

Homologically trivial and annihilator locally C^* -algebras

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ABSTRACT. In this talk, we give a survey of recent results concerning structural properties of some classes of homologically trivial locally C^* -algebras. These include: algebras with projective irreducible Hilbert modules, biprojective and superbiprojective algebras. In particular, we note that (i) each superbiprojective locally C^* -algebra is isomorphic to a direct product of full matrix algebras, i.e., it is contractible, and (ii) each biprojective σ - C^* -algebra is isomorphic to the direct product of a countable family of biprojective C^* -algebras (i.e., of c_0 -sums of full matrix algebras). We present an example of a biprojective locally C^* -algebra that is not isomorphic to a product of biprojective C^* -algebras. We notice the following result: if A is a locally C^* -algebra, generally speaking without an identity, then the condition that all (non-annihilator) irreducible Hilbert A -modules be projective is equivalent to A being an annihilator topological algebra. We describe the structure of annihilator locally C^* -algebras and of biprojective locally C^* -algebras.

This talk is based on joint work with Alexei Pirkovskii.

I shall report on joint work in progress with Alexei Pirkovskii in which we study structural properties of some classes of homologically trivial locally C^* -algebras. These include: algebras with projective irreducible Hilbert modules, biprojective and superbiprojective algebras.

By a *topological algebra* we mean a complete Hausdorff locally convex space equipped with a jointly continuous multiplication. No commutativity or existence of an identity is assumed. The symbol $\widehat{\otimes}$ denotes the complete projective topological tensor product. The *product map* is the continuous linear operator $\pi : A \widehat{\otimes} A \rightarrow A$, well-defined by the assigning ab to $a \otimes b$. By a *Fréchet algebra* we mean a metrizable topological algebra.

Recall that a topological algebra A is an *Arens-Michael algebra* if the topology on A can be defined by a family of submultiplicative seminorms. Recall also that a *locally C^* -algebra* is an Arens-Michael $*$ -algebra whose topology is defined by a family of C^* -seminorms. A C^* -seminorm on A is of course a seminorm p satisfying $p(ab) \leq p(a)p(b)$, $p(a^*) = p(a)$ and $p(a^*a) = p(a)^2$ for all $a, b \in A$. The term “*locally C^* -algebra*” is due to A. Inoue (1971). Locally C^* -algebras are also called σ - C^* -algebras in the case of metrizability. As is known, each locally C^* -algebra is topologically $*$ -isomorphic to a projective limit of C^* -algebras. σ - C^* -algebras are countable projective limits of C^* -algebras.

For any subset S in an algebra A , let $\text{lan}(S)$ and $\text{ran}(S)$ denote the *left* and *right annihilators* of S in A , respectively. So we have

$$\text{lan}(S) = \{a \in A : ab = 0 \text{ for all } b \in S\}, \quad \text{ran}(S) = \{a \in A : ba = 0 \text{ for all } b \in S\}.$$

Let A be a topological algebra. Then A is called an *annihilator algebra* if, for every closed left ideal J and for every closed right ideal R , we have $\text{ran}(J) = 0$ if and only if $J = A$ and $\text{lan}(R) = 0$ if and only if $R = A$. If $\text{lan}(\text{ran}(J)) = J$ and $\text{ran}(\text{lan}(R)) = R$, then A is called a *dual algebra*. It is obvious that a dual algebra is automatically an annihilator algebra.

Dual algebras were defined by Irving Kaplansky in 1948, and annihilator algebras were defined by F. F. Bonsall and A. W. Goldie in 1954. The existence of an annihilator Banach algebra which is not dual was first established by B. E. Johnson in 1967. Recall that every annihilator C^* -algebra is dual.

Recall that the c_0 -sum of a family of C^* -algebras A_ν , $\nu \in \Lambda$, is defined to be the set of all functions f defined on Λ such that

- (1) $f(\nu) \in A_\nu$ for each $\nu \in \Lambda$, and
- (2) for each $\varepsilon > 0$ the set $\{\nu : \|f(\nu)\| \geq \varepsilon\}$ is finite.

