A class of weighted convolution Fréchet algebras

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Overview

Motivation

- For an algebra weight ω on $\mathbb{R}^+ = [0, \infty)$, let
$$L^1(\omega) = \{f : \|f\|_\omega = \int_0^\infty |f(t)|\omega(t) < \infty\}.$$
A commutative Banach algebra with the convolution product.
- Many results about $L^1(\omega)$ (semisimple or radical) [homomorphisms, derivations, ideal structure, ...].
- Some related results about $L^1_{\text{loc}}(\mathbb{R}^+)$ (Ghahramani & McClure, 1992 and Grabiner, 2006).
- $L^1_{\text{loc}}(\mathbb{R}^+) = \bigcap_n L^1[0, n]$ with the projective limit but $L^1_{\text{loc}}(\mathbb{R}^+)$ is still "large"
- Hardly any results about the "small" algebras
$$A(\omega) = \bigcap_n L^1(\omega_n)$$
with the projective limit topology, where $\omega = (\omega_n)$ is an increasing sequence of weights on $\mathbb{R}^+$.

Assumptions on the weights

$\omega = (\omega_n)$ is an increasing sequence of algebra weights on $\mathbb{R}^+$ with

(a) $\omega_n(t) \to \infty$ as $t \to \infty$, $\quad \forall n \in \mathbb{N}$,

(b) $\lim_{t\to\infty} \omega_n(t)^{1/t} = 1$, $\quad \forall n \in \mathbb{N}$,

(c) $\sup_{t\in\mathbb{R}^+} \frac{\omega_{n+1}(t)}{\omega_n(t)} = \infty$, $\quad \forall n \in \mathbb{N}$.

We are most interested in the case where
(d) $\omega_n(t) \to \infty$ as $n \to \infty$, $\quad \forall t \in \mathbb{R}^+$.
[For instance $\omega_n(t) = (1 + t)^n$.]

But the case where $\lim_{n\to\infty} \omega_n(t) < \infty$ has interesting aspects.
[For instance $\omega_n(t) = (1 + t)^{1-1/n}$.]

Weighted convolution Fréchet algebras

$$A(\omega) = \bigcap_n L^1(\omega_n)$$

and

$$B(\omega) = \bigcap_n M(\omega_n)$$

are Fréchet algebras with the increasing sequence of norms

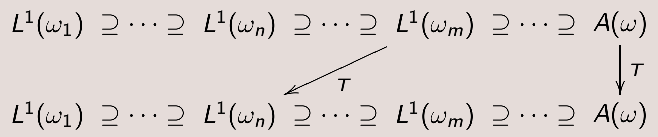
$$\|\mu\|_n = \|\mu\|_{\omega_n} \qquad (\mu \in B(\omega)).$$

Continuous linear maps on $A(\omega)$

Main Lemma

A linear map $T : A(\omega) \to A(\omega)$ is continuous if and only if:

$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} : \quad T \text{ extends continuously } T : L^1(\omega_m) \to L^1(\omega_n) :$$



Similarly, a linear functional $\varphi : A(\omega) \to \mathbb{C}$ is continuous if and only if φ extends continuously to $L^1(\omega_n)$ for some $n \in \mathbb{N}$.

Remarks

- Many results about $A(\omega)$ can be easily deduced from the Main Lemma and the corresponding results about $L^1(\omega)$.
- We will focus on the more "surprising" results.

Basic results deduced from the Main Lemma

- Every character on $A(\omega)$ is continuous (and the characters are given by the Laplace tranform).
- $A(\omega)$ is semisimple.
- For $\mu \in B(\omega)$, let $T_\mu(f) = \mu * f$ $(f \in A(\omega))$. This identifies

  $$\mathrm{Mul}\,(A(\omega)) = B(\omega).$$

  In particular, every multiplier on $A(\omega)$ is continuous.

  Also induces a strong topology on $B(\omega)$.

Bounded approximate identities and factorisation

- $A(\omega)$ has a bounded approximate identity (that is, an approximate identity which is bounded in each $L^1(\omega_n)$).

