Subgroups and invariant C*-subalgebras

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Notation

- Let G be a locally compact group.
- $\mathrm{C}_0(G)$ is the C*-algebra of continuous functions on G vanishing at infinity.
- For $f\colon G \to \mathbf{C}$, the right translation R_s is defined by $R_s f(t) = f(ts)$, $s, t \in G$;
  the left translation by $L_s f(t) = f(st)$.
- A subspace X of $\mathrm{C}_0(G)$ is left invariant if $L_s x \in X$ for every $s \in G$ and $x \in X$.

The result of Lau and Losert

Theorem (Lau–Losert 1990)

There is a one-to-one correspondence between ***non-zero left invariant C*-subalgebras*** *X of $\mathrm{C}_0(G)$ and* ***compact subgroups*** *H of G:*

$$X = \mathrm{C}_0(G/H) = \{\, f \in \mathrm{C}_0(G);\ R_s f = f \quad \forall s \in H \,\},$$

$$H = \{\, s \in G;\ R_s f = f \quad \forall f \in X \,\}.$$

Moreover, H is normal iff X is also right invariant.

Related work

- de Leeuw and Mirkil (1960) considered abelian groups.
- Takesaki and Tatsuuma (1971) linked subgroups with invariant subalgebras of the following Banach algebras: $L^\infty(G)$, $VN(G)$ (group von Neumann algebra), $L^1(G)$, $A(G)$ (Fourier algebra).

Implications

- The result of Lau and Losert can be used for a nice proof of the Kakutani–Kodaira theorem.
- Hu (2005) has shown even a more general version of Kakutani–Kodaira, using Lau–Losert.

Quantum group definition (Kustermans–Vaes)

A (locally compact) quantum group is a C*-algebra A with

- A co-multiplication Γ, that is, a non-degenerate $*$-homomorphism $\Gamma\colon A \to \mathrm{M}(A \otimes A)$ such that

  $$(\mathrm{id} \otimes \Gamma)\Gamma = (\Gamma \otimes \mathrm{id})\Gamma \qquad \text{(co-associativity)}.$$

  We further assume that the linear spans of both $\Gamma(A)(A \otimes 1)$ and $\Gamma(A)(1 \otimes A)$ are dense in $A \otimes A$.
- A faithful left Haar weight: a faithful, KMS-weight ϕ on A such that

  $$\phi\big((\omega \otimes \mathrm{id})\Gamma(a)\big) = \omega(1)\phi(a) \quad (\omega \in A_+^*, a \in A_+, \phi(a) < \infty)$$

  (left invariance).
- A right Haar weight ψ on A.

Classical groups; or the commutative case

Let G be a locally compact group.

- $A = \mathrm{C}_0(G)$
- For $f \in \mathrm{C}_0(G)$ and $s, t \in G$,

$$\Gamma(f)(s,t) = f(st).$$

- $\phi =$ integration w.r.t. the left Haar measure
- $\psi =$ integration w.r.t. the right Haar measure

Duals of classical groups; or the co-commutative case

Let λ be the **left regular representation** of G on $\mathrm{L}^2(G)$:

$$\lambda(s)\xi(t) = \xi(s^{-1}t) \qquad (\xi \in \mathrm{L}^2(G),\ s,t \in G).$$

Integration gives a representation of $\mathrm{L}^1(G)$:

$$\lambda(f) = \int_G f(s)\lambda(s)\,ds \qquad (f \in \mathrm{L}^1(G)).$$

- $A = \mathrm{C}_r^*(G) = \overline{\lambda(\mathrm{L}^1(G))}^{\text{norm}} \subseteq \mathrm{B}(\mathrm{L}^2(G))$
- The co-multiplication $\Gamma\colon \mathrm{C}_r^*(G) \to \mathrm{M}(\mathrm{C}_r^*(G) \otimes \mathrm{C}_r^*(G))$ is determined by

$$\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s) \qquad (s \in G).$$

- $\phi = \psi$ is the Plancherel weight

Left invariant C*-subalgebras of a quantum group

Let (A, Γ) be a quantum group.
The dual space A^* is a Banach algebra w.r.t. the multiplication

$$(\omega, \tau) \mapsto (\omega \otimes \tau)\Gamma.$$

Definition of left invariance

The Banach algebra A^* acts on A:

$$(\omega, a) \mapsto (\omega \otimes \mathrm{id})\Gamma(a) \colon \; A^* \times A \to A.$$

