Spectra and Pseudospectra

Thomas Ransford

Université Laval, Québec

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Plan

1. Introduction to pseudospectra
2. Do pseudospectra determine matrix behavior? (Joint work with Maxime Fortier Bourque, J. London Math. Soc., 2009)

Pseudospectra

Generalization of the notion of spectrum. Applications in:

- atmospheric science
- control theory
- ecology
- hydrodynamic stability
- lasers
- magnetohydrodynamics
- Markov chains
- non-hermitian quantum mechanics
- numerical solutions of differential equations
- rounding error analysis
- $\dots$

General problem

Let A be a linear operator on a Banach space. Determine the evolution of $\|A^n\|$, or of $\|e^{tA}\|$.

We shall concentrate on the case of $\|A^n\|$, where A is an $N \times N$ matrix acting on ℓ^2_N.

First remarks: Let $\rho(A) =$ spectral radius of A.

- $\|A^n\| \geq \rho(A)^n$ for all n, with equality if A normal.
- $\|A^n\|^{1/n} \to \rho(A)$ as $n \to \infty$.

Cautionary example:

Let A be the $N \times N$ matrix

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1/4 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/4 & 0 & 1 \\ & & & 1/4 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_j = \cos\left(\frac{j\pi}{N+1}\right),\ j = 1, \ldots, N$.

In particular $\rho(A) = \cos\left(\frac{\pi}{N+1}\right) < 1$, so $\|A^n\| \to 0$ as $n \to \infty$.

Let's actually compute $\|A^n\| \ldots$

Cautionary example (continued)

We'll consider the case $N = 32$.

n	$\|A^n\|$
1	1.24
2	1.55
4	2.41
10	8.98
20	78.44
40	4442.09
100	485866.04
200	1043599.98
400	544597.34
1000	36339.67
10000	6.63×10^{-14}

Resolvents

Problem: How to estimate $M := \max_{n\geq 0} \|A^n\|$ in general?

One idea: Observe that, for $|z| > 1$,

$$\|(A - zI)^{-1}\| = \Big\|\sum_{n\geq 0} \frac{A^n}{z^{n+1}}\Big\| \leq \sum_{n\geq 0} \frac{M}{|z|^{n+1}} = \frac{M}{(|z| - 1)}.$$

This proves the first half of

Kreiss matrix theorem

$$\max_{n\geq 0} \|A^n\| \geq \sup_{|z|>1} (|z| - 1)\|(A - zI)^{-1}\|$$

$$\max_{n\geq 0} \|A^n\| \leq \sup_{|z|>1} (|z| - 1)\|(A - zI)^{-1}\|.eN$$

Moral: It's useful to look at $\|(A - zI)^{-1}\|$. But how to compute it?

Singular values

The **singular values** of A are the square roots of the eigenvalues of AA^*. We shall denote them by $s_1, \ldots, s_N$.

Singular-value decomposition: We can always write

$$A = U_1 \Sigma U_2,$$

where U_1, U_2 are unitary, and $\Sigma = \mathrm{diag}(s_1, \ldots, s_N)$.

Note:

- $\|A\| = s_{\max}(A)$.
- $\|A^{-1}\| = 1/s_{\min}(A)$.

Two tricks for computing resolvent norms

Trick #1:

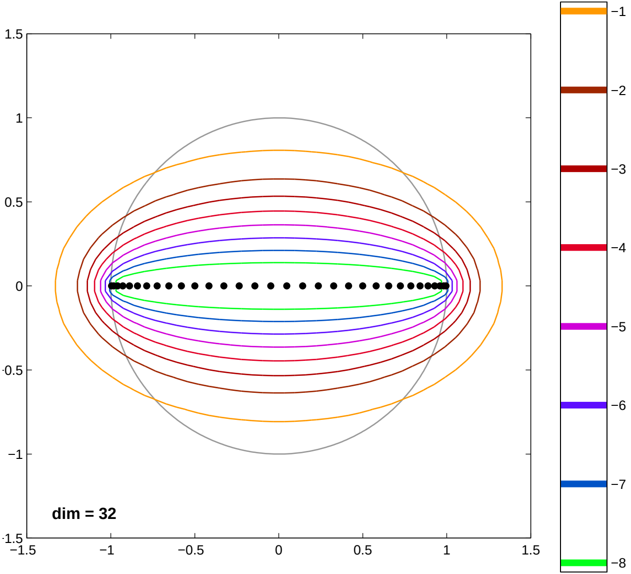
$$\|(A - zI)^{-1}\| = 1/s_{\min}(A - zI).$$

