Q)Pe\|^p + \|(I-P)e\|^p \\ &\overset{\varepsilon}{\lessapprox} \|PQPe\|^p + \|P(I-Q)Pe\|^p + \|(I-P)e\|^p \quad \text{(Corollary 2.4)} \\ &= \|PQPe\|^p + \|P(I-Q)Pe + (I-P)e\|^p \\ &= \|PQPe\|^p + \|(I-PQP)e\|^p \\ &\overset{\varepsilon}{\lessapprox} \|\hat{Q}e\|^p + \|(I-\hat{Q})e\|^p,\end{aligned}$$

which completes the proof of the lemma. □

DEFINITION 3.2. For a Banach space E and $1 \leqq p \leqq \infty$ we set $\pi(E) = \inf\{\delta > 0 \colon \exists$ a nontrivial δ-L^p-projection of $E\}$. (Note that if E is finite-dimensional then $\pi(E) > 0$ if and only if E contains no nontrivial L^p-summand.)

In what follows, n is a cardinal number $1 \leqq n \leqq \aleph_0$ while $\mathbb{N}_n$ will denote the set $\{1, 2, \ldots, n\}$ if $n < \aleph_0$ and $\{1, 2, \ldots\}$ if $n = \aleph_0$.

LEMMA 3.3. *Let $1 \leqq p \leqq \infty$, $p \neq 2$, and let E be a Banach space with $\pi(E) = \delta > 0$. There exists an ε_0, dependent on p and δ, such that if $0 \leqq \varepsilon < \varepsilon_0$ and $Q \colon l_n^p(E) \to l_n^p(E)$ is an ε-L^p-projection, then for some $B \subseteq \mathbb{N}_n$ one has $Q \overset{\varepsilon}{\sim} P_B$, where P_B denotes the projection in $l_n^p(E)$ given by $f \to \chi_B f$.*

Proof. The norm in E is denoted by $\|\cdot\|$ and that in $l_n^p(E)$ by $\|\cdot\|_p$. For fixed $k \in \mathbb{N}_n$ we denote $P_{\{k\}}$ by P_k and define ε_1 by $1 - \varepsilon_1 = \frac{1}{2}[1 + \sqrt{1 - 4(1+\varepsilon)\eta(\varepsilon)}]$, where $\eta(\varepsilon)$ is given by Corollary 2.4. Then, for $k \in \mathbb{N}_n$,

$$\|(P_kQP_k)(P_kQP_k) - P_kQP_k\| \geqq \|P_kQP_k\| - \|P_kQP_k\|^2 \qquad (3.2)$$

which, combined with (3.1) and with the quadratic formula gives, for sufficiently small ε,

$$\left.\begin{aligned}&\|P_kQP_k\| \geqq 1-\varepsilon_1, \quad \text{or}\\ &\|P_kQP_k\| \leqq \varepsilon_1.\end{aligned}\right\} \tag{3.3}$$

We let $E^{(k)}$ denote the restriction of elements of $l_n^p(E)$ to k; $E^{(k)} = \{P_kf : f \in l_n^p(E)\} = \{e \cdot \chi_{\{k\}} : e \in E\}$. Obviously E is isometric with $E^{(k)}$ under the map $e \to e \cdot \chi_{\{k\}}$, and P_kQP_k can be considered as a linear map from $E^{(k)}$ to $E^{(k)}$ whose norm, as an element of $\mathcal{B}(E^{(k)})$, is the same as its norm as an element of $\mathcal{B}(l_n^p(E))$. Thus by Lemma 3.1, for small ε, there exists a projection $\hat{Q}_k : E^{(k)} \to E^{(k)}$ such that $P_kQP_k \overset{\varepsilon}{\sim} \hat{Q}_k$. We thus have, for $e \in E$,

$$\|e \cdot \chi_{\{k\}}\|_p^p \overset{\varepsilon}{\approx} \|\hat{Q}_k e \cdot \chi_{\{k\}}\|_p^p + \|(I-\hat{Q}_k)e \cdot \chi_{\{k\}}\|_p^p$$

where $\hat{Q}_k$ is an ε'-L^p-projection of $E^{(k)}$ onto itself and $\varepsilon' \to 0$ as $\varepsilon \to 0$. Moreover, as the relation between ε and ε' given by Lemma 3.1 is determined only by Theorems 1.1 and 2.1, and is thus independent of the particular L^p-projection P, ε' is here independent of k. As $\pi(E^{(k)}) = \pi(E) = \delta > 0$, for small ε (hence ε') we must have for each fixed k, that $\hat{Q}_k$ is either equal to zero or to the identity operator on $E^{(k)}$.

