

Amenability of $\mathcal{B}(\ell^p)$ for $p \neq 2$

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Definition (B. E. Johnson, 1972)

A Banach algebra \mathfrak{A} is said to be **amenable** if, for every Banach \mathfrak{A} -bimodule E , every bounded derivation $D: \mathfrak{A} \rightarrow E^*$ is inner.

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G :

- 1 G is amenable;
- 2 $L^1(G)$ is amenable.

Grand theme...

For a given class of Banach algebras, determine its amenable members!

Question (B. E. Johnson, 1972)

Is there any infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is amenable? What if E is a Hilbert space?

Philosophical musings...

AMENABLE \approx SMALL

If $\dim E = \infty$, then $\mathcal{B}(E)$ **ought to** be too large to be amenable...

Amenability of $\mathcal{B}(E)$, II

Theorem (S. A. Argyros & R. G. Haydon, 2009)

There is a Banach space E with $E^ = \ell^1$ such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C} \text{id}_E$.*

Corollary

There is an infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is amenable.

Proof.

Let E be the Banach space constructed by Argyros–Haydon. The space $E^* = \ell^1$ has property (\mathbb{A}) , hence so has E . Therefore, $\mathcal{K}(E)$ is amenable, as is $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C} \text{id}_E$. \square

The Hilbert space case

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C^ -algebra \mathfrak{A} :*

- 1 \mathfrak{A} is amenable;
- 2 \mathfrak{A} is nuclear.

Theorem (S. Wassermann, 1976)

The following are equivalent for a von Neumann algebra \mathfrak{M} :

- 1 \mathfrak{M} is nuclear;
- 2 \mathfrak{M} is subhomogeneous.

Corollary

$\mathcal{B}(\ell^2)$ is not amenable.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

Theorem (G. A. Willis, unpublished)

Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

- 1 A quotient of an amenable Banach algebra is again amenable.
- 2 Every (weakly) complemented closed ideal of an amenable Banach algebra is amenable.
- 3 Every amenable Banach algebra has a bounded approximate identity.
- 4 **Pitt's Theorem.** If $p > q$, then $\mathcal{B}(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

Suppose that $p > q$, and note that

$$\mathcal{B}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{B}(\ell^p) & \mathcal{B}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q) \mathcal{K}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix}$$

and

$$\mathcal{K}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{K}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\ \mathcal{K}(\ell^p, \ell^q) & \mathcal{K}(\ell^q) \end{bmatrix},$$

so that

$$\mathcal{C}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{C}(\ell^p) & * \\ 0 & \mathcal{C}(\ell^q) \end{bmatrix}.$$

Then $I := \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ is a complemented ideal of $\mathcal{C}(\ell^p \oplus \ell^q)$, thus is amenable, and thus has a BAI.

But $I^2 = \{0\} \dots$

□

Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

Theorem (C. J. Read, 2006(?))

$\mathcal{B}(\ell^1)$ is not amenable.

Progress since

- 1 Simplification of Read's proof by G. Pisier, 2004.
- 2 Simultaneous proof for the non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$ by N. Ozawa, 2006.

Question

Is $\mathcal{B}(\ell^p)$ amenable for any $p \in (1, \infty) \setminus \{2\}$?

What if $\mathcal{B}(\ell^p)$ were amenable?

Theorem (M. Daws & VR, 2008)

The following are equivalent for a Banach space E and $p \in [1, \infty)$:

- 1 $\mathcal{B}(\ell^p(E))$ is amenable;
- 2 $\ell^\infty(\mathcal{B}(\ell^p(E)))$ is amenable.

Idea

- $\ell^p(\ell^p(E)) \cong \ell^p(E)$
- $\ell^\infty(\mathcal{B}(\ell^p(E))) \cong$ block diagonal matrices in $\mathcal{B}(\ell^p(\ell^p(E)))$

Corollary

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$ and $\ell^\infty(\mathcal{K}(\ell^p))$.

Definition (J. Lindenstrauss & A. Pełczyński, 1968)

Let $p \in [1, \infty]$ and let $\lambda \geq 1$. A Banach space E is called a \mathcal{L}_λ^p -space if, for every finite-dimensional subspace X of E , there is a finite-dimensional subspace $Y \supset X$ of E with $d(Y, \ell_{\dim Y}^p) \leq \lambda$. We call E an \mathcal{L}^p -space if it is an \mathcal{L}_λ^p -space for some $\lambda \geq 1$.

Examples

- 1 All Banach spaces isomorphic to an L^p -space are \mathcal{L}^p -spaces.
- 2 Let $p \in (1, \infty) \setminus \{2\}$. Then $\ell^p(\ell^2)$ and $\ell^2 \oplus \ell^p$ are \mathcal{L}^p -spaces, but not isomorphic to L^p -spaces.

Theorem (M. Daws & VR, 2008)

Let $p \in [1, \infty]$. Then one of the following is true:

- 1 $\ell^\infty(\mathcal{K}(E))$ is amenable for every \mathcal{L}^p -space E with $\dim E = \infty$;
- 2 $\ell^\infty(\mathcal{K}(E))$ is not amenable for any \mathcal{L}^p -space E with $\dim E = \infty$.

