School of Mathematics

FACULTY OF MATHEMATICS AND PHYSICAL SCIENCES

UNIVERSITY OF LEEDS

Multi-normed spaces

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Multi-normed spaces

Definition

A multi-normed space is a Banach space E equipped with a sequence of norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ on the linear spaces $\{E^n : n \in \mathbb{N}\}$ satisfying:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n \leq (\max_{i\in\mathbb{N}_n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n$;

(A3) $\|(x_1, \ldots, x_{n-1}, 0)\|_n = \|(x_1, \ldots, x_{n-1})\|_{n-1}$;

(A4) $\|(x_1, \ldots, x_{n-2}, x, x)\|_n = \|(x_1, \ldots, x_{n-2}, x)\|_{n-1}$.

Example

$\|(x_1, \ldots, x_n)\|_n = \max\{\|x_i\| : i \in \mathbb{N}_n\}$

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Multi-normed spaces

- E is a Banach space.

Proposition

Let $\{\|\cdot\|_n : n \in \mathbb{N}\}$ be a sequence of norms on the spaces $\{E^n : n \in \mathbb{N}\}$. Then E is a multi-normed space if and only if

$$\|Ax\|_m \leq \|A\|_\infty \|x\|_n \quad (A \in \mathbb{M}_{m,n},\ x \in E^n,\ m,n \in \mathbb{N})\,. \qquad \square$$

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□

Tensor norms

Definition

A norm α on the linear space $c_0 \otimes E$ is a *c_0-norm* if:

(i) $\alpha(x \otimes y) = \|x\| \, \|y\|$ for every $x \in c_0$ and $y \in E$, and

(ii) $T \otimes I_E$ is bounded on $(c_0 \otimes E, \alpha)$ with $\|T \otimes I_E\| \leq \|T\|$ for each $T \in \mathcal{B}(c_0)$.

- $\varepsilon(z) \leq \alpha(z) \leq \pi(z)$

Proposition (Daws)

The study of multi-norms over E is equivalent to the study of c_0-norms on $c_0 \otimes E$. □

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Pisier's representation theorem

Example

- L a Banach lattice.
- $J : E \to L$ an isometric embedding.
- The formula

$$\alpha\left(\sum_{j=1}^{n} \delta_j \otimes x_j\right) = \big\| |J(x_1)| \vee \cdots \vee |J(x_n)| \big\|_L \qquad (*)$$

defines a c_0-norm on $c_0 \otimes E$.

Theorem (Pisier (2001))

Let α be a c_0-norm on $c_0 \otimes E$. Then there exists a Banach lattice L and an isometric embedding $J : E \to L$ such that $(*)$ holds. □

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Replace ∞ by p?

- $\|Ax\|_m \leq \|A\|_p \|x\|_n \iff$ ℓ^p-norm on $\ell^p \otimes E$
- $\|Ax\|_m \leq \|A\|_1 \|x\|_n \iff$ (A1)-(A3) +

  (B4) $\|(x_1, \ldots, x_{n-2}, x, x)\|_n = \|(x_1, \ldots, x_{n-2}, 2x)\|_{n-1}$
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The maximum multi-norm

The maximum multi-norm over E is defined by

$$\|(x_1, \ldots, x_n)\|_n^{\max} = \sup \|(x_1, \ldots, x_n)\|_n^{\alpha} \quad (n \in \mathbb{N},\, x_1, \ldots, x_n \in E)\,,$$

where the supremum is taken over all multi-norms $\|\cdot\|_n^{\alpha}$ on E.

- $\|(x_1, \ldots, x_n)\|_n^{\max} = \pi\left(\sum_{i=1}^{n} \delta_i \otimes x_i\right)$

Proposition

For each $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in E^n$, we have

$$\|x\|_n^{\max} = \sup\left\{\left|\sum_{i=1}^{n} \langle x_i, \lambda_i \rangle\right| : \lambda = (\lambda_1, \ldots, \lambda_n) \in (E')^n,\, \mu_{1,n}(\lambda) \le 1\right\}.$$

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Proposition

Let $1 \leq p \leq q < \infty$. For each $n \in \mathbb{N}$ we define a norm on E^n by

$$\|x\|_n^{(p,q)} = \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \right)^{1/q} : \lambda \in (E')^n,\ \mu_{p,n}(\lambda) \leq 1 \right\},$$

where $x = (x_1, \ldots, x_n) \in E^n$. Then the family $\{\| \cdot \|_n^{(p,q)} : n \in \mathbb{N}\}$ is a multi-norm over E, called the weak-(p, q) multi-norm. □

- Let $1 \leq p \leq q < \infty$.
  - Obvious: $\| \cdot \|_n^{(1,q)} \leq \| \cdot \|_n^{(p,q)} \leq \| \cdot \|_n^{(q,q)}$
  - Less obvious: $\| \cdot \|_n^{(p,p)} \leq \| \cdot \|_n^{(q,q)} \leq \| \cdot \|_n^{(1,1)}$

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The weak-(p,p) multi-norm

- E, F Banach spaces, $p \geq 1$. The *Chevet-Saphar p-norm* on $F \otimes E$ is defined by

$$d_p(z) = \inf \left\{ \mu_{p'}(x_1, \ldots, x_n) \left(\sum_{i=1}^{n} \|y_i\|^p \right)^{1/p} : z = \sum_{i=1}^{n} x_i \otimes y_i \right\} .$$

- $(F \otimes E, d_p)' = \mathcal{P}_{p'}(F, E')$.