Now

$$P_k = P_kQP_k + P_k(I-Q)P_k, \tag{3.4}$$

and since arguments analogous to those we have applied to P_kQP_k also apply to $P_k(I-Q)P_k$, for each fixed $k \in \mathbb{N}_n$ each of the operators on the right-hand side in (3.4) is either close to 0, or to the identity operator ($=P_k$) in $E^{(k)}$. Hence, for each k, precisely one of them is close to the identity while the other is close to zero. Thus if ε, hence ε_1, is small, at each $k \in \mathbb{N}_n$ we have $\|P_kQP_k\| \geqq 1-\varepsilon_1$ and $\|P_k(I-Q)P_k\| \leqq \varepsilon_1$, or $\|P_kQP_k\| \leqq \varepsilon_1$ and $\|P_k(I-Q)P_k\| \geqq 1-\varepsilon_1$. Thus the sets $B = \{k \in \mathbb{N}_n : \|P_k(I-Q)P_k\| \leqq \varepsilon_1\}$ and $B' = \{k \in \mathbb{N}_n : \|P_kQP_k\| \leqq \varepsilon_1\}$ are complementary disjoint sets: $\mathbb{N}_n = B \dot{\cup} B'$.

Throughout the remainder of the proof, we assume, as we may, that $\varepsilon_1 < \frac{1}{4}$. Computations exactly analogous to those made in (3.1)–(3.3) show that if $M \subseteq \mathbb{N}_n$ then either $\|QP_M\| \geqq 1-\varepsilon_1$ or $\|QP_M\| \leqq \varepsilon_1$. We set $\mathcal{J} = \{M \subseteq \mathbb{N}_n : \|QP_M\| \leqq \varepsilon_1\}$ and claim, first of all, that $\mathcal{J}$ is closed under finite unions. For if M_1 and M_2 belong to $\mathcal{J}$, then $\|QP_{M_1} + QP_{M_2}\| \leqq 2\varepsilon_1$. Now $\|QP_{M_1} + QP_{M_2}\| = \|QP_{M_1 \cup M_2} + QP_{M_1 \cap M_2}\|$, and since subsets of members of $\mathcal{J}$ are easily seen to belong to $\mathcal{J}$, we have $\|QP_{M_1 \cup M_2}\| \leqq 3\varepsilon_1$. But $\|QP_{M_1 \cup M_2}\|$ is either greater than or equal to $1-\varepsilon_1$ or else is less than or equal to ε_1, and since $\varepsilon_1 < \frac{1}{4}$, we have $\|QP_{M_1 \cup M_2}\| \leqq \varepsilon_1$.

In fact, $\mathcal{J}$ is closed under countable unions. For suppose $M_1 \subseteq M_2 \subseteq M_3 \ldots$ is an increasing sequence of sets in $\mathcal{J}$ and $M_\infty = \bigcup_{k=1}^\infty M_k$. Then $P_{M_k} \to P_{M_\infty}$ in the strong operator topology, so that $QP_{M_k} \to QP_{M_\infty}$. Thus, as the norm in $\mathcal{B}(l_n^p(E))$ is a lower semicontinuous function, $\|QP_{M_\infty}\| \leqq \varepsilon_1$.

Now if $\{k\} \in \mathcal{J}$, i.e. if $\|QP_k\| \leqq \varepsilon_1$, then $\|P_kQP_k\| \leqq \varepsilon_1$, so that $k \in B'$. That is, $\{k : \{k\} \in \mathcal{J}\} \subseteq B'$. Conversely, if $k \in B'$ then $P_k(I-Q)P_k \overset{\varepsilon}{\sim} P_k$. By Corollary 2.4, $P_k(I-Q)P_k \overset{\varepsilon}{\sim} (I-Q)P_k$, so that $QP_k = P_k - (I-Q)P_k$ is small. Thus $B' = \{\text{k} : \{\text{k}\} \in \mathcal{J}\}$, and hence $B' \in \mathcal{J}$.

