Homological dimensions of Köthe algebras

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ℓ^1

$$\left(\ell^1;\ \text{pointwise multiplication}\right)$$

Köthe algebras are locally convex analogues of

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Köthe sequence spaces

Let I be a set (usually countable), and let $P \subset \mathbb{R}_+^I$.

Definition. P is a Köthe set if

(P1) $\forall\, i \in I \quad \exists\, p \in P : \quad p(i) = p_i > 0;$

(P2) $\forall\, p, q \in P \quad \exists\, r \in P : \quad \max\{p_i, q_i\} \le r_i \quad \forall\, i \in I.$

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Given a Köthe set P, the Köthe space $\lambda(P)$ is

$$\lambda(P) = \Big\{ x = (x_i) \in \mathbb{C}^I : \|x\|_p = \sum_i |x_i| p_i < \infty \; \forall p \in P \Big\}$$

This is a complete locally convex space w.r.t. the topology determined by the family of seminorms $\{\| \cdot \|_p : p \in P\}$.

In what follows, we assume that $\boxed{P \text{ is countable.}}$

Then $\lambda(P)$ is a Fréchet space.

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5. $I = \mathbb{N}, \quad 0 < R \le \infty, \quad \alpha = (\alpha_n), \quad 0 \le \alpha_1 \le \alpha_2 \le \ldots \to \infty$, $P = \{(r^{\alpha_k})_{k\in\mathbb{N}} : 0 < r < R\}$. Then $\lambda(P)$ is denoted by $\Lambda_R(\alpha)$ and is called a power series space.

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5. $I = \mathbb{N}, \quad 0 < R \le \infty, \quad \alpha = (\alpha_n), \quad 0 \le \alpha_1 \le \alpha_2 \le \ldots \to \infty$, $P = \{(r^{\alpha_k})_{k\in\mathbb{N}} : 0 < r < R\}$. Then $\lambda(P)$ is denoted by $\Lambda_R(\alpha)$ and is called a power series space. For example:

   5.1) $\alpha_n = \log n \Longrightarrow \Lambda_\infty(\alpha) = s$.
   5.2) $\alpha_n = n^{1/m} \Longrightarrow \Lambda_R(\alpha) \cong \mathscr{O}(\mathbb{D}_R^m)$,
   where $\mathbb{D}_R^m = \{z \in \mathbb{C}^m : |z_i| < R,\ i = 1,\ldots,m\}$.

Köthe algebras

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When is $\lambda(P)$ a subalgebra of $\mathbb{C}^I$?

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Notation. Let P, Q be Köthe sets on I.

$$P \prec Q \iff \forall p \in P \quad \exists C > 0 \quad \exists q \in Q : \quad p_i \le C q_i \quad \forall i \in I.$$

$$P \sim Q \iff P \prec Q \text{ and } Q \prec P.$$

$$P \cdot Q = \{(p_i q_i)_{i \in I} : p \in P,\ q \in Q\}.$$

$$P^2 = P \cdot P \sim \{(p_i^2)_{i \in I} : p \in P\}.$$

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Proposition. *$\lambda(P)$ is a subalgebra of $\mathbb{C}^I$ $\iff$ $\lambda(P)$ is a Fréchet algebra under pointwise multiplication $\iff$ $P \prec P^2$.*

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Remark. In many natural cases we have $p_i \in \{0\} \cup [1, +\infty)$ for all $p \in P$, $i \in I$. This implies that $P \prec P^2$, so $\lambda(P)$ is a Köthe algebra. Moreover, each seminorm $\| \cdot \|_p$ is submultiplicative in this case, so $\lambda(P)$ is locally m-convex.

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Examples: ℓ^1, $\mathbb{C}^{\mathbb{N}}$, s, $\Lambda_R(\alpha)$ $(R \geq 1)$ are Köthe algebras. In particular,

$$\Lambda_R(\{n^{1/m}\}) = (\mathscr{O}(\mathbb{D}_R^m), \text{ Hadamard product }) = \mathscr{H}(\mathbb{D}_R^m).$$

All of them (except $\Lambda_1(\alpha)$) are locally m-convex.

