Fréchet algebras, formal power series, and analytic structure

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Abstract. The notion of *local Riemann algebras* is introduced. By studying the ideal structure of Fréchet algebras, we provide sufficient conditions for the existence of *local analytic structure* in the spectrum of a Fréchet algebra, and, as an application, we characterize local Riemann algebras.

1 Introduction and statement of the Main Theorem.

An important subject in the theory of Fréchet algebras is the question of the existence of analytic structure in spectra. The detailed study of this problem for uniform Fréchet algebras is discussed in [8], especially the work of Brooks, Carpenter, Goldmann and Kramm. It is not always possible to give an analytic structure to the whole spectrum and therefore it is of interest to know conditions which ensure that parts of the spectrum of a Fréchet algebra can be given an analytic structure. Loy in [12] gave a sufficient condition for the existence of local analytic structure in spectra of certain commutative Fréchet algebras.

This paper contains a continuation of the work begun in [15], and we shall feel free to use the terminology and conventions established there. However, all this work is specifically concerned with the determination of sufficient conditions for the existence of local analytic structure in the spectrum of a commutative Fréchet algebra by studying the structure of the algebra. (See the Main Theorem below.) As far as we know, a careful use of the Arens-Michael representation theory, applied to the more general case of Fréchet algebras, is treated for the first time in this paper. As a consequence, we characterize local Riemann algebras (introduced below) by intrinsic properties within the class of Fréchet algebras. Though the present paper is primarily addressed to functional analysts; we hope that complex analysts may also find *sufficient conditions* interesting from an applications point of view.

Throughout the paper, "algebra" will mean a non-zero, complex commutative algebra with identity. We recall that a *Fréchet algebra* is a complete, metrizable locally convex algebra A whose topology may be defined by an increasing sequence $(p_m)_{m\geq 1}$ of submultiplicative seminorms. The basic theory of Fréchet algebras was introduced in [8] and [14]. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which A is given by an inverse limit of Banach algebras A_m (see below). A Fréchet algebra A is called a *uniform Fréchet algebra* if for each $m \geq 1$ and for each $x \in A$, $p_m(x^2) = p_m(x)^2$.

Let A be a Fréchet algebra, with its topology defined by an increasing sequence $(p_m)_{m\geq 1}$ of submultiplicative seminorms. For each m, let $Q_m : A \to A/\ker p_m$ be the quotient map. Then $A/\ker p_m$ is naturally a normed algebra, normed by setting $\|x + \ker p_m\|_m = p_m(x)$ $(x \in A)$. We let $(A_m; \|\cdot\|_m)$ be the completion of $A/\ker p_m$. Then $d_m(x + \ker p_{m+1}) =$

$x + \ker p_m \ (x \in A)$ extends to a norm decreasing homomorphism $d_m : A_{m+1} \to A_m$ such that

$$A_1 \xleftarrow{d_1} A_2 \xleftarrow{d_2} A_3 \longleftarrow \cdots \longleftarrow A_m \xleftarrow{d_m} A_{m+1} \longleftarrow \cdots$$

is an inverse limit sequence of Banach algebras; and bicontinuously $A = \varprojlim(A_m;\, d_m)$. This is called an *Arens-Michael representation* of A. We note that: (i) there is a one-to-one correspondence between the set $M(A)$ of all non-zero continuous complex homomorphisms on A and the closed maximal ideals in A, and (ii) maximal ideals of Fréchet algebras are not, in general, closed (see [8, Example, p. 83]).

As pointed out in [15], the algebra $\mathcal{F}$ is a Fréchet algebra when endowed with the weak topology defined by the projections $\pi_j : \mathcal{F} \to \mathbb{C}$, $j \in \mathbb{Z}^+$, where $\pi_j(\sum_{n=0}^{\infty} \lambda_n X^n) = \lambda_j$. A defining sequence of seminorms for $\mathcal{F}$ is $(p_m^{'})$, where $p_m^{'}(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^{m} \mid \lambda_n \mid \ (m \in \mathbb{N})$. A *Fréchet algebra of power series* is a subalgebra A of $\mathcal{F}$ such that A is a Fréchet algebra containing the indeterminate X and such that the inclusion map $A \hookrightarrow \mathcal{F}$ is continuous (equivalently, the projections π_j, $j \in \mathbb{Z}^+$, are continuous linear functionals on A). Though Fréchet algebras of power series have been considered earlier by Loy in [10] and [11], recently these algebras-and more generally, the power series ideas in general Fréchet algebras-have acquired significance in understanding the structure of Fréchet algebras ([1], [2], [3], [15]). For examples of Fréchet algebras of power series, we refer to [2] and [3].