By an analogous argument, if we let $\mathcal{K} = \{M \subseteq \mathbb{N}_n : \|(I-Q)P_M\| \leqq \varepsilon_1\}$ we find that $B \in \mathcal{K}$. We thus have

$$\begin{aligned} \|P_B - Q\| &= \|P_B - Q(P_B + P_{B'})\| \\ &\leqq \|(I-Q)P_B\| + \|QP_{B'}\| \leqq 2\varepsilon_1 \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

thus concluding the proof. □

Note that the distance between P_B and Q depends only on ε_1, hence only on ε, and is thus independent of the particular ε-L^p-projection Q.

THEOREM 3.4. *Suppose* $1 \leqq p \leqq \infty$, $p \neq 2$, *let* E_1, E_2 *be Banach spaces with* $\pi(E_1)$, $\pi(E_2) \geqq \delta > 0$, *and let* (X_i, Σ_i, μ_i) *be separable measure spaces,* $i = 1, 2$. *There exists an* $\varepsilon > 0$ *dependent on p and* δ *such that, if T is any isomorphism of* $L^p(\mu_1, E_1)$ *onto* $L^p(\mu_2, E_2)$ *with* $\|T\|\,\|T^{-1}\| < 1 + \varepsilon$, *then the measure spaces* (X_1, Σ_1, μ_1) *and* (X_2, Σ_2, μ_2) *are isomorphic. Thus, in particular, if* $E_1 = E_2$ *then* $L^p(\mu_1, E_1)$ *and* $L^p(\mu_2, E_2)$ *are isometric.*

Proof. It is known [**5**, p. 280] that each of the measure spaces involved is isomorphic to one of the following: $[0,1]$, $\mathbb{N}_n$, or $[0,1] \dot{\cup} \mathbb{N}_n$ for some n, $1 \leqq n \leqq \aleph_0$. Thus we may suppose, without loss of generality, that each of the measure spaces has one of the above forms. We must show that:

(a) If μ_1 contains non-zero continuous part then so does μ_2, and

(b) the number of atoms of μ_1 is equal to the number of atoms of μ_2.

We will write $\mu_1 = \mu_{c,1} + \mu_{a,1}$, where $\mu_{c,1}$ denotes the continuous part of μ_1 and $\mu_{a,1}$ the atomic part. If $\mu_{c,1} \neq 0$, $P_{c,1}$ will denote the projection $f \to \chi_{[0,1]} f$, ($P_{c,1} = 0$ if $\mu_{c,1} = 0$) and, if μ_1 has precisely n atoms, $P_{a,1}$ will denote the projection $f \to \chi_{\mathbb{N}_n} f$. Similarly we write $\mu_2 = \mu_{c,2} + \mu_{a,2}$ and let $P_{c,2}$ and $P_{a,2}$ be the corresponding projections in $L^p(\mu_2, E_2)$. For measurable subsets B, as in Lemma 3.3, we may use P_B to denote the L^P-projection $f \to \chi_B f$, and we again write P_k for $P_{\{k\}}$. Throughout the proof, we assume that ε is chosen sufficiently small that the results of Corollary 2.4, Lemma 3.1, and Lemma 3.3 apply.

Thus assume that $\mu_{c,1} \neq 0$. Then $Q_0 = TP_{c,1}T^{-1}$ is an ε-L^p-projection in $L^p(\mu_2, E_2)$ and we set $Q_1 = P_{a,2}Q_0P_{a,2}$. Then by Lemma 3.1, there exists a projection $\hat{Q}_1$ of $L^p(\mu_2, E_2)$ such that $\hat{Q}_1 \overset{\varepsilon}{\sim} Q_1$, and $\hat{Q}_1$ is an ε'-L^p-projection where $\varepsilon' \to 0$ as $\varepsilon \to 0$. We will prove that for sufficiently small ε, $\hat{Q}_1 \overset{\varepsilon}{\sim} 0$ (and hence $\hat{Q}_1 = 0$ since $\hat{Q}_1$ is a projection), so that $Q_1 \overset{\varepsilon}{\sim} 0$.