  [For instance, $e_k(t) = k \cdot 1_{[0,1/k]}$ $(k \in \mathbb{N})$.]
- If $\lim_{n\to\infty} \omega_n(t) = \infty$, $\forall t \in \mathbb{R}^+$, then $A(\omega)$ does not have a *uniformly* bounded approximate identity (that is, with a common bound).
- Craw, 1969: Cohen's factorisation theorem holds for Fréchet algebras with uniformly bounded approximate identites.

  (In particular it holds for $A(\omega)$ if $\lim_{n\to\infty} \omega_n(t) < \infty$, $\forall t \in \mathbb{R}^+$.)
- **Open problem** Do we have factorisation in $A(\omega)$? (If $\lim_{n\to\infty} \omega_n(t) = \infty$, $\forall t \in \mathbb{R}^+$.)

Endomorphisms of $L^1(\omega)$

Theorem (Grabiner, 1980)

Let Φ be a non-zero endomorphism of $L^1(\omega)$. Then:

- Φ extends uniquely to a continuous endomorphism $\widetilde{\Phi}$ of $M(\omega)$.
- $\nu^t = \widetilde{\Phi}(\delta_t)$ $(t \in \mathbb{R}^+)$ defines a semigroup in $M(\omega)$ which is strongly continuous for $t > 0$.
- For $\mu \in M(\omega)$:
$$\widetilde{\Phi}(\mu) = \int_0^\infty \nu^t \, d\mu(t)$$
as a strong Bochner integral in $M(\omega)$.

Standard endomorphisms of $L^1(\omega)$

Theorem (Ghahramani, McClure & Grabiner, 1990)

For a non-zero endomorphisms Φ of $L^1(\omega)$ the following are equivalent:

(a) (ν^t) is strongly continuous in $M(\omega)$ for $t \geq 0$.

(b) Φ is a standard endomorphism
$(L^1(\omega) * f$ dense in $L^1(\omega) \quad \Rightarrow \quad L^1(\omega) * \Phi(f)$ is dense in $L^1(\omega))$.

(c) For all $h \in L^1(\omega)$ there exist $f, g \in L^1(\omega)$ such that $h = \Phi(f) * g$.

(d) $\widetilde{\Phi}$ is strongly continuous.

[plus some more equivalent conditions]

Open problem Are all endomorphisms of $L^1(\omega)$ standard?

Endomorphisms of $A(\omega)$

Corollary (to continuity of characters)

Every endomorphism of $A(\omega)$ is continuous.

From Grabiner's result and the Main Lemma:

Theorem (P, 2009)

Let Φ be a non-zero endomorphism of $A(\omega)$. Then:

- Φ extends uniquely to a continuous endomorphism $\widetilde{\Phi}$ of $B(\omega)$.
- $\nu^t = \widetilde{\Phi}(\delta_t)$ $(t \in \mathbb{R}^+)$ defines a semigroup in $B(\omega)$ which is strongly continuous for $t > 0$.
- For $\mu \in B(\omega)$:
$$\widetilde{\Phi}(\mu) = \int_0^\infty \nu^t \, d\mu(t)$$
as a strong Bochner integral in $B(\omega)$.

Standard endomorphisms of $A(\omega)$

Theorem (P, 2009)

Let Φ be a non-zero endomorphism of $A(\omega)$. Suppose that
$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} : \quad \frac{\omega_m(s)}{\omega_n(s)} \to \infty \quad \text{as } s \to \infty.$$
Then

(a) The semigroup (ν^t) is strongly continuous in $B(\omega)$ for $t \geq 0$.

(b) Φ is standard.

Main ideas in the proof

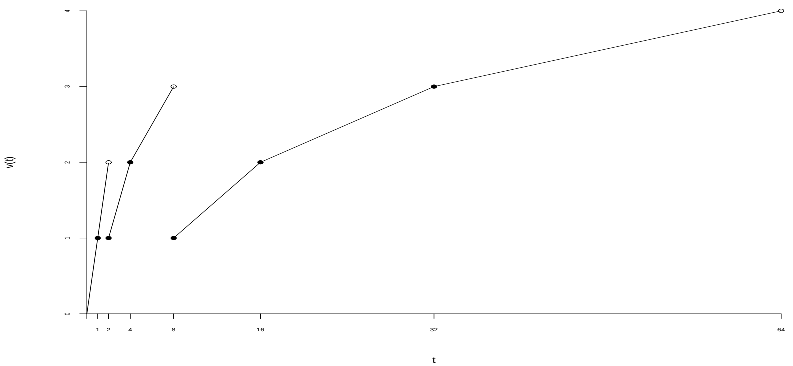
- $\widetilde{\Phi}$ is wk* continuous. ($\simeq$ Grabiner, 2006 for $L^1(\omega)$.)
- $\frac{\omega_n}{\omega_m}$ is a *convergence factor* for ω_m.
Use result of Ghahramani & Grabiner, 2002.