Corollary

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then $\ell^\infty(\mathcal{K}(E))$ is amenable for every \mathcal{L}^p -space E with $\dim E = \infty$.

Question

Is $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?

Theorem (M. Daws & VR, 2008)

The Banach algebra $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ is not amenable for $p = 1, \infty$.

Amenability of $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$

Question

For which Banach spaces E is $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$ amenable?

Theorem

Let E be a Banach space with a shrinking basis $(x_n)_{n=1}^\infty$ such that there is $C > 0$ with

$$\sum_{n=1}^{\infty} \|Sx_n\| \|Tx_n^*\| \leq C N \|S\| \|T\|$$

$$(N \in \mathbb{N}, S \in \mathcal{B}(E, \ell_N^2), T \in \mathcal{B}(E^*, \ell_N^2)).$$

Then $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$ is not amenable.

The setup

The group $SL(3, \mathbb{Z})$ has **Khazhdan's property (T)** and thus, in particular, is finitely generated by g_1, \dots, g_m , say.

Write \mathbb{P} for the set of prime numbers.

Let $p \in \mathbb{P}$, and let Λ_p be the projective plane over $\mathbb{Z}/p\mathbb{Z}$.

Then $SL(3, \mathbb{Z})$ acts on Λ_p through matrix multiplication.

This group action induces a unitary representation

$$\pi_p: SL(3, \mathbb{Z}) \rightarrow \mathcal{B}(\ell^2(\Lambda_p)).$$

Choose $S_p \subset \Lambda_p$ with $|S_p| = \frac{|\Lambda_p| - 1}{2}$ and define a unitary

$\pi_p(g_{m+1}) \in \mathcal{B}(\ell^2(\Lambda_p))$ via

$$\pi_p(g_{m+1})e_\lambda = \begin{cases} e_\lambda, & \lambda \in S_p, \\ -e_\lambda, & \lambda \notin S_p. \end{cases}$$

Ozawa's proof revisited, II

Ozawa's Lemma

It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with the following property: for each $p \in \mathbb{P}$ there are

$\xi_{1,p}, \eta_{1,p}, \dots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\Lambda_p)$ such that $\sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and

$$\begin{aligned} & \left\| \sum_{k=1}^r \xi_{j,p} \otimes \eta_{k,p} - (\pi_p(g_j) \otimes \pi_p(g_j))(\xi_{k,p} \otimes \eta_{k,p}) \right\|_{\ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p)} \\ & \leq \epsilon \left\| \sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \right\|_{\ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p)} \quad (j = 1, \dots, m+1). \end{aligned}$$

Ingredients

- 1 $\text{SL}(3, \mathbb{Z})$ has Khazhdan's property (T).
- 2 The **non-commutative Mazur map** is uniformly continuous.

A key inequality

For $p = 1, 2, \infty$, $N \in \mathbb{N}$, $S \in \mathcal{B}(\ell^p, \ell_N^p)$, and $T \in \mathcal{B}(\ell^{p'}, \ell_N^{p'})$:

$$\sum_{n=1}^{\infty} \|S e_n\|_{\ell_N^p} \|T e_n^*\|_{\ell_N^{p'}} \leq N \|S\| \|T\|.$$

This estimate is no longer true for $p \in (1, \infty) \setminus \{2\}$.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

Theorem

Let E be a Banach space with a shrinking basis $(x_n)_{n=1}^\infty$ such that there is $C > 0$ with

$$\sum_{n=1}^{\infty} \|Sx_n\| \|Tx_n^*\| \leq C N \|S\| \|T\|$$

$$(N \in \mathbb{N}, S \in \mathcal{B}(E, \ell_N^2), T \in \mathcal{B}(E^*, \ell_N^2)).$$

Then $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$ is not amenable.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

Lemma

Let \mathfrak{A} be an amenable Banach algebra, and let $e \in \mathfrak{A}$ be an idempotent. Then, for any $\epsilon > 0$ and any finite subset F of $e\mathfrak{A}e$, there are $a_1, b_1, \dots, a_r, b_r \in \mathfrak{A}$ such that

$$\sum_{k=1}^r a_k b_k = e$$

and

$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\|_{\mathfrak{A} \hat{\otimes} \mathfrak{A}} < \epsilon \quad (x \in F).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

$$\ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)) \subset \ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{K}(\ell^2 \oplus E) =: \mathfrak{A}$$

as “upper left corners”. Let \mathfrak{A} act on

$$\ell^2(\mathbb{P}, \ell^2 \oplus \ell^p) \cong \ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2 -summand of $\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E)$.