As is known, this set is a C^* -algebra with respect to pointwise operations and the norm $\|f\| = \sup_{\nu \in \Lambda} \|f(\nu)\|$.

We recall that a C^* -algebra A is annihilator (or, equivalently, dual) if and only if it is isometrically $*$ -isomorphic to the c_0 -sum of a certain family of algebras $\{\mathcal{K}(H_\nu) : \nu \in \Lambda\}$ of compact operators on H_ν , where all the H_ν are Hilbert spaces.

Let A be a topological algebra and let \mathcal{M}_ℓ be the set of all closed left ideals in A . Then A is called a *left quasi-complemented algebra* if there exists a mapping $q: J \rightarrow J^q$ of \mathcal{M}_ℓ into itself having the following properties:

- (1) $J \cap J^q = 0$ ($J \in \mathcal{M}_\ell$);
- (2) $(J^q)^q = J$ ($J \in \mathcal{M}_\ell$);
- (3) if $J_1 \subset J_2$, then $J_2^q \subset J_1^q$ ($J_1, J_2 \in \mathcal{M}_\ell$).

A left quasi-complemented algebra is called a *left complemented algebra* if it satisfies:

- (4) $J + J^q = A$ ($J \in \mathcal{M}_\ell$).

Right quasi-complemented algebras and *right complemented algebras* are defined analogously. A left and right complemented algebra is called a *complemented algebra*.

Complemented Banach algebras were introduced by B. J. Tomiuk in 1962 and have been studied by various authors. Right quasi-complemented algebras were defined by T. Husain and Pak-Ken Wong in 1972. It is known that a C^* -algebra is complemented if and only if it is dual. A similar result is true for right (left) quasi-complemented algebras.

Let $\{I_\nu : \nu \in \Lambda\}$ be a family of (left, two-sided) ideals in an algebra A . Recall that the smallest (left, two-sided) ideal in A which contains every I_ν is called the *sum* of the ideals I_ν . The sum of the ideals I_ν evidently consists of all finite sums of elements from the ideals I_ν .

If A is a topological algebra, then the closure of the sum of the ideals I_ν is called their *topological sum*. If each I_ν is closed and intersects the topological sum of the remaining ideals in the zero element, then the topological sum is called a *direct topological sum* (Rickart's terminology, 1974).

The following definition is due to Dieudonné (1942). Let A be an algebra. The *left socle*, $\text{Soc}(A)$, of A is the sum of all the minimal left ideals of A .

As is known, the left socle of A is a two-sided ideal. Recall that a C^* -algebra A is annihilator if and only if the left socle of A is dense in A .

THEOREM 1. *Let A be a locally C^* -algebra such that $A = \overline{\text{Soc}(A)}$, and let $\{I_\nu : \nu \in \Lambda\}$ be the collection of minimal closed two-sided ideals of this algebra; we put $P_\nu = \text{lan}(I_\nu)$ ($\nu \in \Lambda$). Then:*

- (1) *A is the direct topological sum of all the I_ν , and moreover, for each $\nu_1, \nu_2 \in \Lambda$ with $\nu_1 \neq \nu_2$, $I_{\nu_1} I_{\nu_2} = 0$;*
- (2) *for each $\nu \in \Lambda$ the algebra I_ν is topologically $*$ -isomorphic to the C^* -algebra $\mathcal{K}(H_\nu)$, where H_ν is a certain Hilbert space;*
- (3) *for each ν , $A = I_\nu \oplus P_\nu$, and moreover, the homomorphism*

$$\psi_\nu: I_\nu \rightarrow A \xrightarrow{\varphi_\nu} A/P_\nu,$$

where

$$\varphi_\nu: A \rightarrow A/P_\nu: a \mapsto a_\nu$$

is the natural epimorphism, is an isomorphism of C^* -algebras;

- (4) *every closed two-sided ideal of A is the intersection of the ideals P_ν that contain it;*
- (5) *every closed two-sided ideal of A is the direct topological sum of the ideals I_ν contained in it;*
- (6) *if $I = \overline{\bigoplus_{\nu \in \Lambda_1} I_\nu}$ is an arbitrary closed two-sided ideal of A , then $A = I \oplus J$, where*

$$J = \text{lan}(I) = \overline{\bigoplus_{\nu \in \Lambda \setminus \Lambda_1} I_\nu},$$

and moreover, the natural homomorphism $\varphi: I \rightarrow A \rightarrow A/J$ is a topological $*$ -isomorphism of locally C^* -algebras;