Theorem (Daws, R)

The weak-(p,p) multi-norm on E is induced by the Chevet-Saphar p'-norm on $c_0 \otimes E$. i.e.,

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The weak-$(1,p)$ multi-norm on $L^1(\Omega)$

Theorem (R)

Let Ω be a measure space, and let $1 \le p < \infty$. Then

$$\|(f_1,\dots,f_n)\|_n^{(1,p)} = \sup_{\mathbf{X}} \left(\sum_{i=1}^{n} \|\chi_{X_i} f_i\|^p \right)^{1/p} \qquad (f_1,\dots,f_n \in L^1(\Omega)) .$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1,\dots,X_n)$ of Ω. □

- In particular, we have

$$\|(f_1,\dots,f_n)\|_n^{(1,1)} = \|(f_1,\dots,f_n)\|_n^{\max} = \| |f_1| \vee \cdots \vee |f_n| \| .$$

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(p, q)-amenability

Definition
Let E be a multi-normed space. A subset $B \subset E$ is multi-bounded if

$$\sup \{\|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in B,\ n \in \mathbb{N}\} < \infty .$$

Definition
Let G be a locally compact group, and let $1 \le p \le q$. Then G is (p, q)-amenable if there exists a mean $\Lambda \in L^1(G)''$ such that the set $\{s \cdot \Lambda : s \in G\}$ is multi-bounded in the weak-(p, q) multi-norm.

- (q, q)-amenable $\implies$ (p, q)-amenable $\implies$ $(1, q)$-amenable.
- $(1, 1)$-amenable $\implies$ (p, p)-amenable $\implies$ (q, q)-amenable.

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Let G be a locally compact group. Then G is $(1,1)$-amenable if and only if G is amenable. □

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The group $\mathbb{F}_2$ is not $(1,p)$-amenable for any $p \geq 1$. □

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The group $\mathbb{F}_2$ is not $(1, p)$-amenable for any $p \geq 1$. $\square$

Injective Banach modules

- Let A be a Banach algebra, and let $E \in A$-**mod** be faithful.
- Then $\mathcal{B}(A, E) \in A$-**mod** with the multiplication

$$(a \cdot T)(b) = T(ba) \quad (a, b \in A,\ T \in \mathcal{B}(A, E)) .$$

- We define the *canonical embedding* $\Pi : E \to \mathcal{B}(A, E)$ by the formula

$$\Pi(x)(a) = a \cdot x \quad (a \in A,\ x \in E) .$$

Definition

The module E is injective if there exists a left A-module morphism $\rho : \mathcal{B}(A, E) \to E$ with $\rho \circ \Pi = I_E$.

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The module E is injective if there exists a left A-module morphism $\rho : \mathcal{B}(A, E) \to E$ with $\rho \circ \Pi = I_E$.

The $L^1(G)$ module $L^p(G)$

- For each $1 < p < \infty$, $L^p(G) \in L^1(G)$-**mod** with the multiplication

$$(a \cdot f)(t) = \int_G a(s) f(s^{-1}t)\, \mathrm{d}m(s) \quad (a \in L^1(G), f \in L^p(G)) .$$

- G amenable $\implies L^p(G)$ injective.

Theorem (R)

Let G be a discrete group, and let $1 < p < \infty$. Then

$$\ell^p(G) \text{ injective} \iff G \ (p,p)\text{-amenable} . \qquad \square$$

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Følner type conditions

Theorem (R)

Let G be a locally compact group, and let $1 < p < \infty$. Then:

$$G\ (1,p)\text{-amenable} \implies G\ p\text{-pseudo-amenable.}$$ □

Definition

Let G be a locally compact group, and let $1 < p < \infty$. Then G is p-pseudo-amenable if there exists $C \geq 1$ such that, for every $n \in \mathbb{N}$ and every finite set $F = \{s_1, \ldots, s_n\} \subset G$, there exists a non-null, compact subset $S \subset G$ with

$$m(FS) \leq Cn^{1-1/p} m(S) .$$

Theorem (Dales & Polyakov (2003))

Let G be a discrete group, and let $1 < p < \infty$. Then:

$$\ell^p(G)\ \text{injective} \implies G\ p\text{-pseudo-amenable} \implies \mathbb{F}_2 \not\subset G.$$ □

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