

Noncommutative Uniform Algebras

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ABSTRACT. We show that a real Banach algebra A such that $\|a^2\| = \|a\|^2$, for $a \in A$ is a subalgebra of the algebra $C_{\mathbb{H}}(X)$ of continuous quaternion valued functions on a compact set X .

1. Introduction

A well known Hirschfeld-Żelazko Theorem [5] (see also [7]) states that for a *complex* Banach algebra A the condition

$$(1.1) \quad \|a^2\| = \|a\|^2, \text{ for } a \in A$$

implies that (i) A is commutative, and further implies that (ii) A must be of a very specific form, namely it must be isometrically isomorphic with a uniformly closed subalgebra of $C_{\mathbb{C}}(X)$. We denote here by $C_{\mathbb{C}}(X)$ the complex Banach algebra of all continuous functions on a compact set X . The obvious question about the validity of the same conclusion for real Banach algebras is immediately dismissed with an obvious counterexample: the non commutative algebra \mathbb{H} of quaternions. However it turns out that the second part (ii) above is essentially also true in the real case. We show that the condition (1.1) implies, for a real Banach algebra A , that A is isometrically isomorphic with a subalgebra of $C_{\mathbb{H}}(X)$ - the algebra of continuous quaternion valued functions on a compact set X . That result is in fact a consequence of a theorem by Aupetit and Zemanek [1]. We will also present several related results and corollaries valid for real and complex Banach and topological algebras. To simplify the notation we assume that the algebras under consideration have units, however, like in the case of the Hirschfeld-Żelazko Theorem, the analogous results can be stated for none unital algebras as one can formally add a unit to such an algebra.

We use standard notation, as can be found for example in [6]. For a real Banach algebra A with a unit e we denote by A^{-1} the set of invertible elements of A ; for $a \in A$ we define the real and the complex spectrum and the corresponding spectral radius by well known formulas:

$$\begin{aligned} \sigma_{\mathbb{R}}(a) &= \{s \in \mathbb{R} : a - se \notin A^{-1}\}, \\ \sigma_{\mathbb{C}}(a) &= \{s + it \in \mathbb{C} : (a - se)^2 + t^2e \notin A^{-1}\}, \\ \varrho_{\mathbb{R}}(a) &= \sup \{|\lambda| : \lambda \in \sigma_{\mathbb{R}}(a)\}, \\ \varrho_{\mathbb{C}}(a) &= \sup \{|\lambda| : \lambda \in \sigma_{\mathbb{C}}(a)\}, \end{aligned}$$

where we put $\varrho_{\mathbb{R}}(a) = 0$ if the set $\sigma_{\mathbb{R}}(a)$ is empty. It is well known [6] that $\sigma_{\mathbb{C}}(a) = \{\lambda \in \mathbb{C} : a - \lambda e \notin A_{\mathbb{C}}^{-1}\}$, where $A_{\mathbb{C}}$ is the complexification of A , and that $\varrho_{\mathbb{C}}(a) = \lim \sqrt[n]{\|a^n\|}$.

2. The Results

THEOREM 1. *Assume A is a real Banach algebra such that $\|a\| \leq C\varrho_{\mathbb{C}}(a)$ for some positive constant C and all $a \in A$. Then A is isomorphic with a uniformly closed subalgebra of $C_{\mathbb{H}}(X)$ for some compact set X .*

To prove the theorem we need the following special case of the main result of [1].

THEOREM 2 (Aupetit-Zemanek). *Let A be a real Banach algebra with unit. If there is a constant β such that $\varrho_{\mathbb{C}}(ab) \leq \beta\varrho_{\mathbb{C}}(a)\varrho_{\mathbb{C}}(b)$ for all a, b in A then for every irreducible representation $\pi : A \rightarrow L(E)$, the algebra $\pi(A)$ is isomorphic with its commutant C_{π} in the algebra $L(E)$ of all linear transformation on E .*

