

Operator algebras associated with subproduct systems

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Definition 0.1 *let \mathcal{S} be an (abelian) semigroup. A subproduct system of Hilbert spaces is a family $X = \{X(s)\}_{s \in \mathcal{S}}$ of Hilbert spaces such that*

(1) $X(0) = \mathbb{C}$.

(2) *For every $s, t \in \mathcal{S}$ there is a coisometric mapping*

$$U_{s,t} : X(s) \otimes X(t) \rightarrow X(s+t).$$

(3) *The maps satisfy the associativity condition*

$$U_{s+t,r}(U_{s,t} \otimes I_{X(r)}) = U_{s,t+r}(I_{X(s)} \otimes U_{t,r}).$$

If all the maps $U_{s,t}$ are unitaries, we get a product system.

Example 0.2 $\mathcal{S} = \mathbb{N}$, $X(1) = H$ (a given Hilbert space) and $X(n) = H^{\otimes n}$. This is a (full) product system.

Example 0.3 $\mathcal{S} = \mathbb{N}$, $X(1) = H$ (a given Hilbert space) and $X(n) = H^{\otimes n}$ (the symmetric tensor product). This is a subproduct system with $U_{m,n}(x \otimes y) = p_{n+m}(x \otimes y)$ where p_{n+m} is the orthogonal projection of $H^{\otimes n}$ onto the symmetric tensor product.

Remark 0.4 *The Hilbert spaces can be replaced by C^* -correspondences (or C^* -bimodules) over a given C^* -algebra.*

The importance of subproduct systems: It can serve as a framework

- (1) To study semigroups of CP maps on von Neumann algebras (and their dilations).
- (2) To study operator algebras generated by tuples of operators satisfying some homogeneous relations.

Semigroups of CP maps

In 1989 Arveson introduced product systems over \mathbb{R}_+ as a tool for studying and classifying E_0 -semigroups (semigroups $\alpha = \{\alpha_t\}_{t \geq 0}$ of $*$ -endomorphisms of $B(H)$ with some continuity property). These product systems were classified into type I, II and III and there are still many questions about their structure. Later, Arveson looked at semigroups of CP maps (completely positive, contractive and normal) on $B(H)$.

Arveson associated to every CP map $\Theta : B(H) \rightarrow B(H)$ a “metric operator space” H_Θ which is a Hilbert space. Given commuting CP maps Θ and Ψ , he showed that there is a natural coisometric map

$$H_\Theta \otimes H_\Psi \rightarrow H_{\Psi\Theta}.$$

Thus, if $\{\Theta_s\}_{s \in \mathcal{S}}$ is a semigroup of CP maps, $\{H_{\Theta_s}\}$ is a subproduct.

Remark 0.5 *When one replaces $B(H)$ by a general von Neumann algebra M , Hilbert spaces are replaced by correspondences over M' .*

An important question about a CP semigroup $\{\Theta_s\}_{s \in \mathcal{S}}$ is the dilation question: Can one find an a semigroup $\{\alpha_s\}$ of *-endomorphisms on some $B(K)$ (with K containing H) such that for every $x \in B(K)$, $P_H \alpha_s(x) P_H = \Theta_s(P_H x P_H)$?

When trying to dilate a CP semigroup on $B(H)$ to a semigroup of *-endomorphisms on (a larger) $B(K)$ one faces the following question.

Question: Given a subproduct system $X = \{X(s)\}$. Can it be embedded in a product system ?

Facts: For $\mathcal{S} = \mathbb{N}$ or \mathbb{N}^2 or \mathbb{R}_+ the answer is always positive. (And, in all these cases, dilation exists). For $\mathcal{S} = \mathbb{N}^3$ we have a counterexample. This counterexample yields an example of 3 commuting CP maps that cannot be simultaneously dilated to a triple of commuting *-endomorphisms (on a larger $B(K)$). For $\mathcal{S} = \mathbb{R}_+^2$ the answer is unknown.

Remark 0.6 *Very recently, Bhat and Mukherjee used subproduct systems as a tool to compute the index of certain product systems. In a related work, Tsirelson has studied subproduct systems with $\dim X(s) = 2$ for every $s \neq 0$.*

Operator Algebras:

Here we set $\mathcal{S} = \mathbb{N}$ and $\dim X(n) < \infty$.

Since $U_{n,m} : X(m) \otimes X(n) \rightarrow X(m+n)$ are coisometric, we can assume that $X(m+n) \subseteq X(m) \otimes X(n)$.

In fact, we can assume that $X(n) \subseteq X(1)^{\otimes n}$ and $U_{n,m} = p_{n+m}$ (the projection of $X(1)^{\otimes m+n}$ onto $X(m+n)$).