Topological Homology

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Let A be a Fréchet algebra.

A left Fréchet A-module is a Fréchet space X together with the structure of a left A-module such that the map $A \times X \to X, \quad (a, x) \mapsto a \cdot x$, is continuous.

Categories of Fréchet A-modules: $\quad A\text{-}\mathbf{mod}, \quad \mathbf{mod}\text{-}A, \quad A\text{-}\mathbf{mod}\text{-}A.$

$$\mathbf{h}_A(X, Y) = \text{ the space of continuous morphisms } X \to Y$$

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The A-module tensor product of $X \in \mathbf{mod}\text{-}A$ and $Y \in A\text{-}\mathbf{mod}$ is

$$X \underset{A}{\widehat{\otimes}} Y = X \mathbin{\widehat{\otimes}} Y / \overline{\mathrm{span}}\{x \cdot a \otimes y - x \otimes a \cdot y : x \in X,\ y \in Y,\ a \in A\}.$$

$$\left\{ \begin{array}{c} \text{continuous linear maps} \\ X \mathbin{\widehat{\otimes}}_A Y \to E \end{array} \right\} = \left\{ \begin{array}{c} \text{continuous } A\text{-balanced bilinear maps} \\ X \times Y \to E \end{array} \right\}$$

Admissible chain complexes

A chain complex

$$X_\bullet = \big(\ldots \leftarrow X_{n-1} \xleftarrow{d_n} X_n \leftarrow X_{n+1} \leftarrow \ldots \big)$$

in A-$\mathbf{mod}$ is admissible if it splits in the category of Fréchet spaces

(i.e., if $X_\bullet$ has a contracting homotopy consisting of continuous linear maps)

$\iff$ $X_\bullet$ is exact, and $\operatorname{Ker} d_n$ is a complemented subspace of X_n for all n.

Projective and flat modules

For $X \in A\text{-}\mathbf{mod}$ we have two covariant functors:

$$\begin{aligned} {}_A\mathbf{h}(X,\,\cdot\,)\colon A\text{-}\mathbf{mod} &\to \mathbf{Vect}, & Y &\mapsto {}_A\mathbf{h}(X,Y),\\ (\,\cdot\,)\mathop{\widehat{\otimes}}_{A} X\colon\ \mathbf{mod}\text{-}A &\to \mathbf{Vect}, & Y &\mapsto Y \mathop{\widehat{\otimes}}_{A} X. \end{aligned}$$

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Definition. X is projective if

$${}_A\mathbf{h}(X,\,\cdot\,)\colon \left\{\begin{array}{c}\text{admissible complexes}\\ \text{in } A\text{-}\mathbf{mod}\end{array}\right\} \mapsto \left\{\begin{array}{c}\text{exact complexes}\\ \text{in } \mathbf{Vect}\end{array}\right\}$$

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Fact: Each projective module is flat.

Resolutions

Given a left Fréchet A-module X, a resolution of X is a chain complex $P_\bullet = (P_n, d_n)_{n\geq 0}$ of left Fréchet A-modules together with a morphism $\epsilon\colon P_0 \to X$ such that the augmented sequence

$$0 \leftarrow X \xleftarrow{\epsilon} P_0 \xleftarrow{d_0} \cdots \leftarrow P_n \xleftarrow{d_n} P_{n+1} \leftarrow \cdots$$

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Fact: A-$\mathbf{mod}$ has enough projectives, i.e., each $X \in A$-$\mathbf{mod}$ has a projective resolution.

The same is true of $\mathbf{mod}$-A and A-$\mathbf{mod}$-A.

Homological dimensions

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Remarks. 1. $\operatorname{dh}_A X = 0 \iff X$ is projective.

2. $\operatorname{w.dh}_A X = 0 \iff X$ is flat.

3. $\operatorname{w.dh}_A X \le \operatorname{dh}_A X$.

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(3) A is biprojective $\iff$ the product map $\pi_A \colon A \mathbin{\widehat{\otimes}} A \to A$ is a retraction of Fréchet A-bimodules.