- (7) *if p is a continuous C^* -seminorm on A , then there exists a subset $\Lambda_1 \subset \Lambda$ such that, for each $a \in A$,*

$$p(a) = \sup_{\nu \in \Lambda_1} \{\|a_\nu\|_{\mathcal{K}(H_\nu)}\}.$$

Suppose now that A is a topological algebra and X is a complete locally convex space which is also a left A -module. Thus there is a bilinear map $(a, x) \mapsto a \cdot x$ from $A \times X$ into X such that $(ab) \cdot x = a \cdot (b \cdot x)$ for $a, b \in A$, $x \in X$. Then X is called a *topological left A -module* if the module map is jointly continuous.

For two such modules X and Y , an *A -module morphism* from X into Y is a continuous linear operator $\varphi: X \rightarrow Y$ which is a module homomorphism.

Similar definitions apply to *topological right A -modules* and *topological A -bimodules*.

Let A_+ denote the unitization of A . Recall that there is the so-called *canonical morphism* $\pi_+: A_+ \widehat{\otimes} X \rightarrow X$ (resp., $\pi_+: A_+ \widehat{\otimes} X \widehat{\otimes} A_+ \rightarrow X$) associated with any topological left A -module (resp., topological A -bimodule) X ; this morphism is defined by $\pi_+(a \otimes x) = a \cdot x$ (resp., $\pi_+(a \otimes x \otimes b) = a \cdot x \cdot b$), $a, b \in A_+$, $x \in X$.

Recall that a topological left A -module (topological A -bimodule) is said to be *projective* if the canonical morphism π_+ has a right inverse in the corresponding category. Recall also that a topological algebra A is said to be *biprojective* if the topological A -bimodule A is projective. A topological algebra A is said to be *contractible* if A is biprojective and has an identity.

For a left A -module X , we denote by $A \cdot X$ the linear span of the set

$$\{a \cdot x : a \in A, x \in X\}.$$

As usual, a topological left A -module X is said to be *essential* if $A \cdot X$ is dense in X . We call a left module X over an algebra A *irreducible* if $A \cdot X \neq 0$ and X contains no non-zero proper submodules. It is obvious that, if X is an irreducible topological left A -module, then it is essential.

Let now A be a locally C^* -algebra. A topological left A -module H is called a *Hilbert A -module* if it has an underlying Hilbert space and, besides, the identity $\langle a \cdot x, y \rangle = \langle x, a^* \cdot y \rangle$ hold. Evidently, the given definition means exactly that the continuous representation of A associated with our module is a $*$ -representation.

THEOREM 2 (Yu. V. Selivanov, 1977). *Let A be a C^* -algebra. Then the following conditions are equivalent:*

- (1) *all irreducible Banach (or, equivalently, irreducible Hilbert) A -modules are projective;*
- (2) *A is an annihilator algebra;*
- (3) *for every closed left ideal I of A , there is a closed left ideal J of A such that $A = I \oplus J$.*

THEOREM 3. *Let A be a locally C^* -algebra. Then the following conditions are equivalent:*

- (1) *all $\overline{\text{irreducible}}$ Hilbert A -modules are projective;*
- (2) *$A = \overline{\text{Soc}}(A)$;*
- (3) *A is the direct topological sum of its minimal closed two-sided ideals each of which is topologically $*$ -isomorphic to a C^* -algebra of the form $\mathcal{K}(H)$, where H is a Hilbert space;*
- (4) *A is an annihilator algebra;*
- (5) *A is a dual algebra;*
- (6) *A is a complemented algebra;*
- (7) *A is a left quasi-complemented algebra.*

Recall that the implications (4) \Rightarrow (5) \Rightarrow (6) in Theorem 3 were proved earlier (in 2005) by Marina Haralampidou (Greece).

Theorem 3 can be strengthened in the case where A is unital.