Remark 0.7 *Suppose we have $X(1), X(2), \dots, X(k)$. Then, to complete it to a product system, we need to choose $X(k+1)$ such that*

$$X(k+1) \subseteq \bigcap_{n_1+\dots+n_m=k+1} X(n_1) \otimes X(n_2) \otimes \dots \otimes X(n_m).$$

*Then we will have a similar condition for choosing $X(k+2)$ etc. If we choose $X(k+1)$ to be equal to the right-hand-side and we continue in a similar way to choose $X(k+2), X(k+3), \dots$, we get **the maximal subproduct with given $X(1)$ through $X(k)$** .*

Example 0.8 *The maximal subproduct with $X(1) = \mathbb{C}^d$ and $X(2) = \mathbb{C}^d \otimes \mathbb{C}^d$ is $X(k) = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$.*

Representations :

Definition 0.9 *$S : X \rightarrow B(H)$ is a representation if $S = \{S_n\}$ is a family of completely contractive maps $S_n : X(n) \rightarrow B(H)$ satisfying “the semigroup condition”:*

$$S_{n+m}(U_{n,m}(x \otimes y)) = S_n(x)S_m(y)$$

for $x \in X(n), y \in X(m)$.

Remark 0.10 *If K is a Hilbert space with o.n.basis $\{e_n\}_{n=1}^m$, then a linear map $T : K \rightarrow B(H)$ is completely contractive iff $(T(e_1), \dots, T(e_m))$ is a row contraction iff*

$$\sum T(e_n)T(e_n)^* \leq I.$$

Example 0.11 *(Arveson) Let X be as in the example above (the symmetric products) and $\{e_i\}$ the (standard) o.n. basis of \mathbb{C}^d . Then S_1 is given by a contractive d -tuple (T_1, \dots, T_d) and for i, j , $U_{1,1}(e_i \otimes e_j - e_j \otimes e_i) = p_2(e_i \otimes e_j - e_j \otimes e_i) = 0$. It follows from the semigroup condition that $T_i T_j - T_j T_i = 0$. In fact, the representations of X are given by commuting contractive d -tuples. (Can be generalized to q -commuting tuples).*

Example 0.12 Fix complex numbers $\{a_{i,j}\}_{i,j=1}^2$ and let X be the maximal subproduct system with $X(1) = \mathbb{C}^2$ and $X(2) = (\sum a_{i,j}e_i \otimes e_j)^\perp \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$. Then the representations are precisely all pairs (T_1, T_2) that are contractive ($T_1T_1^* + T_2T_2^* \leq 1$) and satisfy

$$\sum a_{i,j}T_iT_j = 0.$$

We write A for the matrix $(a_{i,j})$ and X_A for this subproduct system.

Example 0.13 Fix an o.n. basis $\{e_1, e_2\}$ of \mathbb{C}^2 . For a word $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ in $\{1, 2\}$ (that is, $\alpha_i \in \{1, 2\}$) write $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}$ and $|\alpha| = n$. Let $X(n)$ be the span (in $(\mathbb{C}^2)^{\otimes n}$) of all e_α for α of length n that do not contain ‘2’ more than once.

This is a subproduct system whose representations are given by all contractive pairs (T_1, T_2) satisfying $T_2T_1^kT_2 = 0$ for all $k \geq 0$.

In fact, subproduct systems are in one-to-one correspondence with homogeneous ideals of polynomials in non-commuting variables.

More precisely:

Let E be a Hilbert space with (finite) o.n. basis $\{e_i\}_{i=1}^n$ and write $\mathbb{C}\langle(x_i)_{i=1}^n\rangle$ (or $\mathbb{C}\langle\underline{x}\rangle$) for the collection of all polynomials in non-commuting variables.

Given an homogeneous ideal J (2-sided generated by homogeneous polynomials). Get a subproduct system

$$X_J(n) = E^{\otimes n} \ominus \{p(e) : p \in J\}$$

where $p(e) = \sum c_\alpha e_\alpha$ if $p = \sum c_\alpha x_\alpha$.

Conversely,

Given X with $X(1) \subseteq E$, set

$$J^X = \text{span}\{p \in \mathbb{C}\langle\underline{x}\rangle : \exists n > 0, p(e) \in E^{\otimes n} \ominus X(n)\}.$$

Theorem 0.14 *This correspondence is a bijection.*

Note: This is closely related to Popescu’s “Operator theory on non commutative varieties” and to works of Bhat, Bhattacharyya and Dey (on constrained dilations).

