A DISCONTINUOUS FUNCTION DOES NOT OPERATE ON THE REAL PART OF A FUNCTION ALGEBRA

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Let A be a function algebra on a compact Hausdorff space X and let h be a function on an interval I. We say that h operates by composition on $\operatorname{Re} A = \{\operatorname{Re} f : f \in A\}$ if $h \circ u \in \operatorname{Re} A$ whenever $u \in \operatorname{Re} A$ has the range in I. It is an old conjecture that if h operaters by composition on $\operatorname{Re} A$ and h is not affine, then $A = C(X)$. J. Wermer proved the conjecture in the case $h(t) = t^2$ ([4]) and A. Bernard in the case $h(t) = |t|$ ([1]). S. J. Sidney proved that the conclusion holds if h is non-affine and continuously differentiable or if h is "highly non-affine" in a suitable manner [3]. O. Hatari proved the conjecture for h continuous, non-affine and not "highly non-affine" in S. J. Sidney's sense [2]. Thus, the conjecture is verified for any continuous non-affine function h.

The purpose of this note is to prove the conjecture for any noncontinuous function h. In this case one can obtain even more information about A, namely:

Theorem. *A non-continuous function h operates by composition on the real part of a function algebra A if and only if A is finite dimensional.*

Proof. Let A be a function algebra contained in $C(X)$ for some compact Hausdorff set X and let h be a non-continuous real function which operates on $\operatorname{Re} A$. Composing h with a suitable affine function, without loss of generality we can assume that there is a sequence $(\alpha_n)_{n=1}^{\infty}$ tending to 0 and such that $h(\alpha_n) \geqq 1$ for all $n \in \mathbb{N}$ while $h(0) = 0$. Assuming that A is infinite dimensional we get that there is a sequence $(x_n)_{n=1}^{\infty}$ of elements from the Choquet boundary of A and a sequence $(U_n)_{n=1}^{\infty}$ of open pairwise disjoint subsets of X such that $x_n \in U_n$ for $n \in \mathbb{N}$. For a fixed $\varepsilon > 0$ let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n \leqq \varepsilon$, let $(f_n)_{n=1}^{\infty}$ be a sequence of elements of A such that for all $n \in \mathbb{N}$

$$\|f_n\| = 1 = f_n(x_n) \quad \text{and} \quad \sup\{|f(x)| : x \in X \setminus U_n\} \leqq \varepsilon_n\,,$$

and let A_0 be the subalgebra of A generated by the set $\{f_n : n \in \mathbb{N}\}$. We define an equivalence relation on X:

$$x' \sim x'' \equiv f(x') = f(x'') \quad \text{for all } f \text{ in } A_0\,.$$

The set $Y = X/\sim$ is compact and such that $A_0 \subset C(Y) \subset C(X)$. Moreover, the separability of A_0 implies that Y is metrizable. Put $y_n = \pi(x_n)$ where $\pi: X \to X/\sim = Y$ is the natural projection. The set Y is metrizable and compact, so the sequence $(y_n)_{n=1}^{\infty}$ possesses a convergent subsequence; for simplicity of notation we can assume that $y_n \to y_0 \in Y$. We denote by c the Banach space of all infinite convergent sequences with the usual sup-norm, and we define two maps:

$$T: c \to A_0: T((a_1, a_2, \ldots)) = \sum_{n=1}^{\infty} (a_n - \lim a_n) f_n + \lim a_n \, . \, 1 \, ,$$

$$S: A_0 \to c: S(f) = (f(y_n))_{n=1}^{\infty} \, .$$

It is easy to compute that by the definition of $(f_n)_{n=1}^{\infty}$ we have $\|S \circ T - \mathrm{Id}_c\| \leqq 2\varepsilon$. Hence for $\varepsilon < \frac{1}{2}$ the operator S is onto, so there is an $f_0 \in A_0$ such that $f_0(y_n) = \alpha_n$ for all $n \in \mathbb{N}$. Let $g_0 \in A$ be such that $\mathrm{Re}\, g_0 = h \circ \mathrm{Re}\, f_0$ and let (x_α) be a net consisting of elements from the set $\{x_n : n \in \mathbb{N}\}$, convergent to some point $x_0 \in X$. We have

$$x_\alpha \to x_0 \quad \text{and} \quad \pi(x_\alpha) \to y_0 \, , \quad \text{so} \quad \pi(x_0) = y_0 \, ,$$

but

$$\mathrm{Re}\, g_0(x_\alpha) = h \circ \mathrm{Re}\, f_0(x_\alpha) = h \circ \mathrm{Re}\, f_0(y_\alpha) \geqq 1$$

while

$$\mathrm{Re}\, g(x_0) = h \circ \mathrm{Re}\, f_0(x_0) = h \circ \mathrm{Re}\, f_0(y_0) = 0 \, ;$$

this contradicts the continuity of g and therefore completes the proof.

References

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