Function representation of a noncommutative uniform algebra

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Abstract. We construct a Gelfand type representation of a real noncommutative Banach algebra A satisfying $\|f^2\| = \|f\|^2$, for all $f \in A$.

1. Introduction

A uniform algebra A is a Banach algebra such that

$$\left\|f^2\right\| = \|f\|^2 \text{, for all } f \in A. \tag{1.1}$$

The commutative, complex uniform algebras constitute the most classical class of Banach algebras, one which has been intensely studied for decades. It is well known that any such algebra is isometrically isomorphic with a subalgebra of $C_{\mathbb{C}}(X)$ - the algebra of all continuous complex valued functions defined on a compact set X and equipped with the sup-norm topology. Hirschfeld and Żelazko [**5**] proved that the assumption that the algebra is commutative is superfluous - commutativity already follows from (1.1). Algebras of analytic functions serve as standard examples of complex uniform algebras.

Commutative, real uniform algebras have also been studied for years [**9**]. Any such algebra A is isometrically isomorphic with a real subalgebra of $C_{\mathbb{C}}(X)$ for some compact set X; furthermore X can be often divided into three parts X_1, X_2, and X_3 such that $A_{|X_1}$ is a complex uniform algebra, $A_{|X_2}$ consists of complex conjugates of the functions from $A_{|X_1}$, and $A_{|X_3}$ is equal to $C_{\mathbb{R}}(X_3)$. Any commutative, real uniform algebra can be naturally identified with a real subalgebra of

$$C_{\mathbb{C}}(X,\tau) \overset{df}{=} \left\{ f \in C_{\mathbb{C}}(X) : f(\tau(x)) = \overline{f(x)} \text{ for } x \in X \right\}$$

where $\tau : X \to X$ is a surjective homeomorphism with $\tau^2 = id_X$. In case of real uniform algebras the condition (1.1) no longer implies commutativity - the four dimensional algebra of quaternions serves as the simplest counterexample.

There has been very little study of noncommutative real uniform algebras. Only recently it was proved [**8**] that any such algebra is isomorphic with a real subalgebra of $C_{\mathbb{H}}(X)$ - the algebra of all continuous functions defined on a compact set X and taking values in the field $\mathbb{H}$ of quaternions. There is no simple theory of analytic functions of quaternion variables - the standard definition

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$f'(a) = \lim_{q \to 0, q \in \mathbb{H}} (f(a+q) - f(a)) q^{-1}$ is not useful since only linear functions are differentiable (there is an interesting alternative approach to differentiability of such functions - see for example [**10, 12**], that approach is however not applicable for our purpose). Hence there is no obvious example of a non commutative real uniform algebra other then the direct sum of the entire algebra $C_{\mathbb{H}}(X)$ and a commutative real uniform algebra. We prove here that in fact there is no non trivial example and any such algebra is roughly equal to such direct sum; we also obtain a Gelfand type representation of such algebras analogous to $C_{\mathbb{C}}(X, \tau)$. All algebras are assumed to contain a unit.

2. The Results

We first need an easily verifiable elementary description of isomorphisms of the four dimensional real algebra $\mathbb{H}$.

Proposition 1. *A surjective map $T : \mathbb{H} \to \mathbb{H}$ is linear (over the field of real numbers) and multiplicative if and only if*

$$T(a,b,c,d) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} a \\ \begin{bmatrix} b \\ c \\ d \end{bmatrix} \end{bmatrix}, \text{ for } a + bi + cj + dk \in \mathbb{H} \tag{2.1}$$

where M is an isometry of 3 dimensional Euclidean space $\mathbb{R}^3$ preserving the orientation of that space. The space of all such maps forms a 3 dimensional compact connected group which we will denote by $\mathcal{M}$.

We should also notice that if A is a real algebra and $F : A \to \mathbb{H}$ is a multiplicative functional such that $\dim(F(A)) = 2$ then $F(A)$ is isomorphic with $\mathbb{C}$ and the set $\{T \circ F : T \in \mathcal{M}\}$ can be identified with the unit sphere in $\mathbb{R}^3$. Indeed to define an isometry T on $F(A)$ we just have to decide which unit vector of the form $bi + cj + dk$ is to be the image of the imaginary unit of $F(A)$.