A C*-subalgebra $X \subseteq A$ is left invariant if X is a A^*-submodule w.r.t. to this action, ie, if

$$(\omega \otimes \mathrm{id})\Gamma(x) \in X \qquad \forall \omega \in A^*,\ x \in X.$$

Subgroups

Let H be a closed subgroup of a locally compact group G. Dualising $H \hookrightarrow G$ gives a surjection

$$\pi\colon \mathrm{C}_0(G) \to \mathrm{C}_0(H), \quad \pi(f) = f|_H \qquad (f \in \mathrm{C}_0(G)).$$

Now $\mathrm{C}_0(H)$ is a quantum group in its own right and

$$(\pi \otimes \pi)\Gamma_G = \Gamma_H \pi,$$

ie, $(\Gamma_G f)|_{H\times H} = \Gamma_H(f|_H)$ for every $f \in \mathrm{C}_0(G)$.

Subgroups of quantum groups

Definition of subgroup

We say that a quantum group (B, Γ_B) is a (quantum) subgroup of (A, Γ_A) if there is a surjective $*$-homomorphism

$$\pi \colon A \to B$$

such that

$$(\pi \otimes \pi)\Gamma_A = \Gamma_B \pi.$$

Problem

*To show that *something* is subgroup, it needs Haar weights.*

Even with classical groups it is not obvious how to construct the Haar measures of a subgroup without passing through the general construction.

Compact subgroups are easier

We are interested in **compact** subgroups, so the existence of Haar weights can be dealt with the following theorem:

Theorem (Woronowicz)

If A is a ***unital*** *C*-algebra and $\Gamma\colon A \to A \otimes A$ is a co-multiplication such that the linear spans of both $\Gamma(A)(A \otimes 1)$ and $\Gamma(A)(1 \otimes A)$ are dense in $A \otimes A$, then the Haar weights exist for (A, Γ).*

Subgroups from invariant C*-subalgebras

Suppose that (A, Γ) is a quantum group and
X is a non-zero left invariant C*-subalgebra of A.
Recall that, for $A = \mathrm{C}_0(G)$ and $X = \mathrm{C}_0(G/H)$,

$$H = \{\, s \in G;\ R_s f = f \quad \forall f \in X \,\}.$$

Consider instead

$$F_0 = \{\, \omega \in \mathrm{S}(A);\ (\mathrm{id} \otimes \omega)\Gamma(x) = x \quad \forall x \in X \,\}$$

where $\mathrm{S}(A)$ denotes the set of states of A
(ie, positive functionals of norm 1).

Write $F = \operatorname{span} F_0$.

For $A = \mathrm{C}_0(G)$, $X = \mathrm{C}_0(G/H)$, the set F_0 is the probability
measures on G supported by H and $F = \mathrm{M}(H)$.

Subgroups from invariant C*-subalgebras – continued

Put

$$I = \{ a \in A;\ \langle \omega, a^* a \rangle = 0 \quad \forall \omega \in F_0 \}$$

and note that I is a left ideal.
For $A = \mathrm{C}_0(G)$, $X = \mathrm{C}_0(G/H)$,

$$I = \{ f \in \mathrm{C}_0(G);\ f = 0 \text{ on } H \} = F_\perp$$

is a 2-sided ideal and $\mathrm{C}_0(G)/I \cong \mathrm{C}_0(H)$.
So in the classical case

$$X \rightsquigarrow F_0 \rightsquigarrow I \rightsquigarrow A/I$$

associates X with a compact (quantum) subgroup A/I,
and the process can be reversed.

Towards the dual version

Suppose that G is an **amenable** locally compact group.

$$A = C_r^*(G) = C^*(G)$$
$$A^* = B(G) = \text{Fourier–Stieltjes algebra}$$
$$= \{\, \langle\, \rho(\cdot)\xi | \zeta \,\rangle;\ \rho \text{ representation of } G \,\}$$

- The multiplication $(u, v) \mapsto (u \otimes v)\Gamma$ on $B(G)$ is just the pointwise multiplication of functions.
- $B(G)$ acts on $C^*(G)$:

$$u.a := (u \otimes \mathrm{id})\Gamma(a) = (\mathrm{id} \otimes u)\Gamma(a) \quad (u \in B(G), a \in C^*(G));$$

for $f \in L^1(G)$,

$$u.\lambda(f) = \lambda(uf)$$

where uf is the pointwise product.