Suppose for the moment that this has been shown. Then we would have

$$\begin{aligned} TP_{c,1}T^{-1} = Q_0 &= Q_0P_{c,2} + Q_0P_{a,2} \\ &\overset{\varepsilon}{\sim} Q_0P_{c,2} + P_{a,2}Q_0P_{a,2} \quad \text{(Corollary 2.4)} \\ &= Q_0P_{c,2} + Q_1 \overset{\varepsilon}{\sim} Q_0P_{c,2} \end{aligned}$$

so that, for small ε, we would have $P_{c,2} \neq 0$, and hence μ_2 has nonzero continuous part, thus completing the proof of (a).

We will hence show $Q_1 \overset{\varepsilon}{\sim} 0$. If $P_{a,2} = 0$ we are done. Thus suppose that μ_2 has m atoms, $1 \leqq m \leqq \aleph_0$. By Lemma 3.3, if ε, hence ε', is sufficiently small there exists $B \subseteq \mathbb{N}_m$ such that $\hat{Q}_1 \overset{\varepsilon'}{\sim} P_B$. Since $\varepsilon' \to 0$ as $\varepsilon \to 0$ we have, in fact, $\hat{Q}_1 \overset{\varepsilon}{\sim} P_B$. We need to show that $B = \varnothing$.

Suppose, to the contrary, that there exists a point $k_0 \in B$. Set $Q_2 =$

$Q_1P_{k_0} \overset{\varepsilon}{\sim} P_BP_{k_0} = P_{k_0}$, and let

$$\begin{aligned}\check{Q} &= T^{-1}Q_2T = T^{-1}P_{a,2}Q_0P_{a,2}P_{k_0}T\\ &\overset{\varepsilon}{\sim} T^{-1}Q_0P_{a,2}P_{k_0}T \quad \text{(Corollary 2.4)}\\ &= T^{-1}Q_0P_{k_0}T = P_{c,1}T^{-1}P_{k_0}T\\ &\overset{\varepsilon}{\sim} P_{c,1}T^{-1}P_{k_0}TP_{c,1} \quad \text{(Corollary 2.4).}\end{aligned}$$

We thus have

$$Q_2 \overset{\varepsilon}{\sim} P_{k_0} \tag{3.5}$$

and

$$\check{Q} \overset{\varepsilon}{\sim} P_{c,1}T^{-1}P_{k_0}TP_{c,1}, \tag{3.6}$$

the right-hand side of (3.6) being a map of $L^p(\mu_1, E_1)$ into $L^p(\mu_{c,1}, E_1)$.

We next note that P_{k_0} has the property that if Q is any ε-L^p-projection in $L^p(\mu_2, E_2)$, then $QP_{k_0} \overset{\varepsilon}{\sim} 0$ or $QP_{k_0} \overset{\varepsilon}{\sim} P_{k_0}$. For $QP_{k_0} = QP_{a,2}P_{k_0} \overset{\varepsilon}{\sim} P_{a,2}QP_{a,2}P_{k_0}$ (Corollary 2.4), and again by Lemma 3.1 there exists a $\hat{P} \in \mathscr{B}(L^p(\mu_2, E_2))$ with $\hat{P} \overset{\varepsilon}{\sim} P_{a,2}QP_{a,2}$ such that $\hat{P}$ is an ε'-L^p-projection, where $\varepsilon' \to 0$ as $\varepsilon \to 0$. We thus have $QP_{k_0} \overset{\varepsilon}{\sim} \hat{P}P_{k_0}$, and since $\hat{P}$ commutes with $P_{a,2}$, it can be considered as an ε'-L^p-projection of $L^p(\mu_{a,2}, E_2)$ to itself. Hence by Lemma 3.3, there exist a subset $D \subseteq \mathbb{N}_m$ such that $\hat{P} \overset{\varepsilon'}{\sim} P_D$. As $\varepsilon' \to 0$ as $\varepsilon \to 0$, we have, in fact, $\hat{P} \overset{\varepsilon}{\sim} P_D$ and hence

$$QP_{k_0} \overset{\varepsilon}{\sim} P_DP_{k_0} = \begin{cases} P_{k_0}, & \text{if} \quad k_0 \in D,\\ 0, & \text{if} \quad k_0 \notin D.\end{cases} \tag{3.7}$$