Definition 1. *We say that $A \subset C(X)$ separates the points of X if for any $x_1, x_2 \in X$ there is $f \in A$ such that $f(x_1) \neq f(x_2)$. We say that $A \subset C(X)$ strongly separates the points of X if for any $x_1, x_2 \in X$ there is $f \in A$ such that $f(x_1) \neq f(x_2) = 0$.*

Definition 2. *A real algebra A is fully noncommutative if any nonzero linear and multiplicative functional $F : A \to \mathbb{H}$ is surjective.*

Notice that A is not fully noncommutative iff $\{F(a) : a \in A\}$ is isomorphic with $\mathbb{R}$ or with $\mathbb{C}$ for some $\mathbb{H}$-valued multiplicative functional F on A.

Definition 3. *Let $\Phi : \mathcal{M} \to Hom(X) : \Phi(T) = \Phi_T$ be a homomorphism of the group $\mathcal{M}$ onto a group of homeomorphisms of a compact set X. We define*

$$C_{\mathbb{H}}(X, \Phi) \stackrel{df}{=} \{f \in C_{\mathbb{H}}(X) : f \circ \Phi_T(x) = T \circ f(x) \text{ for } x \in X, T \in \mathcal{M}\}.$$

For a commutative real or complex uniform algebra A the standard way to represent A as a subalgebra of $C_{\mathbb{C}}(X)$ is to consider the space $X = \mathfrak{M}(A)$ of all complex valued linear-multiplicative functionals and use the Gelfand transformation:

$$A \ni a \longmapsto \hat{a} \in C(X) : \hat{a}(x) \stackrel{df}{=} x(a). \tag{2.2}$$

For a real algebra A and for any $x \in \mathfrak{M}(A)$ also the complex conjugate $\bar{x} = \tau(x)$ of x is an element of $\mathfrak{M}(A)$ hence the representation of A as a subalgebra of $C_{\mathbb{C}}(X, \tau)$. An important feature of such representation is that $^\wedge$ maps the set of invertible elements of A exactly onto the subset of $\hat{A}$ consisting of functions that do not vanish on X. We will construct a similar representation for a noncommutative uniform algebra.

Theorem 1. *Assume A is a real uniform algebra then there is a compact set X and an isomorphism $\Phi : \mathcal{M} \to Hom(X)$ such that A is isometrically isomorphic with a subalgebra $\hat{A}$ of $C_{\mathbb{H}}(X, \Phi)$. Furthermore $a \in A$ is invertible if and only if the corresponding element $\hat{a} \in \hat{A}$ does not vanish on X.*

If A is fully noncommutative then $\hat{A} = C_{\mathbb{H}}(X, \Phi)$.

We have obvious candidates for X, Φ, and the map $^\wedge$:

$$X \overset{df}{=} \mathfrak{M}_{\mathbb{H}}(A) \quad = \quad \text{quaternion valued real-linear and multiplicative functionals}$$

$$\Phi_T(x) \overset{df}{=} T \circ x, \qquad \hat{a}(x) \overset{df}{=} x(a). \tag{2.3}$$

It is clear that the map $^\wedge$ is a homomorphism of A into $C_{\mathbb{H}}(X, \Phi)$, we need to show that it is an isometry, that it preserves the set of non invertible elements, and that $\hat{A} = C_{\mathbb{H}}(X, \Phi)$ in the fully noncommutative case. So without loss of generality we may assume that A is separable, or equivalently countably generated.

Before proving the theorem it will be useful to notice that for $x \in X$ we may encounter exactly three distinct cases - the set $\{T \circ x : T \in \mathcal{M}\}$ may be

- equal to the singleton $\{x\}$ if $x(A) = \mathbb{R}$, or
- homeomorphic with the unit sphere in $\mathbb{R}^3$ if $x(A)$ is a 2 dimensional commutative subalgebra of $\mathbb{H}$ isomorphic with $\mathbb{C}$, or
- homeomorphic with $\mathcal{M}$ if $x(A) = \mathbb{H}$.

