Unbounded operators in the context of C^*-algebras

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Będlewo, 23.07.2009

Affiliation relation η

Murray, von Neumann:	W*-algebra context	1936
Baaj, Julg:	C^*-algebra context	1980
SLW	C^*-algebra context	1990

In many situations it is desirable to extend W*- or C^*-algebra by including some unbounded elements (and extend non-unital C^*-algebra by including some invertible elements). These additional elements are *affiliated* to the considered algebra. After Murray, von Neumann we use symbol η to denote the affiliation relation.

Warning:

$$\eta_{W^*} \neq \eta_{\mathrm{C}^*}$$

Commutative C^*-algebras

Any commutative C^*-algebra is of the form $C_0(\Lambda)$, where Λ is a locally compact space. In practice however we often deals with functions f that are continuous, but do not tend to 0 at infinity.They do not belong to $C_0(\Lambda)$. Instead we say that they are affiliated with $C_0(\Lambda)$:

$$f \ \eta \ C_0(\Lambda),$$

$$C(\Lambda) = C_0(\Lambda)^\eta.$$

Group C^*-algebras

To any locally compact group G one assigns a C^*-algebra $C^*(G)$ whose representations are in one to one correspondence with unitary representations of G. Elements of G do not belong to $C^*(G)$. Instead they are affiliated to $C^*(G)$:

$$G \subset C^*(G)^\eta.$$

If G is a Lie group then elements of Lie algebra $\mathfrak{g}$ are affiliated to $C^*(G)$

$$\mathfrak{g} \subset C^*(G)^\eta.$$

Some elements of enveloping algebra are affiliated to $C^*(G)$: If $(X_1, X_2, \ldots, X_n)$ is a basis of $\mathfrak{g}$ then

$$X_1^2 + X_2^2 + \cdots + X_n^2 \;\eta\; C^*(G).$$

Observables in quantum physics

C^*-algebra A of observables consists of bounded elements. The most important physical observables are unbounded: position, momentum, energy, angular momentum, number of particles etc. They are affiliated to A:

$$x, p, H, M, N, \dots\ \eta\ A$$

Finite-dimensional representations of non-compact quantum groups

Let $G = (A, \Delta)$ be a locally compact quantum group, e.g. quantum $SL(2, \mathbb{R})$. The fundamental 2-dimensional representation of G is a matrix of the form

$$u = \begin{pmatrix} \alpha \, , \beta \\ \gamma \, , \delta \end{pmatrix}.$$

For quantum $SU(2)$, $\alpha, \beta, \gamma, \delta \in A$, For quantum $SL(2, \mathbb{R})$, matrix elements are unbounded, so $\alpha, \beta, \gamma, \delta \; \eta \; A$.

What affiliated elements are?

- Consider embeding:

  $$\begin{array}{ccc} A & \longrightarrow & B(A) \\ a & \longmapsto & (x \mapsto ax) \end{array}$$

  We identify elements of a C^*-algebra with (some) bounded operators acting on A (by left multiplication).
- Elements affiliated to A will be identified with (some) unbounded densely defined closed operators acting on A (Unbounded multipliers):

  $$A^\eta \subset \left\{ \begin{array}{c} \text{Closed, densely defined} \\ \text{operators acting on } A \end{array} \right\}$$

Definition

T $\qquad$ $D(T)$ is a dense linear subset of A.

$$\text{Graph } T = \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} : x \in D(T) \right\} \subset A \oplus A$$

$$(\text{Graph } T)^{\perp} = \left\{ \begin{pmatrix} c \\ d \end{pmatrix} : \begin{matrix} c^* x + d^* Tx = 0 \\ \text{for all } x \in D(T) \end{matrix} \right\}$$

Definition

$$A^{\eta} = \left\{ T : \begin{matrix} (\text{Graph } T) + (\text{Graph } T)^{\perp} = A \oplus A \\ \left\{ d : \begin{pmatrix} c \\ d \end{pmatrix} \in (\text{Graph } T)^{\perp} \right\} \text{ is dense in } A \end{matrix} \right\}$$