It thus follows that if Q is any ε–L^p-projection in $L^p(\mu_2, E_2)$ then, as $Q_2 = Q_1P_{k_0}$, we have

$$\begin{aligned} QQ_2 &\overset{\varepsilon}{\sim} QP_{k_0} \quad \text{(by (3.5))}\\ &\overset{\varepsilon}{\sim} \begin{cases} P_{k_0}\\ 0\end{cases} \quad \text{or (by (3.7))}\\ &\overset{\varepsilon}{\sim} \begin{cases} Q_2\\ 0\end{cases} \quad \text{or (by (3.5)).}\end{aligned}$$

Hence either

(i) $T^{-1}QQ_2T = (T^{-1}QT)(T^{-1}Q_2T) = (T^{-1}QT)\check{Q}$ should be close to $T^{-1}Q_2T = \check{Q} \overset{\varepsilon}{\sim} P_{c,1}T^{-1}P_{k_0}TP_{c,1}$ (by (3.6)), which because of (3.5) and the definition of $\check{Q}$ is, for small ε, a map of norm approximately one taking $L^p(\mu_{c,1}, E_1)$ to itself, or else

(ii) $(T^{-1}QT)\check{Q}$ is close to the zero map.

Also, as Q is an arbitrary ε-L^p-projection of $L^p(\mu_2, E_2)$, we may take $Q = TP_MT^{-1}$ where M is any measurable subset of $[0, 1]$. Take $f \in L^p(\mu_1, E_1)$ with $\|f\|_p < 2$ and $\|P_{c,1}T^{-1}P_{k_0}TP_{c,1}f\|_p = 1$, and then choose $M \subseteq [0, 1]$ such that $\|P_MP_{c,1}T^{-1}P_{k_0}TP_{c,1}f\|_p = \frac{1}{2}$ and $\|(I - P_M)P_{c,1}T^{-1}P_{k_0}TP_{c,1}f\|_p = \frac{1}{2}$. Since, by (3.6), $P_M\check{Q} \overset{\varepsilon}{\sim} P_MP_{c,1}T^{-1}P_{k_0}TP_{c,1}$ then, for small ε, $P_M\check{Q}$ is neither close to $\check{Q}$ nor to 0, and this contradiction shows that in fact the set B is empty. Thus $Q_1 \overset{\varepsilon}{\sim} 0$ as we wished to show, and the proof of (a) is complete.

To prove (b), we first note that $TP_{c,1}T^{-1}$ is an ε-L^p-projection, so that, again by Lemma 3.1, $P_{c,2}TP_{c,1}T^{-1}P_{c,2}\overset{\varepsilon}{\sim}\hat{P}_0$, where $\hat{P}_0$ is a projection in $\mathcal{B}(L^p(\mu_2, E_2))$, which can be considered as a projection of $L^p(\mu_{c,2}, E_2)$. Thus, by the last sentence of the previous paragraph, we have $Q_0 = TP_{c,1}T^{-1}\overset{\varepsilon}{\sim}TP_{c,1}T^{-1}P_{c,2}\overset{\varepsilon}{\sim}P_{c,2}TP_{c,1}T^{-1}P_{c,2}\overset{\varepsilon}{\sim}\hat{P}_0$. We claim that, for small ε, $\hat{P}_0 = P_{c,2}$. For if we suppose the contrary, we would have $\hat{P}_0P_{c,2} = \hat{P}_0$, and

$$\|P_{c,2} - \hat{P}_0\| \geqq 1. \tag{3.8}$$

Then

$$(T^{-1}\hat{P}_0T)(T^{-1}P_{c,2}T) = T^{-1}\hat{P}_0T. \tag{3.9}$$

We let $\hat{Q}_0$ be the projection of $L^p(\mu_{c,1}, E_1)$, which is related to $T^{-1}P_{c,2}T$ in a manner analogous to that in which $\hat{P}_0$ is related to $TP_{c,1}T^{-1}$. Since