Consequently the set $\mathfrak{M}_{\mathbb{H}}(A)$ can be divided into three parts X_1, X_2, X_3:

- $X_1 = \{x \in X : T \circ x = x \text{ for all } T \in \mathcal{M}\} = \{x \in X : x(A) = \mathbb{R}\}$ - that set is equal to the subset of X consisting of the points where all the functions from $\hat{A}$ are real valued;
- $X_2 = \{x \in X : \dim x(A) = 2\}$ - that set is a union of disjoint copies of the unit sphere in $\mathbb{R}^3$, the algebra $\hat{A}$ restricted to $X_1 \cup X_2$ is commutative;
- $X_3 = \{x \in X : x(A) = \mathbb{H}\}$ - that set is a union of disjoint copies of $\mathcal{M}$ and the algebra $\hat{A}$ restricted to X_3 is fully noncommutative.

That is analogous to the commutative case when a real-linear multiplicative functional x takes only real values if $\tau(x) = x$, or takes all complex values if $\tau(x) \neq x$ and the set $\mathfrak{M}(A)$ can be divided into just two parts.

Put

$$A_{\mathcal{R}} \overset{df}{=} \left\{ \begin{array}{c} a \in A : \hat{a} \text{ is constant on } X_1 \cup X_2 \\ \text{and on } \{T \circ x : T \in \mathcal{M}\} \text{ for each } x \in X_3 \end{array} \right\},$$

and

$$\hat{A}_{\mathcal{R}} = \{\hat{a} \in C_{\mathbb{R}}(X) : a \in A\}.$$

We prove the Theorem in several short steps; the steps also provide a much more detail description of the representation of A:

(1) the map $^\wedge$ defined by (2.3) is injective and A is semisimple,

(2) $\hat{A}_{\mathcal{R}} = \left\{ \begin{array}{c} f \in C_{\mathbb{R}}(X) : f \text{ is constant on } X_1 \cup X_2 \\ \text{and on } \{T \circ x : T \in \mathcal{M}\} \text{ for each } x \in X_3 \end{array} \right\}$, furthermore $\|a\| = \|\hat{a}\|$ for any $a \in A_{\mathcal{R}}$,

(3) $X_1 \cup X_2$ is a weak peak set for $\hat{A}$ so any function from $\hat{A}_{|X_1 \cup X_2} \overset{df}{=} \left\{ f_{|X_1 \cup X_2} \in C(X_1 \cup X_2) : f \in \hat{A} \right\}$ has a norm preserving extension to a function in $\hat{A}$; $\mathfrak{M}\left(\hat{A}_{|X_1 \cup X_2}\right) = X_1 \cup X_2$,

(4) $\{f \in C_{\mathbb{H}}(X, \Phi) : f(x) = 0 \text{ for } x \in X_1 \cup X_2\} \subset \hat{A}$,

(5) $\hat{A} = \left\{ f \in C_{\mathbb{H}}(X, \Phi) : f_{|X_1 \cup X_2} \in \hat{A}_{|X_1 \cup X_2} \right\}$,

(6) the map $^\wedge$ is an isometry and $\hat{A}_{|X_1 \cup X_2}$ is a complete algebra,

(7) any noninvertible element of A is contained in a kernel of a functional from $\mathfrak{M}_{\mathbb{H}}(A)$; $\hat{A} = C_{\mathbb{H}}(X, \Phi)$ if A is fully noncommutative.

The first part follows from Th 3 of [**8**], we provide a independent proof here for completeness. We will need the following special case of the main result of [**1**].

Theorem 2 (Aupetit-Zemanek). *Let A be a real Banach algebra with unit. If $\lim \sqrt[n]{\|a^n\|} = \|a\|$ for all a in A then for every irreducible representation $\pi : A \to L(E)$, the algebra $\pi(A)$ is isomorphic with its commutant C_π in the algebra $L(E)$ of all linear transformation on E.*

Proof. **Part 1**. Since the commutant C_π is a normed real division algebra ([**2**] p. 127) it is isomorphic with $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Let X be the set of all irreducible representations of A; for $x \in X$ and $a \in A$ put $\hat{a}(x) = x(a) \in \mathbb{H}$. The map $^\wedge$ is an isomorphisms of A into the algebra of $\mathbb{H}$-valued functions on X. Let σ be the strongest topology on X such that all the functions $\hat{a}$, $a \in A$, are continuous. Assume (X, σ) is not compact and let x_0 be a point from $\beta X \backslash X$, the operator $A \ni a \to \hat{a}(x_0) \in \mathbb{H}$ is an irreducible representation on A so $x_0 \in X$, contradiction.