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Theorem (Helemskii). *Let A be a Fréchet algebra.*

1. *If A is biprojective, then* $\operatorname{db} A \le 2$.
2. *If A is biflat, then* $\operatorname{w.dg} A \le 2$.
3. *If A is biflat and projective in* A-**mod** *or in* **mod**-A, *then* $\operatorname{w.db} A \le 2$.

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Remark. If A is a biflat Banach algebra, then $\mathrm{w.db}\, A \le 2$ (Helemskii).

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Problem. Let $X \in A$-$\mathbf{mod}$ and $Y \in \mathbf{mod}$-A be flat Fréchet A-modules.
Is $X \mathbin{\widehat{\otimes}} Y$ flat in A-$\mathbf{mod}$-A? (Yes, if either X or Y is projective; yes, if A is a Banach algebra and X, Y are Banach modules).

Some conditions on P

1. Condition (U)

Let $\lambda(P)$ be a Köthe algebra on I.

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Example. $\mathscr{H}(\mathbb{D}_1)$ is unital; $f(z) = 1/(1-z)$ is the identity element.

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Example. $\mathbb{C}^I$ is unital.

Example. $\mathscr{H}(\mathbb{D}_1)$ is unital; $f(z) = 1/(1-z)$ is the identity element.

Example. $\Lambda_R(\alpha)$ is not unital unless $R = 1$. In particular, $\mathscr{H}(\mathbb{D}_R)$ is not unital if $R > 1$; s is not unital.

Some conditions on P

2. Condition (N)

(N) $$\boxed{\forall p \in P \quad \exists\, q \in P \quad \exists\, \alpha \in \ell^1 \ :\ p \leq \alpha q}$$

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Theorem (Grothendieck, Pietsch)**.** *P satisfies* **(N)** $\iff$ *$\lambda(P)$ is nuclear.*

Nuclear locally convex spaces

Let X, Y be Banach spaces. A linear map $T\colon X \to Y$ is nuclear if

$$T(x) = \sum_n \lambda_n f_n(x) y_n \qquad (x \in X),$$

where $\{f_n\} \subset X^*$ and $\{y_n\} \subset Y$ are bounded sequences and $\sum_n |\lambda_n| < \infty$.

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Let X, Y be locally convex spaces. A linear map $T\colon X \to Y$ is nuclear if it factors as

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d] & Y \\ E \arrow[r, "S"'] & F \arrow[u] \end{tikzcd}$$

where E, F are Banach spaces and S is a nuclear map.

A locally convex space X is nuclear if each continuous linear map from X to a Banach space is nuclear.

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(N) $$\boxed{\forall p \in P \quad \exists\, q \in P \quad \exists\, \alpha \in \ell^1 \; : \; p \le \alpha q}$$

Theorem (Grothendieck, Pietsch). *P satisfies* **(N)** $\iff$ $\lambda(P)$ *is nuclear.*

Notation:

$$\lambda^\infty(P) = \Big\{ x = (x_i) \in \mathbb{C}^I : \|x\|_p^\infty = \sup_i |x_i| p_i < \infty \; \forall p \in P \Big\}$$

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Examples. $\Lambda_\infty(\alpha)$ is nuclear $\iff$ $\sup_n \frac{\log n}{\alpha_n} < \infty$.

If $R < \infty$, then $\Lambda_R(\alpha)$ is nuclear $\iff$ $\lim_n \frac{\log n}{\alpha_n} = 0$.

For instance, s and $\mathscr{H}(\mathbb{D}_R)$ are nuclear.

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Theorem 1. *Let* $A = \lambda(P)$ *be a Köthe algebra. The following conditions are equivalent:*

1. P *satisfies* **(B)***;*
2. A *is biprojective;*
3. A *is biflat;*
4. A *is flat in* A-$\mathbf{mod}$*;*
5. *the product map* $A \mathbin{\widehat{\otimes}} A \to A$, $a \otimes b \mapsto ab$ *is onto.*

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Example. ℓ^1 is biprojective (Helemskii, 1972); $\Lambda_R(\alpha)$ is biprojective $\iff$ either $R = 1$ or $R = \infty$. For instance, s, $\mathscr{H}(\mathbb{C})$, $\mathscr{H}(\mathbb{D}_1)$ are biprojective. Moreover, $\mathscr{H}(\mathbb{D}_1)$ is unital and hence contractible.