We recall that the spectrum $M(A)$ (with the Gel'fand topology) has an *analytic disc* at $\phi \in M(A)$ if there are a disc D in the complex plane and a continuous injection $f : D \to M(A)$ such that $f(0) = \phi$ and $\hat{x} \circ f \in \mathrm{Hol}(D)$ for all $x \in A$. A uniform Fréchet algebra A is called *Riemann algebra* if it is topologically and algebraically isomorphic to the Fréchet algebra $\mathrm{Hol}(X)$ of all holomorphic functions on some Riemann surface X (see [8, 19.5.1]). Note that $\mathrm{Hol}(D)$ is a semisimple Fréchet algebra of power series with a *power series generator* z [2, Example 1.4], which is a *planar* Riemann algebra [9, §4]. We call a Fréchet algebra A *a local Riemann algebra* if a non-empty part of $M(A)$ can be given the structure of a Riemann surface in such a way that the completion in the compact-open topology of the algebra of Gel'fand transforms of elements of A, restricted to this part, is the Fréchet algebra of all holomorphic functions on this Riemann surface. Similarly, we can also define *local Stein algebras* in the several variable case.

Let $x \in A$ and let R_x denote the linear operator $A \to A$ of right- multiplication by x. A non-zero element $x \in A$ is called a *strong topological divisor of zero* in A if R_x is not an isomorphism into, i.e. a linear homeomorphism of A onto Ax [14, Definition 11.1]. A non-zero element $x \in A$ is a *topological divisor of zero* in A if, for each sequence (p_m) of seminorms defining the Fréchet topology of A, there exists $m \in \mathbb{N}$ such that x_m is a topological divisor of 0 in A_m [14, Definition 11.2]. We remark that the two notions of topological divisor of zero agree for normed algebras.

We now state the main result on analytic structure to be proved in the paper. We remark that every principal maximal ideal is a closed maximal ideal in A, by [14, Theorem 5.4] (in the unital case).

MAIN THEOREM. *Let A be a Fréchet algebra, with its topology defined by a sequence (p_m) of norms. Suppose that A has a principal maximal ideal At, that ϕ (corresponding to At) is not isolated in $M(A)$; and that t has the property that there exists $n \in \mathbb{N}$ such that t_m is not a topological divisor of zero in A_m for all $m \geq n$. Then:*

(i) *$A/\bigcap_{n\geq 1} At^n$ is a semisimple Fréchet algebra of power series;*

(ii) *there is an analytic disc at ϕ;*

(iii) *for each $x \in A$, $\hat{x}$ vanishes on a neighbourhood of ϕ if and only if $x \in \bigcap_{n\geq 1} At^n$.* □

The paper ends with some remarks on the hypotheses of the theorem, in particular, with counterexamples, showing that the assumptions, considered on ϕ, t and (p_m), cannot be dropped. We provide a characterization of local Riemann algebras as a corollary to the main theorem, and also provide examples of local Riemann algebras. In particular, the algebra $A^\infty(\Gamma)$, Γ the unit circle, is an interesting example of a local Riemann algebra, which allows us to remark that the presence of the condition on t (which arises very naturally) also exhibits a significant difference between the Banach algebra case and the Fréchet algebra case. (See [13, Theorem 4], [5, Theorem 1 and p. 304], [6, Lemma 2.1], 4.2 (a), and 4.4 for more details.)

The author hopes that the present work will encourage some people to invest some time and energy in order to make progress on the theory of local Stein (Riemann) algebras, that is, on the refinement of the theory of Stein (Riemann) algebras. As evidenced by [8, Part 3] in the case of uniform Fréchet algebras, this project is also interesting, since the author believes that the study of local Stein algebras should play an important role in enhancing research about the theories of several complex variables and PDE.

2 Fréchet algebras of power series.

The proof of the main theorem, presented in the next section, is broken up into several technical results of some independent interest.