Trick #2:

Decompose $A = U^* T U$, with U unitary, T upper triangular. Then

$$s_{min}(A - zI) = s_{\min}(T - zI) \quad \text{for all } z \in \mathbb{C}.$$

Level curves of $\|(A - zI)^{-1}\|$ for $A = \text{tridiag}(32, \frac{1}{4}, 0, 1)$



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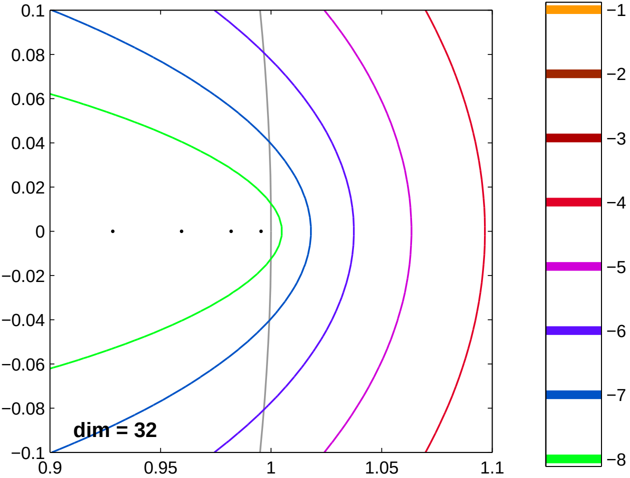

Pseudospectra

The ϵ-**pseudospectrum** of A is

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} : \|(A - zI)^{-1}\| > 1/\epsilon\}.$$

- $\sigma_\epsilon(A) \downarrow \sigma(A)$ as $\epsilon \downarrow 0$.
- $\sigma_\epsilon(A) = \bigcup\{\sigma(A + E) : \|E\| < \epsilon\}$.
- $\sigma_\epsilon(A)$ is easy to compute.
- Many phenomena can be explained using pseudospectra.

Reference:
L. N. Trefethen, M. Embree, *Spectra and Pseudospectra*,
Princeton University Press, 2005.

Do pseudospectra determine matrix behavior?

Question: Suppose that A, B have identical pseudospectra, i.e.

$$\|(A - zI)^{-1}\| = \|(B - zI)^{-1}\| \quad \text{for all } z \in \mathbb{C}.$$

- Must we have $\|A^n\| = \|B^n\|$ for all n?
- Must A, B be unitarily equivalent?

Answers:

- **Yes**, if $N = 1, 2$.
- **No**, if $N \geq 4$.
- **Good news:** we always have $1/2 \leq \|A\|/\|B\| \leq 2$.
- **Bad news:** given submultiplicative sequences (α_n) and (β_n), there exist A, B with identical pseudospectra such that

$$\|A^n\| = \alpha_n \quad \text{and} \quad \|B^n\| = \beta_n \qquad (2 \leq n \leq (N-3)/2).$$

Super-identical pseudospectra

A, B have identical pseudospectra iff

$$\|(A - zI)^{-1}\| = \|(B - zI)^{-1}\| \quad \text{for all } z \in \mathbb{C}.$$

This is equivalent to

$$s_{\min}(A - zI) = s_{\min}(B - zI) \quad \text{for all } z \in \mathbb{C}.$$

Definition:

A, B have **super-identical pseudospectra** if, for $k = 1, \ldots, N$,

$$s_k(A - zI) = s_k(B - zI) \quad \text{for all } z \in \mathbb{C}.$$

- Does this condition imply that $\|A^n\| = \|B^n\|$ for all n?
- Does it imply that A, B are unitarily equivalent?