To show that $^\wedge$ is injective we need to show that for any $0 \neq a \in A$ there is an irreducible representation π with $\pi(a) \neq 0$. Fix $0 \neq a \in A$ and let A_a be the closed subalgebra of A generated by all the elements of the form $q(a)$, where q is a rational function with real coefficients and with poles outside the spectrum of a. Notice that A_a is a commutative uniform algebra such that so $radA_a = \{0\}$. If $b \in A^{-1} \cap A_a$ then b^{-1} as given by a rational function is in A_a, that means that $A^{-1} \cap A_a = A_a^{-1}$. Hence by [**11**] p. 476, $A_a \cap radA \subset radA_a = \{0\}$, so $a \notin radA$ and, since a was arbitrary, A is semisimple.

Part 2. Put

$$\mathcal{R} = \left\{ (fg - gf)^2 \in C_{\mathbb{H}}(X) : f, g \in \hat{A} \right\}.$$

For arbitrary quaternions $w_p = a_p + b_p i + c_p j + d_p k, p = 1, 2$ we have

$$\begin{aligned}
(w_1 w_2 - w_2 w_1)^2 &= (2(b_1c_2 - b_2c_1)ij + 2(b_1d_2 - b_2d_1)ik + 2(c_1d_2 - c_2d_1)jk)^2 \\
&= 4((b_1c_2 - b_2c_1)k - (b_1d_2 - b_2d_1)j + (c_1d_2 - c_2d_1)i)^2 \\
&= -4\left((b_1c_2 - b_2c_1)^2 + (b_1d_2 - b_2d_1)^2 + (c_1d_2 - c_2d_1)^2\right) \\
&\in C_{\mathbb{R}}(X),
\end{aligned}$$

so $\mathcal{R}$ contains only real valued functions. Let $\sim$ be an equivalence relation on X defined by

$$x_1 \sim x_2 \text{ iff } h(x_1) = h(x_2) \text{ for all } h \in \mathcal{R},$$

let $X_0 = X_{/\sim}$ be the quotient space, and let $\pi : X \to X_0$ be the natural projection. For $x \in X$ there is a function in $\mathcal{R}$ not vanishing at x if and only if $x \in X_3$, furthermore on X_3 we have $x_1 \sim x_2$ if and only if $\ker x_1 = \ker x_2$, or equivalently if and only if $\{T \circ x_1 : T \in \mathcal{M}\} = \{T \circ x_2 : T \in \mathcal{M}\}$. By the Stone-Weierstrass Theorem $\hat{A}_{\mathcal{R}}$ can be identified with a dense subset of $C_{\mathbb{R}}(X_0)$, we shall show that in fact it is equal to the entire $C_{\mathbb{R}}(X_0)$.