Dual version of Lau–Losert

Recall the notion of support (due to Eymard) for an element x in $\mathrm{C}^*(G)$:

$$s \in \operatorname{supp} x \iff \forall U \text{ nhood of } s,\, \exists u \in \mathrm{A}(G) \text{ with } \operatorname{supp} u \subseteq U \text{ such that } \langle x, u \rangle \neq 0.$$

Theorem

There is a one-to-one correspondence between ***non-zero invariant C*-subalgebras*** *X of $\mathrm{C}^*(G)$ and* ***open subgroups*** *H of G:*

$$X = \mathrm{C}^*(H) = \{\, x \in \mathrm{C}^*(G);\ \operatorname{supp} x \subseteq H \,\},$$

$$H = \{\, s \in G;\ s \in \operatorname{supp} x \textit{ for some } x \in X \,\} = \bigcup_{x \in X} \operatorname{supp} x.$$

Relation to the earlier construction

Let $H \leq G$ be open and $X = \mathrm{C}^*(H)$. Then

$$F_0 = \{\, u \in \mathrm{P}_1(G);\ u.x = x \quad \forall x \in X \,\}$$

where $\mathrm{P}_1(G) =$ continuous positive definite functions on G with $u(e) = 1$. In this case

$$F_0 = \{\, u \in \mathrm{P}_1(G);\ u = 1 \text{ on } H \,\}.$$

Lemma

Every continuous positive definite function u on G with $u = 1$ on H is constant on every left and right coset of H.

It follows that

$$F = \{\, u \in \mathrm{B}(G);\ u = \text{const on left and right cosets of } H \,\}.$$

Normality of H

Recall that

$$I = \{\, a \in A;\ \langle\, \omega, a^* a \,\rangle = 0 \quad \forall \omega \in F_0 \,\}.$$

Theorem

The open subgroup H is normal iff I is a 2-sided ideal.
In this case, $I = F_\perp$ and $\mathrm{C}^*(G)/I \cong \mathrm{C}^*(G/H)$.

Remark

This shows that, in the case of general quantum groups, some form of "normality" is needed to make I a 2-sided ideal and A/I a compact quantum group.

Review of the classical case and its dual

commutative case	**co-commutative case**
$A = \mathrm{C}_0(G)$	$A = \mathrm{C}^*(G)$
H compact subgroup	H open subgroup
$X = \mathrm{C}_0(G/H)$ left invariant	$X = \mathrm{C}^*(H)$ 2-sided invariant
I 2-sided ideal	I left ideal
X 2-sided invariant iff H normal	I 2-sided ideal iff H normal

Remark

Recall subgroup duality for abelian groups:

$$H^\perp \cong (G/H)\widehat{\ } \qquad \text{and} \qquad \widehat{H} \cong \widehat{G}/H^\perp.$$

So H is open iff $H^\perp$ is compact and H is compact iff $H^\perp$ is open.

More notation for quantum groups

- (A, Γ) a quantum group
- $X \subseteq A$ a non-zero left invariant C*-subalgebra
- We may assume that $A \subseteq \mathrm{B}(H)$ where $H = \mathrm{L}^2(A, \phi)$.
- Denote the **compact operators** on H by $\mathrm{B}_0(H)$.
- There is a **multiplicative unitary** $W \in \mathrm{B}(H \otimes H)$ such that

$$\Gamma(a) = W^*(1 \otimes a)W \qquad (a \in A).$$

- (A, Γ) is co-amenable if it has a bounded co-unit ϵ:

$$(\mathrm{id} \otimes \epsilon)\Gamma(a) = (\epsilon \otimes \mathrm{id})\Gamma(a) = a \qquad (a \in A).$$

Quantum normality

We say that X is normal (or should we say co-normal?) if

$$W(x \otimes 1)W^* \in \mathrm{M}(X \otimes \mathrm{B}_0(H)) \quad \forall x \in X.$$

Vaes and Vainerman (2003) have a similar definition in the von Neumann algebra setting.

Examples

- For $A = \mathrm{C}_0(G)$, every X is normal.
- For $A = \mathrm{C}^*(G)$, $X = \mathrm{C}^*(H)$ is normal iff H is normal.

Normal version for quantum groups

Theorem

Let (A, Γ) *be a co-amenable quantum group.*
If $X \subseteq A$ *is a normal, left invariant C*-subalgebra, then*

- I *is a 2-sided ideal*
- $I = F_\perp$ (*recall* $F = \text{span}\, F_0$)
- A/I *is a compact quantum group.*