Elementary properties

Proposition

- *$D(T)$ is a right ideal in A.*
- *$T(xa) = T(x)a$ for all $x \in D(T)$ and $a \in A$.*

If A is unital ($I \in A$) then a dense subset of A contains invertible elements. Therefore $D(T) = A$ and for any $x \in A$ we have: $Tx = T(Ix) = T(I)x$. It shows that T is the left multiplication by $T(I)$. This way we proved

Proposition

$$\Bigl(A - \mathrm{unital}\Bigr) \Longrightarrow \Bigl(A^\eta = A\Bigr)$$

Any continuos function on a compact space is bounded.

Multiplier algebra $M(A)$

$$M(A) = \{T \in A^\eta : \|T\| < \infty\}$$

$M(A)$ is a unital C^*-algebra, $A \subset M(A)$, A is an essential ideal in $M(A)$.

Example

Let G be a locally compact group and $C^*(G)$ be its C^*-algebra. Then elements of G are bounded elements affiliated to $C^*(G)$. Therefore we have an inclusion $G \subset M(C^*(G))$.

Adjoint element

$$T \ \eta \ A \Longrightarrow T^* \ \eta \ A$$

Definition

$$\left(\begin{array}{c} y \in D(T^*) \\ z = T^*y \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} x^*z = (Tx)^*y \\ \text{for any } x \in D(T) \end{array}\right).$$

$$x^*(T^*y) = (Tx)^*y$$

for any $x \in D(T)$ and $y \in D(T^*)$. Clearly

$$(T^*)^* = T.$$

z-transform

Let $T \ \eta \ A$. Then T^*T is a selfadjoint element affiliated to A and using the functional calculus one may introduce bounded multiplier

$$z_T = T\left(I + T^*T\right)^{-\frac{1}{2}} \in M(A).$$

We have: $\|z_T\| \leq 1$ and

$$T = z_T\left(I - z_T^* z_T\right)^{-\frac{1}{2}}$$

$$\Bigl(T \ \eta \ A\Bigr) \Longleftrightarrow \left(\begin{array}{c}\text{There exists } z \in M(A) \text{ with } \|z\| \leq 1 \\ \text{such that } (I - z^*z)A \text{ is dense in } A \\ \text{and } T : (I - z^*z)^{\frac{1}{2}}x \longmapsto zx\end{array}\right).$$

Examples

Let Λ be a locally compact space and $A = C_0(\Lambda)$. Then $M(A) = C_{\mathrm{bounded}}(\Lambda)$ and $A^\eta = C(\Lambda)$.

Let $A = \mathcal{K}(H)$ be the algebra of all compact operators acting on a Hilbert space H. Then $M(A) = B(H)$ and $A^\eta = \mathcal{C}(H)$ is the set of all closed, densely defined operators acting on H. So A^η is not an algebra (even not a vector space).

If $A \subset B(H)$ then $M(A) \subset B(H)$ and $A^\eta \subset \mathcal{C}(H)$

$$M(A) = \left\{ a \in B(H) : \begin{array}{c} ax, xa \in A \\ \text{for all } x \in A \end{array} \right\}$$

Perturbation theory

Theorem

$$\left(\begin{array}{c} T\ \eta\ A \\ a \in M(A) \end{array}\right) \implies (T + a\ \eta\ A)\,,$$

$$\left(\begin{array}{c} T\ \eta\ A \\ a, a^{-1} \in M(A) \end{array}\right) \implies (Ta, aT\ \eta\ A)\,.$$

Operator theory in a new setting

By operator theory I mean the theory of closed densely defined operators acting on a Hilbert space. Formula $\mathcal{C}(H) = \mathcal{K}(H)^{\eta}$ shows that this theory is related to the particular C^*-algebra $\mathcal{K}(H)$ of all compact operators. What if $\mathcal{K}(H)$ is replaced by an arbitrary separable C^*-algebra? The following topics have been considered:

- Functional calculus of normal operators,
- Infinitesimal generators of one-parameter groups of unitary operators,
- Nelson theory (when the representation of a Lie algebra integrates to the representation of the Lie group),
- Selfadjoint extensions of symmetric operators,
- Positive selfadjoint extensions of positive operators.