$A_{\mathcal{R}}$ is a commutative real uniform algebra so it is isometric with a closed subalgebra of $C_{\mathbb{R}}(\mathfrak{M}(A_{\mathcal{R}}))$; X_0 can be naturally identified with a closed subset of $\mathfrak{M}(A_{\mathcal{R}})$. Assume $X_0 \neq \mathfrak{M}(A_{\mathcal{R}})$, let $y_0 \in \mathfrak{M}(A_{\mathcal{R}}) \backslash X_0$. Since $\mathfrak{M}(A_{\mathcal{R}})$ is equipped with the weak * topology there are $a_1, ..., a_n \in A_{\mathcal{R}}$ such that $\hat{a}_1(y_0) = ... = \hat{a}_n(y_0) = 0$ and $\max\{|\hat{a}_j(y)| : j = 1, ..., n\} \geq 1$ for any $y \in X_0$. Hence $\hat{a}_0 \overset{df}{=} \hat{a}_1^2 + ... + \hat{a}_n^2 \in A_{\mathcal{R}}$ is strictly positive on $X_0 = X_{/\sim}$, and since A is semisimple a_0 is an invertible element of $A_{\mathcal{R}} \subset A$ contrary to the fact it is in the kernel of the functional y_0. The contradiction shows that $X_0 = \mathfrak{M}(A_{\mathcal{R}})$ and consequently $A_{\mathcal{R}} = C_{\mathbb{R}}(X_0)$. Assume there is $a \in A_{\mathcal{R}}$ such that $\|a\| = 1 > \|\hat{a}\|$. Since $\|\hat{a}^n\| \to 0$ while $\|a^n\| = 1$ it follows that the range of $^\wedge$ is not complete - that contradiction shows that $^\wedge$ is an isometry on $A_{\mathcal{R}}$.

Part 3. We recall that a subset K of the maximal ideal space of a function algebra is called a weak peak set iff for any open neighborhood U of K there is a function f in that algebra such that $f(x) = 1 = \|f\| > |f(x')|$ for any $x \in K$ and any $x' \notin U$. Since $C_{\mathbb{R}}(X_0) \simeq A_{\mathcal{R}} \subset A$ it follows that $X_1 \cup X_2$ is a weak peak set. Assume $a \in A$ with $\sup\{|\hat{a}(x)| : x \in X_1 \cup X_2\} = 1$. Since since $\mathcal{M}$ consists of isometries $\frac{1}{\max\{1,|\hat{a}|\}}$ is constant on the sets $\{T \circ x : T \in \mathcal{M}\}$ for $x \in X_3$ as well as on the set $X_1 \cup X_2$. Hence $\frac{1}{\max\{1,|\hat{a}|\}} \in \hat{A}$ so $\hat{a} \cdot \frac{1}{\max\{1,|\hat{a}|\}} \in \hat{A}$; that function coincides with $\hat{a}$ on $X_1 \cup X_2$ and has norm 1. It is important to notice for further reference that since $^\wedge$ is an isometry on $A_{\mathcal{R}}$ we have $\left\| a \cdot \frac{1}{\max\{1,|a|\}} \right\| \leq \|a\| \left\| \frac{1}{\max\{1,|a|\}} \right\| = \|a\|$ so not only $a \cdot \frac{1}{\max\{1,|a|\}}$ is a norm preserving extension of a with respect to the spectral norm but also with respect to the original norm of the algebra A.

The maximal ideal space of $A_{|X_1 \cup X_2}$ is equal to $X_1 \cup X_2$ since any linear and multiplicative functional on $A_{|X_1 \cup X_2}$ gives a functional on as well A.

Part 4. Fix $x_0 \in X_3$ and let $f_1, f_2, f_3, f_4 \in A$ be such that $f_1(x_0) = 1, f_2(x_0) = i, f_3(x_0) = j, f_4(x_0) = k$. Let U_0 be an open neighborhood of x_0 such that $|f_p(x) - f_p(x_0)| < 1/4$ for $x \in U_0$ and $p = 1, 2, 3, 4$. Let $g_0 \in C_{\mathbb{H}}(X, \Phi)$ and assume the support of g_0 is contained in $V_0 \overset{df}{=} \pi^{-1}(\pi(U_0))$. For any $x \in U_0$, the numbers $f_1(x), f_2(x), f_3(x), f_4(x)$ can be seen as linearly independent vectors in $\mathbb{H} \simeq \mathbb{R}^4$ so there are unique real valued functions h_p defined on U_0 such that

$$g_0(x) = \sum_{p=1}^{4} h_p(x) f_p(x) \text{ for } x \in U_0. \tag{2.4}$$

Let $x \in U_0$ and let $x' = T \circ x$ be another point of $\pi^{-1}(\pi(x))$, where $T \in \mathcal{M}$. By the definition of $C_{\mathbb{H}}(X, \Phi)$ we have

$$g_0(x') = T \circ g_0(x) = \sum_{p=1}^{4} h_p(x) T \circ f_p(x) = \sum_{p=1}^{4} h_p(x) f_p(x').$$