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PROOF OF THEOREM 1. It is clear that our condition $\|a\| \leq C\varrho_{\mathbb{C}}(a)$ implies that $\varrho_{\mathbb{C}}(ab) \leq \gamma\varrho_{\mathbb{C}}(a)\varrho_{\mathbb{C}}(b)$, with $\gamma = C^2$. Since the commutant C_{π} is a normed real division algebra ([3] p. 127) it is isomorphic with \mathbb{R}, \mathbb{C} , or \mathbb{H} . Hence by Theorem 2 any irreducible representation of A in an algebra of linear maps on a Banach space is equivalent to a representation into \mathbb{H} ; let X be the set of all such equivalent classes of irreducible representations of A . For $x \in X$ and $a \in A$ put $J(a)(x) = x(a) \in \mathbb{H}$. The map J is an isomorphism of A into the algebra of \mathbb{H} -valued functions on X . One could introduce a compact topology on X making all functions $J(a)$ continuous referring to the structure space topology on the space of all irreducible representations; we would like to give a more direct and elementary argument.

Let τ be the topology on X generated by $J(a)$, $a \in A$, that is the weakest topology such that all the functions $J(a)$, $a \in A$, are continuous. By the definition of X , for any two distinct points $x_1, x_2 \in X$ there is an $a \in A$ such that $J(a)(x_1) \neq J(a)(x_2)$, so τ is Hausdorff. By [4] (Ex. 1.5.E(a), p. 72) a topological Hausdorff space is completely regular if and only if its topology is generated by a family of real valued functions. Since an \mathbb{H} -valued function is continuous if and only if all four of its coordinates are X is completely regular and consequently has a maximum compactification βX [4]. Assume $\beta X \setminus X$ is not empty and let $x_0 \in \beta X \setminus X$. The operator $A \ni a \rightarrow (J(a))(x_0) \in \mathbb{H}$ is an irreducible representation on A so $x_0 \in X$, the contradiction shows that X is compact.

To show that J is injective we need to show that A is semisimple. Fix $0 \neq a_0 \in A$ and let A_0 be the closed subalgebra of A generated by all the elements of the form $q(a_0)$, where q is a rational function with real coefficients and with poles outside $\sigma_{\mathbb{C}}(a_0)$. Notice that A_0 is a commutative Banach algebra such that, by our assumption, the spectrum of each nonzero element is not trivial, so $\text{rad}A_0 = \{0\}$. If $b \in A^{-1} \cap A_0$ then b^{-1} as given by a rational function is in A_0 , that means that $A^{-1} \cap A_0 = A_0^{-1}$. Hence by [8] p. 476, $A_0 \cap \text{rad}A \subset \text{rad}A_0 = \{0\}$, so $a_0 \notin \text{rad}A$ and, since a_0 was arbitrary, A is semisimple. \square

COROLLARY 1. Assume A is a real Banach algebra such that $\|a\|^2 \leq C\|a^2\|$ for some positive constant C and all $a \in A$. Then A is isomorphic with a uniformly closed subalgebra of $C_{\mathbb{H}}(X)$ for some compact set X ; if $C = 1$ then the isomorphism is also an isometry.

COROLLARY 2. Assume A is a real m -convex Hausdorff algebra with the topology given by a family $\{p_{\alpha} : \alpha \in \Lambda\}$ of m -convex seminorms. If for every $\alpha \in \Lambda$ there is a constant C_{α} such that $(p_{\alpha}(a))^2 \leq C_{\alpha}p_{\alpha}(a^2)$ for all $a \in A$ then A is a subalgebra of $C_{\mathbb{H}}(X)$ for some Hausdorff set X .

PROOF. Fix $\alpha \in \Lambda$ and let A_{α} be the completion of the normed algebra $A/\ker p_{\alpha}$; we denote by q_{α} the norm on A_{α} generated by p_{α} . Since p_{α} is continuous we have $(q_{\alpha}(f))^2 \leq C_{\alpha}q_{\alpha}(f^2)$ for all $f \in A_{\alpha}$. By the previous Corollary A_{α} is a uniformly closed subalgebra of $C_{\mathbb{H}}(X_{\alpha})$ for some compact set X_{α} . We now obtain the promised representation of A as a subalgebra of $C_{\mathbb{H}}(X)$ by taking a projective limit of the algebras $A_{\alpha} \subset C_{\mathbb{H}}(X_{\alpha})$. \square

THEOREM 3. Assume A is a real Banach algebra such that $\|a\| \leq C\varrho_{\mathbb{R}}(a)$ for some positive constant C and all $a \in A$. Then A is isomorphic with the algebra $C_{\mathbb{R}}(X)$ for some compact set X .