Functional calculus of normal operators

Theorem

Let $T \ \eta \ A$ *such that* $T^*T = TT^*$ *and* $\Lambda = \operatorname{Sp} T \subset \mathbb{C}$. *Then there exists unique morphism* $\varphi_T \in \operatorname{Mor}(C_0(\Lambda), A)$*:*

$$\varphi_T : f \longmapsto f(T)$$

such that $\varphi_T(\mathrm{id}_\Lambda) = \mathrm{id}_\Lambda(T) = T$.

Infinitesimal generators of one-parameter groups of unitary operators

Theorem

Let $(U_t)_{t\in\mathbb{R}}$ *be a strictly continuous one-parameter group of unitary elements of* $M(A)$ *and* T *be an operator acting on* A *in the following way:*

$$Tx = \lim_{t\to 0} \frac{U_t x - x}{it}$$

with $D(T)$ *consisting of all* $x \in A$ *for which the above limit exists in norm topology. Then* T *is a selfadjoint element affiliated to* A *and*

$$U_t = e^{itT}.$$

Some formulae involving z-transform

$$z_{T^*} = (z_T)^*$$

$$(T \subset S) \Longleftrightarrow \left(z_S \left(I - z_T^* z_T\right)^{\frac{1}{2}} = \left(I - z_S z_S^*\right)^{\frac{1}{2}} z_T \right)$$

An element $T \;\eta\; A$ is symmetric if $T \subset T^*$. Let z be the z-transform of a symmetric operator. Then

$$z^* \left(I - z^* z\right)^{\frac{1}{2}} = \left(I - z^* z\right)^{\frac{1}{2}} z$$

Selfadjoint extensions of symmetric operators

Theorem

Let $T\ \eta\ A$ be symmetric, $z = z_T$ and E^+ and E^- be elements of $M(A)$ defined by

$$E^{\pm} = z^*z \pm i\left(z\,(I - z^*z)^{\frac{1}{2}} - (I - z^*z)^{\frac{1}{2}}\,z^*\right) - zz^*.$$

Then

- *E^+ and E^- are orthogonal projections.*
- *T is selfadjoint if and only if $E^{\pm} = 0$.*
- *The set of selfadjoint extensions of T is in one to one correspondence with the set of $v \in M(A)$ such that $v^*v = E^+$ and $vv^* = E^-$.*

Friedrichs extention

$T \ \eta \ A$ is called positive if $x^*Tx \geq 0$ for all $x \in D(T)$. We endow $D(T)$ with the norm

$$|||x||| = \sqrt{\|x^*(T+I)x\|}.$$

Let $\tilde{D}(T) \subset A$ be the completion of $D(T)$ with respect to this norm, $D_F = \tilde{D}(T) \cap D(T^*)$ and $T_F = T^*|_{D_F}$. Then $T \subset T_F$.

Problem

Is T_F affiliated to A? The answer is "yes" if $LM(A) = M(A)$.

Category of C*-algebras

$$\alpha \in \mathrm{Mor}(A,B) \iff \left(\begin{array}{c} \alpha \text{ is a } {}^*\text{−algebra homomorphism} \\ A \longrightarrow M(B) \\ \text{such that } \alpha(A)B \text{ is dense in } B \end{array} \right).$$

Examples

- *Continuous map* $\Lambda \xrightarrow{f} \Lambda'$ *produces (by inverse image)* $f_* \in \mathrm{Mor}(C_0(\Lambda'), C_0(\Lambda))$.
- *Group homomorphism* $G \xrightarrow{\varphi} H$ *extends to* $\varphi \in \mathrm{Mor}(C^*(G), C^*(H))$.
- $A \ni a \xmapsto{\alpha} a \otimes I \in M(A \otimes B)$ *is a morphism* $\alpha \in \mathrm{Mor}(A, A \otimes B)$.