Hence the functions $h_p, p = 1, 2, 3, 4$ are constant on the sets of the form $\pi^{-1}(\pi(x))$ and consequently can be naturally extended to V_0, furthermore they can be extended to the entire set X by assigning value zero outside V_0. Since the coefficients with respect to a fixed basis in $\mathbb{R}^4$ are continuous functions of a vector in $\mathbb{R}^4$, the functions $h_p, p = 1, 2, 3, 4$ belong to $A_{\mathcal{R}} \subset A$. Hence $g_0 \in A$.

We proved that for any point x_0 in X there is a neighborhood V_0 of $\pi^{-1}(\pi(x))$ such that $A_{|V_0} = C_{\mathbb{H}}(X, \Phi)_{|V_0}$. Let $V_s = \pi^{-1}(\pi(U_s)), s \in S$ be an open cover of X consisting of such sets. Since A is separable and X_3 is a subset of its dual space equipped with the weak * topology, X_3 is metrizable and consequently paracompact. Let $\sum_{s\in S} \alpha_s = 1$ be a locally finite partition of unity in $C_{\mathbb{R}}(X_3)$ subordinated to the cover $\pi(U_s), s \in S$ of X_0 ([**3**], page 375). For any $g \in C_{\mathbb{H}}(X, \Phi)$ such that $g(x) = 0$ for $x \in X_1 \cup X_2$ we have $g(x) = \sum_{s\in S} \alpha_s \circ \pi(x) g(x)$ where $\operatorname{supp}\alpha_s \circ \pi g \subset V_s$ so $g \in A$.

Part 5. Assume $f \in C_{\mathbb{H}}(X, \Phi)$ is such that $f_{|X_1\cup X_2} \in \hat{A}_{|X_1\cup X_2}$. Let $g \in A$ be such that $g_{|X_1\cup X_2} = f_{|X_1\cup X_2}$. Since $f - g \in \{f \in C_{\mathbb{H}}(X, \Phi) : f(x) = 0 \text{ for } x \in X_1 \cup X_2\}$ by the previous step $f = g + (f - g) \in A$.

Part 6. Put $I = \{a \in A : \hat{a}(x) = 0 \text{ for } x \in X_1 \cup X_2\}$ and let $B = A/I$ be the quotient algebra. Notice that B is a commutative Banach algebra with $\mathfrak{M}(B) = X_1 \cup X_2$. Assume there is $a + I \in B$ such that $\|a + I\| = \|a\| > 2$ and $\|\hat{a}\| \leq 1$. By Part 3 we may assume that $\|\hat{a}\| = \left\|\hat{a}_{|X_1\cup X_2}\right\|$. Suppose that $\left\|a^2 + I\right\| < 1$ so there is $g \in I$ with $\left\|a^2 + g\right\| < 1$, by Part 3 again we may assume that $\hat{a}^2 + \hat{g}$ is smaller than 1 on X and consequently $\|g\| = \|\hat{g}\| < 2$. We have

$$\left\|a^2 + g\right\| < 1, \|g\| < 2, \left\|a^2\right\| = 4,$$

which is impossible. The above proves that

$$\left\|b^2\right\| \geq \frac{1}{4} \|b\|^2, \text{ for all } f \in B.$$

so B is isomorphic with a complete uniform algebra which must be equal to $\hat{A}_{|X_1\cup X_2}$; by the same arguments as at the end of Part 3 we conclude that $\wedge$ is in fact an isometry.

Part 7. An obvious consequence of Part 5 since $\mathfrak{M}\left(\hat{A}_{|X_1\cup X_2}\right) = X_1 \cup X_2$. $\square$

COROLLARY 1. *Assume A is a fully non commutative closed subalgebra of $C_{\mathbb{H}}(Y)$. Then $A = C_{\mathbb{H}}(Y)$ if and only if A strongly separates the points of Y.*

We do not assume here that A contains all constant functions; if we assume that the constant functions i, j, k are in A then the above version of the Stone-Weierstrass Theorem easily follows from following fact: $f \in A$ implies that $\operatorname{Re} f = (f - ifi - jfj - kfk)/4 \in A$ ([**6**]).