The condition on ω in the previous theorem

Remark The condition $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: \frac{\omega_m(t)}{\omega_n(t)} \to \infty$ as $t \to \infty$ is strictly stronger than the assumption $\sup_{t\in\mathbb{R}^+} \frac{\omega_{n+1}(t)}{\omega_n(t)} = \infty,\ \forall n \in \mathbb{N}$:

Let ω be an unbounded weight which does *not* satisfy $\lim_{t\to\infty} \omega(t) = \infty$. Then let $\omega_n = \omega^n$.

For instance, let $\omega(t) = 2^{v(t)}$ with $v(t)$ as below:



Open problems about endomorphisms of $A(\omega)$

Are the conditions

(c) For all $h \in L^1(\omega)$ there exist $f, g \in L^1(\omega)$ such that $h = \Phi(f) * g$.

(d) $\widetilde{\Phi}$ is strongly continuous.

equivalent to (a) and (b) above?

Remarks

- We do have (c) $\Rightarrow$ (d) $\Rightarrow$ (a)-(b).
- The equivalence would follow from factorisation in $A(\omega)$ as a Fréchet $A(\omega)$-module under the action $f \cdot g = \Phi(f) * g$ $(f, g \in A(\omega))$.

  (In particular it holds if $\lim_{n\to\infty} \omega_n(t) < \infty,\ \forall t \in \mathbb{R}^+$.)

Derivations on $L^1(\omega)$

- No non-zero derivations on semisimple $L^1(\omega)$ (Johnson, 1969).
- Derivations on $L^1(\omega)$ are automatically continuous (Jewell & Sinclair, 1976).

Theorem (Ghahramani, 1980)

A linear operator D on $L^1(\omega)$ is a derivation if and only if there is a measure μ on $\mathbb{R}^+$ with

$$\sup_{t\in\mathbb{R}^+} \frac{t}{\omega(t)} \int \omega(t+s)\, d|\mu|(s) < \infty$$

such that

$$D(f) = (Xf) * \mu \qquad (f \in L^1(\omega))$$

where

$$(Xf)(t) = tf(t) \qquad (t \in \mathbb{R}^+, \quad f \in L^1(\omega))$$

Continuity of derivations on $A(\omega)$

Open problem Are derivations on $A(\omega)$ automatically continuous?

- Standard method does not work since $A(\omega)$ is not *locally bounded*.
- If we attempt a closed graph approach:

$$f_n \to 0 \quad \text{with} \quad D(f_n) \to g$$

then we obtain

$$D(\mu * f_n) \to \mu * g \qquad \forall \mu \in B(\omega).$$

Can this be used to prove $g = 0$ and hence that D is continuous?

Derivations on $A(\omega)$

Observation $(Xf)(t) = tf(t) \quad (t \in \mathbb{R}^+,\ f \in A(\omega))$
defines a derivation on $A(\omega)$ if and only if

$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} : \quad \sup_{t \in \mathbb{R}^+} \frac{t\omega_n(t)}{\omega_m(t)} < \infty. \qquad (*)$$

Theorem (P, 2009)

(a) Suppose that $(*)$ is satisfied. Then

$$D_\mu(f) = (Xf) * \mu \qquad (f \in A(\omega))$$

defines a continuous derivation on $A(\omega)$ for every $\mu \in B(\omega)$ and conversely every continuous derivation on $A(\omega)$ has this form.

(b) If condition $(*)$ is not satisfied, then there are no non-zero, continuous derivations on $A(\omega)$.

Remark In contrast to Ghahramani's result, our condition $(*)$ only involves the weights (ω_n); not the measure $\mu \in B(\omega)$.