Functorial properties

Any $\alpha \in \mathrm{Mor}(A, B)$ extends to a mapping $A^\eta \xrightarrow{\alpha} B^\eta$.

Definition

Let $\alpha \in \mathrm{Mor}(A, B)$ and $T \; \eta \; A$. By definition $\alpha(T)$ is the closed operator acting on B such that

- *$\alpha(D(T))B$ is an essential domain of $\alpha(T)$ and*
- *$\alpha(T)\alpha(a)b = \alpha(T(a))b$ for any $a \in D(T)$ and $b \in B$.*

Remark: Let $B = \mathcal{K}(H)$. Then $M(B) = B(H)$, $\mathrm{Rep}(A, H) = \mathrm{Mor}(A, B)$, $B^\eta = \mathcal{C}(H)$ and

$$\pi : A^\eta \longrightarrow \mathcal{C}(H)$$

for any $\pi \in \mathrm{Rep}(A, H)$.

Natural topologies

- On A: uniform (norm) topology,
- On $M(A)$: strict topology: $a_\lambda \longrightarrow 0$ iff $\|a_\lambda x\|$ and $\|x a_\lambda\|$ tend to 0 for any $x \in A$,
- On A^η: topology of almost uniform convergence: $T_\lambda \longrightarrow T_\infty$ iff z_{T_λ} tends to z_{T_∞} strictly,
- On $\mathcal{C}(H) = \mathcal{K}(H)^\eta$: the natural topology as defined above,
- On $\mathrm{Mor}(A, B)$: $\alpha_\lambda \longrightarrow \alpha_\infty$ iff for any $a \in A$, $\alpha_\lambda(a)$ tends to $\alpha_\infty(a)$ strictly,
- On $\mathrm{Rep}(A, H) = \mathrm{Mor}(A, \mathcal{K}(H))$: the natural topology as defined above.

Tensoring with commutative C^*-algebra

Theorem

With the natural topologies we have:

$$\begin{aligned} C_0(\Lambda) \otimes A &= C_0(\Lambda, A), \\ M\left(C_0(\Lambda) \otimes A\right) &= C_{\mathrm{bounded}}(\Lambda, M(A)), \\ \left(C_0(\Lambda) \otimes A\right)^\eta &= C(\Lambda, A^\eta), \\ \mathrm{Mor}(A, C_0(\Lambda) \otimes B) &= C(\Lambda, \mathrm{Mor}(A, B)). \end{aligned}$$

C^*-algebra generated by affiliated elements

Theorem

Let A be a C^*-algebra and $T_1, T_2, \ldots, T_N \;\eta\; A$.

$$\Psi : \begin{array}{ccc} \mathrm{Rep}(A,H) & \longrightarrow & \mathcal{C}(H)^N \\ \Cup & & \Cup \\ \pi & \longmapsto & (\pi(T_1), \pi(T_2), \ldots, \pi(T_N)) \end{array}$$

Then the following conditions are equivalent:

- *The image of Ψ is closed in $\mathcal{C}(H)^N$ and Ψ is a homeomorphism from $\mathrm{Rep}(A,H)$ onto the image.*
- *For any $\pi \in \mathrm{Rep}(A,H)$ and C^*-algebra $B \subset B(H)$ we have:*

$$\left(\begin{array}{c} \pi(T_i) \;\eta\; B \\ i = 1, 2, \ldots, N \end{array}\right) \Longrightarrow \Big(\pi \in \mathrm{Mor}(A,B)\Big)$$

We say that A is generated by $T_1, T_2, \ldots, T_N \,\eta\, A$, if the conditions in the previous Theorem are satisfied. Since no effective procedure leading from $T_1, T_2, \ldots, T_N$ to A is known, it is better to say that the set $\{T_1, T_2, \ldots, T_N\}$ is total in A.