For a fully non commutative real uniform algebra A its maximal ideal space $\mathfrak{M}_{\mathbb{H}}(A)$ consists of disjoint copies of $\mathcal{M}$, and is locally homeomorphic with a Cartesian product of $\mathcal{M}$ with another set. Quite often we also have a global decomposition: $\mathfrak{M}_{\mathbb{H}}(A) = Y \times \mathcal{M}$. However such global decomposition can not always be achieved; not even its analog in the commutative case. In the commutative case when $A \subset C_{\mathbb{C}}(X, \tau)$ and we can divide X into three parts X_1, X_2, and X_3 such that $A_{|X_1}$ is a complex algebra, $A_{|X_2}$ consists of complex conjugates of the functions from $A_{|X_1}$, and $A_{|X_3}$ is equal to $C_{\mathbb{R}}(X_3)$. It is easy to select $X_3 = \{x \in X : \tau(x) = x\}$, we may also easily select one point form each equivalence class $x_1 \sim x_2$ iff $\tau(x_1) =$

x_2 to get X_1 and we can form X_2 from the remaining points. However we may not be able to make the sets X_1, X_2 closed so $A_{|X_1}, A_{|X_2}$ may not be uniform algebras - see for example $A = \left\{ f \in C_{\mathbb{C}}\left(S^1\right) : f\left(e^{it}\right) = \overline{f\left(e^{i(t+\pi)}\right)} \text{ for } 0 \leq t \leq \pi \right\}$ where S^1 is a unit circle. This is the reason we represent a real commutative uniform algebra as a subalgebra of $C_{\mathbb{C}}(X, \tau)$ rather then, what may seem more appealing, a direct sum of $A_{|X_1}, A_{|X_2}$, and $A_{|X_3}$.

We proved that the algebra $\mathbb{H}$ has the following property:

> Any subalgebra A of $C_{\mathbb{H}}(X) = C_{\mathbb{R}}(X) \otimes \mathbb{H}$ that strongly separates the points of X and such that $\{f(x) : x \in X\} = \mathbb{H}$ for all $x \in X$ is equal to $C_{\mathbb{H}}(X)$.

We also know (Stone-Weierstrass Theorem) that the algebra $\mathbb{R}$ has the same property. It would be interesting to find other algebras with the same property.

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References

[1] B. Aupetit and J. Zemanek, *On the real spectral radius in real Banach algebras*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 26 (1978) 969-973.

[2] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973

[3] R. Engelking, *General Topology*, Mathematical Monographs, Vol. 60, PWN—Polish Scientific Publishers, Warsaw, 1977. 626 pp.

[4] S. Heinrich, *Ultrapowers in Banach space theory*, J. Reine Angew. Math., 313 (1980), 72-104.

[5] R. A. Hirschfeld and W. Żelazko, *On spectral norm Banach algebras.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 195–199.

[6] J. Holladay, *A note on the Stone-Weierstrass theorem for quaternions.* Proc. Amer. Math. Soc. 8 (1957), 656–657.

[7] K. Jarosz , *Ultraproduct and small bound perturbations*, Pacific J. Math., 148 (1991) 81-88.

[8] K. Jarosz, *Noncommutative Uniform Algebras*, Studia Math. 162 (2004), 213-218.

[9] S. H. Kulkarni and B. V. Limaye, *Real Function Algebras*, Monographs and Textbooks in Pure and Applied Math., 168, Marcel Dekker, Inc., New York, 1992.

[10] S. V. Lüdkovsky and F. van Oystaeyen, *Differentiable functions of quaternion variables.* Bull. Sci. Math. 127 (2003), no. 9, 755–796.

[11] T. Palmer, *Banach Algebras and the General Theory of *-Algebras. vol. I. Algebras and Banach algebras*, Encyclopedia of Mathematics and its Applications, 49, Cambridge Univ. Press, Cambridge, 1994.

[12] A. Sudbery, *Quaternionic analysis.* Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 2, 199–224.

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