$$T^{-1}\hat{P}_0T\overset{\varepsilon}{\sim}P_{c,1} \tag{3.10}$$

we would have $P_{c,1}(T^{-1}P_{c,2}T)\overset{\varepsilon}{\sim}P_{c,1}$ by (3.9), while by (3.8) and (3.10) $\|P_{c,1} - T^{-1}P_{c,2}T\|\overset{\varepsilon}{\gtrapprox}K$, where K is a constant $\geqq 1$. Hence

$$P_{c,1}\hat{Q}_0\overset{\varepsilon}{\sim}P_{c,1} \tag{3.11}$$

and

$$\|P_{c,1} - \hat{Q}_0\|\overset{\varepsilon}{\gtrapprox}K. \tag{3.12}$$

But (3.11) implies (since $P_{c,1}$ is the identity in $L^p(\mu_{c,1}, E_1)$) that $\hat{Q}_0 = P_{c,1}$, while (3.12) says that $\hat{Q}_0 \neq P_{c,1}$. This contradiction proves our contention that $P_{c,2} = \hat{P}_0$ for small ε, and thus

$$TP_{c,1}T^{-1}\overset{\varepsilon}{\sim}P_{c,2}, \tag{3.13}$$

from which it follows that

$$TP_{a,1}T^{-1}\overset{\varepsilon}{\sim}P_{a,2}. \tag{3.14}$$

Now from (3.13) $TP_{c,1}\overset{\varepsilon}{\sim}P_{c,2}T$ so that $P_{c,2}TP_{c,1}\overset{\varepsilon}{\sim}P_{c,2}T$. Similarly $P_{a,2}TP_{a,1}\overset{\varepsilon}{\sim}P_{a,2}T$ and hence

$$T = P_{c,2}T + P_{a,2}T\overset{\varepsilon}{\sim}P_{c,2}TP_{c,1} + P_{a,2}TP_{a,1}. \tag{3.15}$$

Let S denote the operator on the right-hand side in (3.15), $S := P_{c,2}TP_{c,1} + P_{a,2}TP_{a,1}$ and suppose, as we may, that the distance between T and S is less than $\frac{1}{2}$ and that $\|T^{-1}\| < 2$. It then follows that $\|T^{-1}T - T^{-1}S\| < 1$, proving that S is surjective. Hence each of the maps $P_{c,2}TP_{c,1}\colon L^p(\mu_{c,1}, E_1)\to L^p(\mu_{c,2}, E_2)$ and $P_{a,2}TP_{a,1}\colon L^p(\mu_{a,1}, E_1)\to L^p(\mu_{a,2}, E_2)$ must be surjective. Also, by (3.13) and (3.14), each of the latter two maps can easily be seen to be injective. Thus T gives rise to an isomorphism of small bound from $L^p(\mu_{c,1}, E_1)$ onto $L^p(\mu_{c,2}, E_2)$ and a similar map from $l^p_n(E_1)$ to $l^p_m(E_2)$. Hence, to end the proof, we may assume that μ_1 and μ_2 are both purely atomic, and $T\colon l^p_n(E_1)\to l^p_m(E_2)$.

For any $k \in \mathbb{N}_n$ we let $Q_k = TP_kT^{-1}$. Then Q_k is an ε-L^p-projection of $l^p_m(E_2)$ so that, by Lemma 3.3, there is a subset $B_k \subseteq \mathbb{N}_m$ such that $\|Q_k - P_{B_k}\| \leqq \varepsilon_2$, where by the note following the proof of that lemma, ε_2 is independent of k and tends to zero as $\varepsilon\to 0$. If $k_1 \neq k_2$ we have $\|Q_{k_1} + Q_{k_2}\|\overset{\varepsilon}{\approx}1$ so that $\|P_{B_{k_1}} + P_{B_{k_2}}\|\overset{\varepsilon}{\approx}1$. Now if $B_{k_1}\cap B_{k_2} \neq \varnothing$ we would have $\|P_{B_{k_1}} + P_{B_{k_2}}\| = 2$, so that consequently $m \geqq n$. By a symmetric argument, with the roles of T and T^{-1} interchanged, we have $n \geqq m$, thus establishing (b) and concluding the proof of the theorem. ☐

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