Some conditions on P

4. Condition (M)

(**M**) There exist complex matrices $\alpha = (\alpha_{ij})_{i,j\in I}$ and $\beta = (\beta_{ij})_{i,j\in I}$ such that

(**M**1) $\alpha_{ij} + \beta_{ij} = 1 \quad (i, j \in I)$;
(**M**2) $\forall p \in P \quad \exists\, C > 0 \quad \exists\, q \in P \quad \forall j \in \mathbb{N} \quad \sup_i |\alpha_{ij}| p_i p_j \le C q_j^2$;
(**M**3) $\forall p \in P \quad \exists\, C > 0 \quad \exists\, q \in P \quad \forall i \in \mathbb{N} \quad \sup_j |\beta_{ij}| p_j p_i \le C q_i^2$.

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Remark. Condition (**M**) is often satisfied automatically. For example, P satisfies (**M**) whenever $I = \mathbb{N}$ and $p_i \le p_{i+1}$ for all $p \in P$, $i \in \mathbb{N}$. Therefore $\Lambda_R(\alpha)$ satisfies (**M**) whenever $R > 1$. In particular, s and $\mathscr{H}(\mathbb{D}_R)$ $(R > 1)$ satisfy (**M**).

Weak dimensions

Theorem 2. *Let* $A = \lambda(P)$ *be a Köthe algebra. Then*

$$\operatorname{w.dg} A = \operatorname{w.db} A = \begin{cases} 0, & P \textit{ satisfies } (\mathbf{U}). \\ & \\ & \\ & \end{cases}$$

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Here A *acts on* $\mathbb{C}$ *trivially, i.e.,* $a \cdot \lambda = 0$ *for all* $a \in A$, $\lambda \in \mathbb{C}$.

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“Strong” (projective) dimensions

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Here $\bar{P}$ is defined as follows. For each $p \in P$ define $\bar{p} \in \mathbb{R}_+^I$ by $\bar{p}_i = \min\{p_i, 1\}$, and let $\bar{P} = \{\bar{p} : p \in P\}$. The Köthe space $\lambda(\bar{P})$ is a Fréchet $\lambda(P)$-module under pointwise multiplication. For example, if $p_i \geq 1$ for all $p \in P$, $i \in I$ (so that $\lambda(P)$ is locally m-convex), then $\lambda(\bar{P}) = \ell^1$.

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Examples

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Example 5. If $1 < R < \infty$, then $\operatorname{dg}\Lambda_R(\alpha) = \operatorname{db}\Lambda_R(\alpha) = \operatorname{w.dg}\Lambda_R(\alpha) = \operatorname{w.db}\Lambda_R(\alpha) = \infty$.

Examples

Example 6. Let $I = \mathbb{N} \times \mathbb{N}$. For each $i, j, k \in \mathbb{N}$ we define

$$p_{ij}^{(k)} = \begin{cases} 2^{(kj)^i}(i+j)^k & (i \le k), \\ (i+j)^k & (i > k). \end{cases}$$

Set $p^{(k)} = (p_{ij}^{(k)})_{i,j \in \mathbb{N}}$, and consider the Köthe set $P = \{p^{(k)}\}_{k \in \mathbb{N}}$. Then P satisfies (**B**) and (**N**), but does not satisfy (**M**). Therefore

$$\operatorname{dg} \lambda(P) = \operatorname{db} \lambda(P) = \operatorname{dh}_{\lambda(P)} \ell^1 = 2,$$
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Approximate contractibility and around

Definition (Ghahramani and Loy (2004); Lawson and Read (2008))**.** A Fréchet algebra A is approximately contractible if, for each Fréchet A-bimodule X, every derivation $D\colon A \to X$ is approximately inner, i.e., D is the limit of a pointwise convergent net of inner derivations.