Some reformulations

By definition, A, B have super-identical pseudospectra iff

$$s_k(A - zI) = s_k(B - zI) \qquad (z \in \mathbb{C},\ k = 1, \ldots, N).$$

This is equivalent to

$$\sigma\Big((A - zI)(A^* - \overline{z}I)\Big) = \sigma\Big((B - zI)(B^* - \overline{z}I)\Big) \qquad (z \in \mathbb{C}).$$

This is equivalent to

$$\operatorname{tr}\Big([(A{-}zI)(A^*{-}\overline{z}I)]^k\Big) = \operatorname{tr}\Big([(B{-}zI)(B^*{-}\overline{z}I)]^k\Big) \quad (z \in \mathbb{C},\ k \geq 0).$$

This is also equivalent to the same condition, but with $1 \leq k \leq N$.

Some easy consequences

Theorem 1

Let F be a uniqueness set for polynomials in $z, \overline{z}$ of bidegree N, N. Then A, B have super-identical pseudospectra iff

$$s_k(A - zI) = s_k(B - zI) \qquad (z \in F,\ k = 1, \dots, N).$$

Theorem 2

If A, B have super-identical pseudospectra then, for every polynomial p,

$$\frac{1}{\sqrt{N}} \le \frac{\|p(A)\|}{\|p(B)\|} \le \sqrt{N}.$$

An example

For $0 < \alpha < \beta \leq \pi/4$, define

$$A := \begin{pmatrix} 0 & \sec\alpha & 0 & 1 \\ 0 & 0 & \sec\beta\csc\beta & 0 \\ 0 & 0 & 0 & \csc\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and let B be the same matrix with the roles of α, β interchanged. Then A, B have super-identical pseudospectra, but

$$\|A^2\|/\|B^2\| = \cos\alpha/\cos\beta \neq 1.$$

Consequences:

- In Theorem 2, cannot replace $\sqrt{N}$ by 1 (but maybe by $\sqrt{2}$?).
- Super-identical pseudospectra $\not\Rightarrow$ unitary equivalence.

Super-identical pseudospectra and unitary equivalence

Specht's theorem: A, B are unitarily equivalent iff

$$\mathrm{tr}(w(A, A^*)) = \mathrm{tr}(w(B, B^*)) \quad \text{for all words } w.$$

Recall that A, B have super-identical pseudospectra iff

$$\mathrm{tr}\Big([(A-zI)(A^*-\overline{z}I)]^k\Big) = \mathrm{tr}\Big([(B-zI)(B^*-\overline{z}I)]^k\Big) \quad (z \in \mathbb{C},\ k \geq 0).$$

Algebraicity theorem

Given a word w, the polynomial $\mathrm{tr}(w(X, Y))$ is algebraic over the algebra generated by $\{\mathrm{tr}([(X - zI)(Y - \overline{z}I)]^k) : k \geq 0,\ z \in \mathbb{C}\}$.

Corollary (Discreteness theorem)

'Almost every' s-i-p equivalence class is a union of a finite number of unitary equivalence classes. The number is bounded by a constant depending only on N.

Conclusion

- Modeling of experiments by linear dynamical systems gives rise to the problem of estimation of $\|A^n\|$ or of $\|e^{tA}\|$.
- For non-normal matrices, standard eigenvalue analysis tells only part of the story, and can sometimes even be misleading.
- We can get more information by looking at level curves of the resolvent (pseudospectra). These can be rapidly computed using singular values.
- Pseudospectra are not sufficient to determine matrix behavior. The notion of super-identical pseudospectra leads to more satisfactory results.
- How to implement this notion in practice?

DZIĘKUJĘ!