The algebras \mathcal{A}_X

Define the X -Fock space

$$\mathfrak{F}_X = \bigoplus_{n=0}^{\infty} X(n).$$

On it, for every $x \in X(n)$, define the “shift” operator

$$S^X(x)y = U_{n,m}(x \otimes y)$$

for $y \in X(m)$. S^X is a representation of X in $B(\mathfrak{F}_X)$.

Definition 0.15 \mathcal{A}_X is the unital Banach algebra generated by the operators $S^X(x)$ (for $x \in X$). It is also the unital Banach algebra generated by $S_i^X := S^X(e_i)$ where $\{e_i\}$ is an onb for $X(1)$.

Let X be a subproduct system and let J be the homogeneous ideal in $\mathbb{C}\langle(x_i)_{i=1}^n\rangle$ associated to it. Write

$$Z(J) = \{\underline{T} = (T_i)_{i=1}^n \in (B(H)^n)_1 : \forall p \in J, p(\underline{T}) = 0\}.$$

Note: $(S_i^X) \in Z(J)$ where $X = X_J$ and $S_i^X = S^X(e_i)$.

$$Z(J) = \{\underline{T} = (T_i)_{i=1}^n \in (B(H)^n)_1 : \forall p \in J, p(\underline{T}) = 0\}$$

Theorem 0.16 \mathcal{A}_X is the universal algebra generated by a tuple in $Z(J^X)$. That is, it is generated by such a tuple and if \mathcal{B} is the operator algebra generated by some \underline{T} in $Z(J^X)$, there is a unique unital c.c. homomorphism ϕ from \mathcal{A}_X onto \mathcal{B} mapping S_i^X to T_i .

Thus: When $X = X_J$, the tuple $\underline{S}^X := (S_1^X, S_2^X, \dots, S_n^X)$ serves as a model for tuples in $Z(J)$. Another result along these lines is the following.

Theorem 0.17 *Let \underline{T} be in $Z(J)$ and let K be an infinite dimensional, separable Hilbert space. Then for all $r > \|\underline{T}\|_{cb}$, \underline{T} is unitarily equivalent to a piece of $\underline{S}^X \otimes rI_K$.*

Moreover, $\|\underline{T}\|_{cb}$ is equal the infimum of r such that \underline{T} is such a piece.

Corollary 0.18 *(von Neumann inequality) For every $\underline{T} \in Z(J^X)$ and polynomials p, q of n non-commuting variables,*

$$\|p(T_1, \dots, T_n)q(T_1, \dots, T_n)^*\| \leq \|p(S_1^X, \dots, S_n^X)q(S_1^X, \dots, S_n^X)^*\|.$$

Note: An important tool used in the proofs of the last results is Popescu's Poisson kernel.

Corollary 0.19 *The character space of \mathcal{A}_X is (assume $\dim X(1) = n$)*

$$\mathcal{M}_X = \{z \in \overline{\mathbb{B}_n} : p(z_1, z_2, \dots, z_n) = 0 \forall p \in J^X\}.$$

Remark 0.20 (1) *For the product system $\{X(1)^{\otimes n}\}$ we get Popescu's noncommutative disc algebras. When Hilbert spaces are replaced by correspondences: Tensor Algebras.*

(2) *One can also study the C^* -algebras $C^*(\mathcal{A}_X)$ and its quotient by $K(\mathcal{F}_X)$ generalizing the Cuntz algebra (or Cuntz-Pimsner algebras) and their Toeplitz extensions. For the symmetric-subproduct system this was done by Arveson. We show that Matsumoto's subshift C^* -algebras can be obtained in this way.*

Definition 0.21 *Two supproduct systems are isomorphic ($X \cong Y$) if there is a family of unitaries*

$W_n : X(n) \rightarrow Y(n)$ *satisfying*

$$W_{n+m} \circ U_{n,m}^X = U_{n,m}^Y \circ (W_n \otimes W_m).$$

Theorem 0.22 *The following are equivalent*

(1) $X \cong Y$

(2) \mathcal{A}_X and \mathcal{A}_Y are isometrically isomorphic via an isomorphism ϕ that preserves Ω (the generator of $X(0) = \mathbb{C}$); that is, $\langle \phi(a)\Omega, \Omega \rangle = \langle a\Omega, \Omega \rangle$ for $a \in \mathcal{A}_X$.

Moreover, in this case, ϕ is graded; that is $\phi(S_\alpha^X) \in \text{span}\{S_\beta^Y : |\beta| = |\alpha|\}$. ($S_\alpha^X = S_{\alpha_1}^X S_{\alpha_2}^X \cdots S_{\alpha_n}^X$ where $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$).