COROLLARY 1. *Let A be a unital locally C^* -algebra. Then the following conditions are equivalent:*

- (1) *all $\overline{\text{irreducible}}$ Hilbert A -modules are projective;*
- (2) *$A = \overline{\text{Soc}}(A)$;*
- (3) *A is an annihilator algebra;*
- (4) *A is topologically $*$ -isomorphic to the cartesian product of a certain family of full matrix algebras.*

We point out that Corollary 1 is a strengthening of the following theorem.

THEOREM 4 (M. Fragoulopoulou, 2001). *Let A be a locally C^* -algebra. Then the following conditions are equivalent:*

- (1) *A is contractible;*
- (2) *all topological left A -modules are projective;*
- (3) *A is topologically $*$ -isomorphic to the cartesian product of a certain family of full matrix algebras.*

Recall that a topological algebra A is biprojective if and only if the product map $\pi: A \widehat{\otimes} A \rightarrow A$ is a retraction in the category of topological A -bimodules.

The class of biprojective algebras was introduced by A. Ya. Helemskii in connection with the study of the cohomology groups, $\mathcal{H}^n(A, X)$, and of other homological characteristics of Banach and topological algebras.

We recall some examples of biprojective algebras.

EXAMPLE 1 (A. Ya. Helemskii, 1972). The Banach sequence algebras c_0 and ℓ^1 (with coordinatewise product) are biprojective.

EXAMPLE 2 (A. Ya. Helemskii, 1972). If G is a compact group, then the group algebras $L^1(G)$ and $C^*(G)$ are biprojective.

EXAMPLE 3 (Yu. V. Selivanov, 1996). If G is a compact Lie group, then the Fréchet algebra $\mathcal{E}(G) = C^\infty(G)$ of smooth (= infinite differentiable) functions on G with convolution product is biprojective.

EXAMPLE 4 (A. Yu. Pirkovskii, 2001). Let $\alpha = (\alpha_n)$, $n \in \mathbb{N}$, be a non-decreasing sequence of positive numbers with $\lim_n \alpha_n = +\infty$. Then the power series spaces $\Lambda_1(\alpha)$ and $\Lambda_\infty(\alpha)$ are biprojective Fréchet algebras with respect to pointwise product. In particular, the algebra s of rapidly decreasing sequences is biprojective.

EXAMPLE 5 (Yu. V. Selivanov, 1976–1996). Let (E, F) be a pair of complete locally convex spaces endowed with a jointly continuous bilinear form

$$\langle \cdot, \cdot \rangle: E \times F \rightarrow \mathbb{C}$$

that is not identically zero. The space $A = E \widehat{\otimes} F$ is then a topological algebra (actually, an Arens-Michael algebra) with respect to the multiplication defined by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1 \rangle x_1 \otimes y_2 \quad (x_i \in E, y_i \in F).$$

This algebra is biprojective.

In particular, if E is a Banach space with the approximation property, then the algebra $A = E \widehat{\otimes} E^*$ is isomorphic to the algebra $\mathcal{N}(E)$ of nuclear operators on E , and so $\mathcal{N}(E)$ is biprojective.

The following result gives the complete description of finite-dimensional biprojective algebras.

THEOREM 5 (O. Yu. Aristov, 2008). *Let A be a finite-dimensional biprojective algebra. Then A is isomorphic to the cartesian product of a finite number of algebras of the type $E \widehat{\otimes} F$, where $(E, F, \langle \cdot, \cdot \rangle)$ is a pair of finite-dimensional spaces together with a non-zero bilinear form.*

We now turn to biprojective C^* -algebras, σ - C^* -algebras (i.e., metrizable locally C^* -algebras) and arbitrary locally C^* -algebras.

THEOREM 6 (A. Ya. Helemskii and Yu. V. Selivanov, 1972–1977). *A C^* -algebra is biprojective if and only if it decomposes into the c_0 -sum of a family of full matrix algebras¹.*

THEOREM 7 (A. Yu. Pirkovskii and Yu. V. Selivanov, 2007). *Every biprojective σ - C^* -algebra is topologically $*$ -isomorphic to the cartesian product $\prod_{n=1}^{\infty} B_n$, where each B_n is a C^* -algebra isomorphic to a c_0 -sum of full matrix algebras.*

¹As is known, these algebras also can be characterized as annihilator C^* -algebras with finite-dimensional minimal ideals.