PROOF. Since $\|a\| \leq C\varrho_{\mathbb{R}}(a) \leq C\varrho_{\mathbb{C}}(a)$, it follows from Th. 1 that $A \subset C_{\mathbb{H}}(X)$ for some compact set X . We show that for any $f \in A$ we actually have $f \in C_{\mathbb{R}}(X)$. Let A_f be the closed subalgebra of A generated by f , since A_f is commutative we have $A_f \subset C_{\mathbb{C}}(X)$. For any $g \in A_f$ we have $\sigma_{\mathbb{R}}(g) \subset \{\text{Re } g(x) : x \in X\}$ and consequently by our assumption $\|g\| \leq C\|\text{Re } g\|$. Hence for any sequence (g_n) in A_f if $(\text{Re } g_n)$ is a Cauchy sequence in $C_{\mathbb{R}}(X)$, then (g_n) is a Cauchy sequence in $A_f \subset C_{\mathbb{C}}(X)$ so $\text{Re } A$ is uniformly closed. By Th. 2.3.3 from [6] $A_f = C(X, \tau)$ for some topological involution $\tau : X \rightarrow X$. Assume there is a point x_0 in X such that $\tau(x_0) \neq x_0$ and let U be a neighborhood of x_0 such that $\tau(U) \cap U = \emptyset$. Let $h_0 \neq 0$ be a real valued continuous function with support in U and define

$$h_1(x) = \begin{cases} ih_0(x) & \text{for } x \in U \\ -ih_0(\tau^{-1}(x)) & \text{for } x \in \tau(U) \\ 0 & \text{for } x \in X - (U \cup \tau(U)). \end{cases}$$

We have $h_1 \in C(X, \tau) = A_f$ however $\varrho_{\mathbb{R}}(h_1) = 0$ which contradicts our assumption. Hence $\tau = id_X$ and consequently f is real valued. \square

COROLLARY 3. If K is a compact subset of the complex plane not fully contained in the real line then for any $C > 0$ there is a polynomial p with real coefficients such that

$$\sup\{|\text{Re}(p(z))| : z \in K\} < 1 \text{ while } \sup\{|p(z)| : z \in K\} > C.$$

PROOF. Let A be the uniform closure of the subalgebra of $C_{\mathbb{R}}(K)$ consisting of all polynomials with real coefficients. Let $z_0 \in K \setminus \mathbb{R}$. Since $\overline{f(z_0)} = f(\overline{z_0})$ for all $f \in A$ it follows that A is not of the form $C_{\mathbb{R}}(X)$, for any compact set X . By the last Theorem for any $C > 0$ there is $f \in A$ such that

$$\sup \{ |\operatorname{Re} f(z)| : z \in K \} = \varrho_{\mathbb{R}}(f) < 1 \text{ while } \sup \{ |f(z)| : z \in K \} = \varrho_{\mathbb{C}}(f) > C,$$

and since polynomials with real coefficients are dense in A we get the Corollary. \square

COROLLARY 4. *If A is a real algebra then there is a constant C such that*

$$\varrho_{\mathbb{C}}(a) \leq C \varrho_{\mathbb{R}}(a), \text{ for all } a \in A$$

if and only if

$$\sigma_{\mathbb{R}}(a) = \sigma_{\mathbb{C}}(a), \text{ for all } a \in A.$$

PROOF. One implication is obvious. To prove the other assume there is $a_0 \in A$ such that $K = \sigma_{\mathbb{C}}(a_0) \not\subseteq \mathbb{R}$ and let p be a polynomial with real coefficient given by the previous Corollary. We have $\varrho_{\mathbb{C}}(p(a_0)) = \sup \{ |p(z)| : z \in K \} > C$ while $\varrho_{\mathbb{R}}(p(a_0)) \leq \sup \{ |\operatorname{Re} p(z)| : z \in K \} < 1$ contrary to our assumptions. \square