Presentations of C^*-algebras

Presentation: The way of introducing a C^*-algebra in terms of generators and relations.
Consider a sequence of symbols (called generators) $T_1, T_2, \dots T_N$ subject to a set of relations $\mathcal{R}$. Let $\mathrm{Solutions}(\mathcal{R})$ be the subset of $\mathcal{C}(H)^N$ consisting of all N-tuples of closed operators that satisfy the relations $\mathcal{R}$. We look for a C^*-algebra A with affiliated elements $T_1, T_2, \dots T_N \ \eta \ A$ such that the mapping

$$\Psi : \pi \longmapsto \Big(\pi(T_1), \pi(T_2), \dots, \pi(T_N)\Big)$$

is a homeomorphism from $\mathrm{Rep}(A; H)$ onto $\mathrm{Solutions}(\mathcal{R})$. By the theorem, A is unique.

Example: Quantum $SU(2)$

$$S_qU(2) \quad \rightsquigarrow \quad \boxed{\begin{array}{cc} \text{Generators :} & \alpha, \gamma \\ \text{Relations :} & \left\{\begin{array}{c} \alpha^*\alpha + \gamma^*\gamma = I \\ \alpha\alpha^* + q^2\gamma^*\gamma = I \\ \alpha\gamma = q\gamma\alpha \\ \alpha\gamma^* = q\gamma^*\alpha \\ \gamma\gamma^* = \gamma^*\gamma \end{array}\right. \\ \text{Comultiplication :} & \left\{\begin{array}{c} \Delta\alpha = \alpha \otimes \alpha - q\gamma^* \otimes \gamma \\ \Delta\gamma = \gamma \otimes \alpha + \alpha^* \otimes \gamma \end{array}\right. \end{array}}$$

Example: Quantum $E(2)$

$$E_q(2) \quad \rightsquigarrow \quad \boxed{\begin{array}{cc} \text{Generators :} & v, n \\ \text{Relations :} & \left\{ \begin{array}{c} v \text{ is unitary} \\ n \text{ is normal} \\ vn = qnv \\ \operatorname{Sp}|n| \subset q^{\mathbb{Z}} \cup \{0\} \end{array} \right. \\ \text{Comultiplication :} & \left\{ \begin{array}{c} \Delta v = v \otimes v \\ \Delta n = v \otimes n + n \otimes v^* \end{array} \right. \end{array}}$$

Example: Quantum $SU(1,1)$

$$S_qU(1,1) \rightsquigarrow \boxed{\begin{array}{cc} \text{Generators}: & \alpha, \gamma \\ \text{Relations}: & \left\{\begin{array}{c} \alpha^*\alpha - \gamma^*\gamma = I \\ \alpha\alpha^* - q^2\gamma^*\gamma = I \\ \alpha\gamma = q\gamma\alpha \\ \alpha\gamma^* = q\gamma^*\alpha \\ \gamma\gamma^* = \gamma^*\gamma \end{array}\right. \\ \text{Comultiplication}: & \left\{\begin{array}{c} \Delta\alpha = \alpha \otimes \alpha + q\gamma^* \otimes \gamma \\ \Delta\gamma = \gamma \otimes \alpha + \alpha^* \otimes \gamma \end{array}\right. \end{array}}$$

Problem

$$A \;\rightsquigarrow\; \boxed{\begin{array}{ll} \text{Generators :} & \qquad T \\[2ex] \text{Relations :} & \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \end{array}}$$

Does there exist a C^*-algebra A and an element $T \;\eta\; A$ such that

$$\Psi : \quad \begin{array}{ccc} \mathrm{Rep}(A, H) & \longrightarrow & \mathcal{C}(H) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \pi & \longmapsto & \pi(T) \end{array}$$

is a homeomorphism?