Set $e = (P_p)_{p \in \mathbb{P}}$. Then e is an idempotent in \mathfrak{A} with

$$e\mathfrak{A}e = \ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the Lemma there are thus $a_1, b_1, \dots, a_r, b_r \in \mathfrak{A}$ such that $\sum_{k=1}^r a_k b_k = e$ and

$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C+1)(m+1)} \quad (x \in F),$$

where

$$F := \{(\pi_p(g_j))_{p \in \mathbb{P}} : j = 1, \dots, m+1\}.$$

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$T_p(q, n) := \sum_{k=1}^r P_p a_k(e_q \otimes e_n) \otimes P_p^* b_k^*(e_q^* \otimes e_n^*) \in \ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p)$$

and

$$S_p(q, n) := \sum_{j=1}^r P_p a_k(e_q \otimes x_m) \otimes P_p^* b_k^*(e_q^* \otimes x_m^*) \in \ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p).$$

Sketched proof of the Theorem (continued)

It follows that

$$\begin{aligned} \sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} & \|T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j))T_p(q, n))\| \\ & + \|S_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j))S_p(q, n))\| \\ & \leq \frac{\epsilon}{m+1} |\Lambda_p| \end{aligned}$$

for $j = 1, \dots, m+1$ and $p \in \mathbb{P}$.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$\begin{aligned} & \sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \|T_p(q, n)\| + \|S_p(q, n)\| \\ & \geq \sum_{n=1}^{\infty} \sum_{k=1}^r \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{n=1}^{\infty} \sum_{k=1}^r \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \\ & = \text{Tr} \sum_{k=1}^r b_{k,p} P_p a_{k,p} \\ & = \text{Tr} \sum_{k=1}^r P_p a_{k,p} b_{k,p} \\ & = \text{Tr} P_p \\ & = |\Lambda_p|. \end{aligned}$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$ and

$$\|T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j))T_p(q, n))\| \leq \epsilon \|T_p(q, n)\|$$

for $j = 1, \dots, m + 1$ **or** with $S_p(q, n) \neq 0$ and

$$\|S_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j))S_p(q, n))\| \leq \epsilon \|S_p(q, n)\|$$

for $j = 1, \dots, m + 1$.

Either case violates **Ozawa's Lemma**.

Definition

Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T: E \rightarrow F$ is called **p -summing** if the amplification $\text{id}_{\ell^p} \otimes T: \ell^p \otimes E \rightarrow \ell^p \otimes F$ extends to a bounded map from $\ell^p \check{\otimes} E$ to $\ell^p(F)$. The operator norm of $\text{id}_{\ell^p} \otimes T: \ell^p \check{\otimes} E \rightarrow \ell^p(F)$ is called the **p -summing norm** of T and denoted by $\pi_p(T)$.

Theorem (Y. Gordon, 1969)

$$\pi_p(\text{id}_{\ell_N^2}) \sim N^{\frac{1}{2}}$$

for all $p \in [1, \infty)$.

Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$ such that

$$\sum_{n=1}^{\infty} \|S e_n\|_{\ell_N^2} \|T e_n^*\|_{\ell_N^2} \leq C N \|S\| \|T\|$$

$$(N \in \mathbb{N}, S \in \mathcal{B}(\ell^p, \ell_N^2), T \in \mathcal{B}(\ell^{p'}, \ell_N^2)).$$

Proof.

Identify (algebraically)

$$\mathcal{B}(\ell^p, \ell_N^2) = \ell^{p'} \check{\otimes} \ell_N^2 = \ell^{p'} \otimes \ell_N^2 = \ell^{p'}(\ell_N^2)$$

$$\mathcal{B}(\ell^{p'}, \ell_N^2) = \ell^p \check{\otimes} \ell_N^2 = \ell^p \otimes \ell_N^2 = \ell^p(\ell_N^2).$$

Note that

$$\begin{aligned} \sum_{n=1}^{\infty} \|S e_n\|_{\ell_N^2} \|T e_n^*\|_{\ell_N^2} &\leq \|S\|_{\ell^{p'}(\ell_N^2)} \|T\|_{\ell^p(\ell_N^2)}, && \text{by Hölder,} \\ &\leq \pi_{p'}(\text{id}_{\ell_N^2}) \pi_p(\text{id}_{\ell_N^2}) \|S\| \|T\| \\ &\leq C N \|S\| \|T\|, && \text{by Gordon.} \quad \square \end{aligned}$$

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary

Let $p \in (1, \infty)$ and let E be an \mathcal{L}^p -space with $\dim E = \infty$. Then $\ell^\infty(\mathcal{K}(E))$ is not amenable.

Theorem

Let $p \in (1, \infty)$, and let E be an \mathcal{L}^p -space. Then $\mathcal{B}(\ell^p(E))$ is not amenable.

Proof.

If $\mathcal{B}(\ell^p(E))$, then so is $\ell^\infty(\mathcal{B}(\ell^p(E)))$ as is $\ell^\infty(\mathcal{K}(\ell^p(E)))$.
Impossible! □

Corollary

Let $p \in (1, \infty)$. Then $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p[0, 1])$ are not amenable.

V. Runde, $\mathcal{B}(\ell^p)$ is never amenable. arXiv:0907.3984.