COROLLARY 2. *For a σ - C^* -algebra A the following conditions are equivalent:*

- (1) *A is biprojective;*
- (2) *A is topologically $*$ -isomorphic to the cartesian product of a countable family of biprojective C^* -algebras.*

The next result shows that the above result does not hold for non-metrizable locally C^* -algebras.

THEOREM 8. *There exists a biprojective locally C^* -algebra that is not isomorphic to the cartesian product of biprojective C^* -algebras (i.e., of c_0 -sums of full matrix algebras).*

For each function $s: \mathbb{N} \rightarrow \mathbb{N}$, consider the set

$$M_s = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq s(i)\},$$

and define

$$A = \left\{ a = (a_{ij}) \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}} : \lim_{\substack{(i,j) \rightarrow \infty \\ (i,j) \in M_s}} |a_{ij}| = 0 \quad \forall s \in \mathbb{N}^{\mathbb{N}} \right\}.$$

Clearly, A is a $*$ -subalgebra of $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$. Moreover, A is a locally C^* -algebra with respect to the family of C^* -seminorms

$$\|a\|_s = \sup_{(i,j) \in M_s} |a_{ij}| \quad (s \in \mathbb{N}^{\mathbb{N}}).$$

This locally C^* -algebra is biprojective, but it is not isomorphic to a product of c_0 -sums of full matrix algebras.

THEOREM 9. *Let A be a locally C^* -algebra. Then the following conditions are equivalent:*

- (1) *A is biprojective;*
- (2) *A is the direct topological sum of its minimal closed two-sided ideals each of which is topologically $*$ -isomorphic to a full matrix algebra;*
- (3) *A is an annihilator algebra with finite-dimensional minimal ideals.*

The initial stimulus to select biprojective algebras was the following result.

THEOREM 10 (A. Ya. Helemskii). *Let A be a biprojective topological algebra. Then for every topological A -bimodule X and for every $n \geq 3$ we have $\mathcal{H}^n(A, X) = 0$.*

Recall that a topological algebra A is said to be *superbiprojective* if A is biprojective and $\mathcal{H}^2(A, X) = 0$ for each topological A -bimodule X .

EXAMPLE 6 (A. Yu. Pirkovskii, 2001). If G is an infinite compact Lie group, then the Fréchet algebra $\mathcal{E}(G)$ of smooth functions on G with convolution product is superbiprojective.

As to Banach algebras we have the following.

THEOREM 11 (Yu. V. Selivanov, 2001). *Let A be a superbiprojective semisimple Banach algebra with the approximation property. Then A is isomorphic to the direct sum of a finite number of full matrix algebras and, in particular, it is contractible.*

Now let us turn to locally C^* -algebras. The following theorem is another strengthening of Theorem 4 of M. Fragoulopoulou.

THEOREM 12. *Let A be a superbiprojective locally C^* -algebra. Then A is topologically $*$ -isomorphic to the cartesian product of a family of full matrix algebras.*

COROLLARY 3. *For a locally C^* -algebra A the following are equivalent:*

- (1) *A is superbiprojective;*
- (2) *A is contractible.*

The proof of the next theorem is analogous to the proof of Theorem 7.

THEOREM 13. *Every annihilator σ - C^* -algebra is topologically $*$ -isomorphic to the cartesian product of a countable family of annihilator C^* -algebras.*

COROLLARY 4. *Let A be a σ - C^* -algebra. Then the following conditions are equivalent:*

- (1) *all irreducible Hilbert A -modules are projective;*
- (2) *$A = \overline{\text{Soc}(A)}$;*
- (3) *A is an annihilator algebra;*
- (4) *A is topologically $*$ -isomorphic to the cartesian product $\prod_{n=1}^{\infty} B_n$, where each B_n is a C^* -algebra isomorphic to the c_0 -sum of a certain family of algebras of the form $\mathcal{K}(H)$, where H is a Hilbert space.*

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