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Theorem (GLZ, 2008; LR, 2008)**.** *A locally m-convex Fréchet algebra A is approximately contractible iff A is approximately amenable (i.e., for each Fréchet A-bimodule X, every derivation $D\colon A \to X^*$ is approximately inner).*

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Examples. (1) All amenable lmc Fréchet algebras are approximately contractible;
(2) The c_0-sum of a family of amenable Banach algebras is approximately contractible [GL];
(3) Some Banach sequence algebras are approximately contractible [Dales, LZ (2006); GLZ (2008)]; ℓ^p is not [DLZ];
(4) $A(G)$ is approximately contractible if G is an amenable discrete countable group [G., Stokke (2007)]; $A(\mathbb{F}_2)$ is not [Choi, GZ (2009)];
(5) Some ℓ^1-semigroup algebras are approximately contractible [CGZ (2009)];
(6) Some Köthe algebras (e.g., s) are approximately contractible [LR].

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Problem. Is the converse true?

Conjecture [Dales, Loy, Zhang (2006)]. If A is a Banach sequence algebra, then there is a relationship between the approximate contractibility of A and the property

$$\boxed{A = A^2},$$

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Question. Let $A = \lambda(P)$. When is $A = A^2$?

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Question. Let $A = \lambda(P)$. When is $A = A^2$?

Clearly, if $A = A^2$, then the product map $A \widehat{\otimes} A \to A$ is onto, which is equivalent to the biprojectivity of A (condition (**B**)).

Approximate contractibility and around

Definition. Let $\lambda(P)$ be a Köthe space. The Köthe-Toeplitz dual of $\lambda(P)$ is

$$\lambda(P)^\times = \Big\{x = (x_i) \in \mathbb{C}^I : \|x\|_y = \sum_i |x_i y_i| < \infty \ \forall y \in \lambda(P)\Big\}.$$

In other words, $\lambda(P)^\times = \lambda(P^\times)$, where $P^\times = \lambda(P)_{\geq 0}$.

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Theorem 5. *Let $A = \lambda(P)$ be a biprojective Köthe algebra. Consider the following conditions:*

(1) *A is nuclear (i.e., P satisfies* **(N)***);*
(2) *$A^\times$ is nuclear (i.e., $P^\times$ satisfies* **(N)***);*
(3) *$A = A^2$.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$. If $p_i \geq 1$ for all $p \in P$, $i \in I$, then all three conditions are equivalent and imply that A is approximately contractible.

Approximate contractibility and around

Corollary. *Let* $A = \lambda(P)$ *be a biprojective Köthe algebra. Suppose that* $I = \mathbb{N}$ *and that* $1 \le p_i \le p_{i+1}$ *for all* $p \in P$, $i \in \mathbb{N}$. *Then the following conditions are equivalent and imply that* A *is approximately contractible:*

(1) A *is nuclear (i.e.,* P *satisfies* **(N)***);*
(2) $A^\times$ *is nuclear (i.e.,* $P^\times$ *satisfies* **(N)***);*
(3) $A = A^2$*;*
(4) *There exists* $p \in P$ *such that* $\sup_n \frac{\log n}{\log p_n} < \infty$.

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Corollary. *Let $A = \lambda(P)$ be a biprojective Köthe algebra. Suppose that $I = \mathbb{N}$ and that $1 \le p_i \le p_{i+1}$ for all $p \in P$, $i \in \mathbb{N}$. Then the following conditions are equivalent and imply that A is approximately contractible:*

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Example. All nuclear $\Lambda_\infty(\alpha)$ (e.g., s and $\mathscr{H}(\mathbb{C})$) are approximately contractible.

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Example. All nuclear $\Lambda_\infty(\alpha)$ (e.g., s and $\mathscr{H}(\mathbb{C})$) are approximately contractible.

Problem. Suppose that $\lambda(P)$ is approximately contractible. Does it imply that $\lambda(P)$ is biprojective? Does it imply any of the conditions (1)–(4) of the above corollary?