A class of examples: Degree-2 relation algebras Recall: To every 2×2 matrix A we associated a subproduct system X_A and therefore an algebra \mathcal{A}_A . X_A is defined by $X(1) = \mathbb{C}^2$, $X(2) = (\sum a_{i,j} e_i \otimes e_j)^\perp$ and the rest by maximality. The algebra is universal with respect to a contractive pair (T_1, T_2) satisfying $\sum a_{i,j} T_i T_j = 0$. (A general degree-2 relation.)

Theorem 0.23 *The following are equivalent:*

- (1) \mathcal{A}_A and \mathcal{A}_B are isometrically isomorphic.
- (2) $X_A \cong X_B$.
- (3) There are some $\lambda \in \mathbb{C}$ and a unitary 2×2 matrix U such that $B = \lambda U^t A U$.

Moreover, if $r(A + A^t) > 0$, then every completely isometric automorphism of \mathcal{A}_A preserves ω_Ω and, thus, is graded.

Note: This classifies all the algebras that are universal for a single degree-2 relation.

Lemma 0.24 *We have*

$$\mathcal{M}_A = \{z \in \overline{\mathbb{B}}_2 : \sum a_{i,j} z_i z_j = 0\}$$

Thus, the topology of the character space \mathcal{M}_A depends on $r(A + A^t)$.

(i) If $r(A + A^t) = 0$, $\mathcal{M}_A \cong \overline{\mathbb{B}}_2$.

(ii) If $r(A + A^t) = 1$, $\mathcal{M}_A \cong \overline{\mathbb{D}}$.

(iii) If $r(A + A^t) = 2$, \mathcal{M}_A is homeomorphic to two discs pasted together at the origin.

Thus, if $\mathcal{A}_A \cong \mathcal{A}_B$, then $r(A + A^t) = r(B + B^t)$.

Cases:

The case $r = 0$: Either $A = 0$ (and \mathcal{A}_A is Popescu's noncommutative disc algebra) or A is a multiple of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (and the algebra is the commutative algebra generated by Arveson's 2-shift). They are not isomorphic.

The case $r = 2$. An isomorphism induces a homeomorphism of the character spaces and, thus, preserves the 0 (which is the character ω_Ω).

The case $r = 1$

Here it can be reduced to the family $\{A_q : q \geq 0\}$

where $A_q = \begin{pmatrix} 1 & q \\ -q & 0 \end{pmatrix}$.

Write \mathcal{A}_q for \mathcal{A}_{A_q} . This is the algebra universal w.r.to the relation $T_1^2 + qT_1T_2 - qT_2T_1 = 0$.

Proposition 0.25 (1) *If $q \neq r$, the algebras \mathcal{A}_q and \mathcal{A}_r are not isometrically isomorphic.*

(2) *If $q = r$, every completely isometric automorphism is graded (and comes from an isomorphism of the subproduct system).*

About the proof of (1): Every isomorphism of \mathcal{A}_q onto \mathcal{A}_r induces an homeomorphism from \mathcal{M}_q onto \mathcal{M}_r (both are homeomorphic to the disc). The set of the images of 0 will be written $\mathcal{O}(0; q, r)$. One can show that this set is invariant under rotations and does not contain 0. The proof now proceeds by studying this set and its preimage.

The proof of (2) is more involved.

Another class: The q -commuting algebras

A matrix Q will be said to be admissible if $q_{ii} = 0$ and $q_{ij} = q_{ji}^{-1} \neq 0$ for $i \neq j$.

Write $E = \mathbb{C}^d$, let Q be a $d \times d$ admissible matrix and define X_Q by: $X_Q(1) = E$,

$$X_Q(2) = (\text{span}\{e_i \otimes e_j - q_{ij}e_j \otimes e_i : i \neq j\})^\perp \subseteq E \otimes E$$

and by maximality. Then $\mathcal{A}_Q = \mathcal{A}_{X_Q}$ is universal with respect to the relations $T_i T_j = q_{ij} T_j T_i$.

Theorem 0.26 *Let Q and R be two admissible $d \times d$ matrices such that $q_{ij}, r_{ij} \neq 1$ for all i, j . Then \mathcal{A}_Q is isometrically isomorphic to \mathcal{A}_R if and only if $X_Q \cong X_R$ if and only if there is a permutation $\sigma \in S_d$ such that $R = U_\sigma Q U_\sigma^{-1}$. In this case the isometric isomorphisms between \mathcal{A}_Q and \mathcal{A}_R are precisely those of the form*

$$S_i^Q \mapsto \lambda_i S_{\sigma(i)}^R,$$

where $\lambda_1, \dots, \lambda_d$ are any complex numbers on the unit circle.