The above results show that certain inequalities between the real spectrum, the complex spectrum, and the norm in a topological algebra can provide a very strong conclusion concerning the structure of the algebra. Another example of a similar situation is provided by a Kulkarni Theorem [6]: if $\|a\|^2 \leq \|a^2 + b^2\|$ for all a, b with $ab = ba$ in a real algebra A , then the algebra is commutative. It is natural to ask what conclusion could be obtained from an assumption that ab is related by a specific inequality to ba , for arbitrary elements of a noncommutative topological algebra. Some results in this direction can be obtained by using the well known technique (see e.g. [2] p. 330) of analyzing the entire function

$$\lambda \mapsto \exp(\lambda a) b \exp(-\lambda a) \in A.$$

For example if A is a complex m -convex Hausdorff algebra with the topology given by a family $\{p_{\alpha} : \alpha \in \Lambda\}$ of m -convex seminorms, and if for any $\alpha \in \Lambda$ there is a constant C_{α} such that

$$p_{\alpha}(ab) \leq C_{\alpha} p_{\alpha}(ba) \text{ for all } a, b \in A$$

then A is commutative. Indeed in this case for any λ and α we have $p_{\alpha}(\varphi(\lambda)) \leq C_{\alpha} p_{\alpha}(b) < \infty$ so the function $\varphi(\lambda) = e + (ab - ba)\lambda + \dots \lambda^2 + \dots$ is bounded in all the algebras A_{α} , $\alpha \in \Lambda$, where A_{α} is the completion of the normed algebra $A/\ker p_{\alpha}$. Since the continuous linear functionals on A_{α} separate the points of this algebra we get $ab - ba \in \ker p_{\alpha}$, and consequently $ab = ba$. The are two crucial elements in the proof: we consider the complex case, and there is enough continuous linear functionals to separate the points. Hence basically the same proof works for the complex ample p -algebras or Mackey complete locally normed algebras. However it is an open problem whether similar result can be obtained for complex topological algebras without continuous functionals, for example for the class of p -algebras, or for real algebras.

Below we state another problem asking if an almost commutative multiplication must be close to a commutative one.

CONJECTURE 1. *Assume A is a Banach algebra, $\varepsilon > 0$ and that*

$$(2.1) \quad \|ab - ba\| \leq \varepsilon \|a\| \|b\|, \quad a, b \in A.$$

Then there is a commutative multiplication \times on A with $\|ab - a \times b\| \leq \varepsilon' \|a\| \|b\|$ where $\varepsilon' = \varepsilon'(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Notice that the condition (2.1) does not imply that the original multiplication is commutative. Indeed if we take an arbitrary Banach algebra A we can define a new multiplication \cdot on the Banach space A by $a \cdot b = \frac{\varepsilon}{2} ab$. The Banach algebra (A, \cdot) satisfies (2.1) but need not be commutative. The situation does not change if we add an assumption that A has a unit - we can add it artificially to (A, \cdot) . This example does not however contradict the conjecture since \cdot is close to the zero multiplication.

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References

- [1] B. Aupetit and J. Zemanek, *On the real spectral radius in real Banach algebras*, Bull.Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 26 (1978) 969-973.
- [2] V. K. Balachandran, *Topological Algebras*, North-Holland Math. Studies Vol. 185, 2000
- [3] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973
- [4] R. Engelking, *General Topology*, PWN, Warsaw, 1977
- [5] R. A. Hirschfeld and W. Zelazko, *On spectral norm Banach algebras*, Bull.Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 16 (1968) 195-199.
- [6] S. H. Kulkarni and B. V. Limaye, *Real Function Algebras*, Marcel Dekker, Inc. Monographs and textbooks in pure and applied mathematics: 168, 1992.
- [7] C. Le Page, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris Sér. A-B 265 (1967)
- [8] T. Palmer, *Banach Algebras and the General Theory of *-Algebras*. vol. I. Algebras and Banach algebras, Encyclopedia of Mathematics and its Applications, 49, Cambridge Univ. Press, Cambridge, 1994.

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