Let M be a closed ideal of a Fréchet algebra A. Then $\overline{M^n}$ for each $n \geq 1$ and $\bigcap_{n\geq 1} \overline{M^n}$ are also closed ideals of A. Here, for each $n \geq 1$, M^n is the ideal generated by products of n elements in M. We now state our two vital technical lemmas (see [15, p. 127]), recalling the Arens-Michael representations of M, $\overline{M^n}$ for each $n \geq 1$ and $\bigcap_{n\geq 1} \overline{M^n}$, and their quotient Fréchet algebras $A/\overline{M^n}$ for each $n \geq 1$ and $A/\bigcap_{n\geq 1} \overline{M^n}$.

LEMMA 2.1. *Let M be a closed ideal of A. Then the Arens-Michael isomorphism $A \cong \varprojlim(A_m; d_m)$ induces isomorphisms:*

(i) $M \cong \varprojlim(M_m; \overline{d_m})$;

(ii) $\overline{M^n} \cong \varprojlim(\overline{M_m^n}; \overline{d_m})$ $(n \geq 1)$;

(iii) $\bigcap_{n\geq 1} \overline{M^n} \cong \varprojlim(\bigcap_{n\geq 1} \overline{M_m^n}; \overline{d_m})$.

(Here $\overline{d_m} = d_m |_{I_{m+1}} : I_{m+1} \to I_m$, where $I_m = \overline{Q_m(I)}$ (closure in A_m), whenever I is a closed ideal in A.) □

LEMMA 2.2. *With the above notation, the Arens-Michael isomorphism $A \cong \varprojlim(A_m; d_m)$ induces isomorphisms:*

(i) $A/\overline{M^n} \cong \varprojlim(A_m/\overline{M_m^n};\, \tilde{d_m})$ $(n \geq 1)$;

(ii) $A/\bigcap_{n\geq 1}\overline{M^n} \cong \varprojlim(A_m/\bigcap_{n\geq 1}\overline{M_m^n};\, \tilde{d_m})$.

(Here $\tilde{d_m} : A_{m+1}/\overline{M_{m+1}^n} \to A_m/\overline{M_m^n}$ is the homomorphism induced by d_m.) □

In a special case, we have the following proposition, which is in a stronger form of [15, Proposition 2.3]. Since the method of proof is used in the proof of the main theorem, we present the proof here for the reader's convenience.

PROPOSITION 2.3. *Let $(A, (p_m))$ be a commutative, unital Fréchet algebra with the Arens-Michael isomorphism $A \cong \varprojlim(A_m; d_m)$, and let M be a non-nilpotent, closed maximal ideal of A such that: (i) $\bigcap_{n\geq 1}\overline{M^n} = \{0\}$ and (ii) dim$(M/\overline{M^2}) = 1$. Then there exists $t \in M$ such that $\overline{M^n} = \overline{M^{n+1}} \oplus \mathbb{C}\, t^n$ for each $n \geq 1$. Assume further that each p_m is a norm. Then, for each sufficiently large m, M_m is a non-nilpotent maximal ideal of A_m such that: (a) $\bigcap_{n\geq 1}\overline{M_m^n} = \{0\}$ and (b) dim$(\overline{M_m^n}/\overline{M_m^{n+1}}) = 1$ for each n.*

Proof. We clearly have $\overline{M^{n+1}} \neq \overline{M^n} \neq \{0\}$ for each n.

Since $\dim(M/\overline{M^2}) = 1$, there exists $t \in M$ such that $M = \overline{M^2} \oplus \mathbb{C}\, t$, so that $\overline{M^n} = \overline{M^{n+1}} + \mathbb{C}\, t^n$ for each $n \in \mathbb{N}$. If it were the case that $t^n \in \overline{M^{n+1}}$ for some $n \in \mathbb{N}$, then $\overline{M^l} = \overline{M^n}$ for all $l \geq n$, and so $\overline{M^n} = \bigcap_{l\geq 1}\overline{M^l} = \{0\}$, a contradiction of the fact that M is non-nilpotent. Thus

$\overline{M^n} = \overline{M^{n+1}} \oplus \mathbb{C}\, t^n$ for each $n \in \mathbb{N}$.

Assume further that each p_m is a norm. First, suppose that M_m (closure in A_m) is nilpotent for some m. Then there exists $n \in \mathbb{N}$ such that $M_m^n = \{0\}$. But then $M \subset M_m$ implies that M is nilpotent, a contradiction of the fact that M is non-nilpotent. Thus it is clear that M_m is not nilpotent for each m, and so, by Lemma 2.1 (ii), we have $\overline{Q_m(M)^n} = \overline{M_m^n} \neq \{0\}$ for all n, m. Also, since $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$, we have $\bigcap_{n\geq 1} \overline{M_m^n} = \{0\}$ for each m, by Lemma 2.1 (iii) and [7, Corollary A.1.25]. Since M is a closed maximal ideal of A, we have $A = M + \mathbb{C}$. Thus $Q_m(M) + \mathbb{C} = M + \mathbb{C}$ is dense in A_m, and so also is $M_m + \mathbb{C}$. Since M_m is closed in A_m, we have $A_m = M_m + \mathbb{C}$. As it is not true that $M_m = A_m$ for infinitely many $m \in \mathbb{N}$, this proves that M_m is a maximal ideal of A_m for each sufficiently large m. Repeating this argument for $M = \overline{M^2} \oplus \mathbb{C}\, t$ and using the fact that $\overline{Q_m(\overline{M^2})} = \overline{M_m^2}$, we obtain $\dim(M_m/\overline{M_m^2}) = 1$ for each sufficiently large m.

Following the argument given in the previous paragraphs, for sufficiently large m, we also obtain $\dim(\overline{M_m^n}/\overline{M_m^{n+1}}) = 1$ for all n. $\square$

Let A be a Fréchet algebra of power series. Then A is an integral domain. Set $M = \ker \pi_0$. Then M is a closed maximal ideal of A and M is not nilpotent. Note that π_0 is a continuous projection on A, which is also a complex homomorphism on A. Further, $\overline{M^n} \subset \ker \pi_{n-1}$ for each $n \geq 1$, so that $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$. The following result generalizes this argument. We recall that a Fréchet algebra of power series A satisfies *condition (E)* if there is a sequence (γ_n) of positive reals such that $(\gamma_n^{-1} \pi_n)$ is an equicontinuous family [13], and, by [15, Theorem 3.6], Fréchet algebras of power series (except $\mathcal{F}$ itself) satisfy this condition.

THEOREM 2.4. *Let A be a Fréchet algebra, $\theta : A \to B$ a homomorphism of A onto a Fréchet algebra of power series $B\,(\neq \mathcal{F})$. Then A contains a non-nilpotent closed maximal ideal M such that $\bigcap_{n\geq 1} \overline{M^n} = \ker\, \theta$.* □

REMARK. We note that the range of θ is not one-dimensional, so, by [15, Theorem 4.1], θ is continuous, and hence one can follow the proof given in [13, Theorem 1] for the Banach algebra case. If we replace B by $\mathcal{F}$ in the above theorem, then the theorem is no longer true since there are Banach algebras which have discontinuous epimorphisms onto $\mathcal{F}$ [7, Theorem 5.5.19]. In fact, the above theorem significantly generalizes Theorem 1 of [13], that is, a Banach algebra of power series in the codomain could be replaced by a Fréchet algebra of power series ($\neq \mathcal{F}$); it also provides a necessary condition for the existence of a non-surjective homomorphism from a Fréchet algebra into $\mathcal{F}$ (Thomas provided necessary conditions for the existence of an epimorphism from a Fréchet algebra onto $\mathcal{F}$ (see [16, §2] for more details)).

If we delete (i) from Theorem 3.1 of [15] and strengthen (ii) of that theorem to $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for all n, then we have the following theorem; we will merely sketch a proof.

THEOREM 2.5. *Suppose that A has a closed maximal ideal M such that $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for each n. Then $A/\bigcap_{n\geq 1}\overline{M^n}$ is a Fréchet algebra of power series.*

Proof. Following the argument given in the proof of [15, Theorem 3.1], we have a homomorphism $\Psi : x \mapsto \sum_{i=0}^{\infty} \pi_i(x)\, t^i$ from A onto an algebra of formal power series with kernel $\bigcap_{n\geq 1}\ker \pi_n = \bigcap_{n\geq 1}\overline{M^n}$. The inclusion $\bigcap_{n\geq 1}\ker \pi_n \subseteq \bigcap_{n\geq 1}\overline{M^n}$ is clear. For the reverse, suppose that $x \in \bigcap_{n\geq 1}\overline{M^n}$ and that x does not belong to $\bigcap_{n\geq 1}\ker \pi_n$. Let k be the least index such that $\pi_k(x) \neq 0$. Then $x = \pi_k(x)\, t^k + y_k$, where $y_k \in \overline{M^{k+1}}$. So $t^k \in \overline{M^{k+1}}$, a contradiction.

For $x \in A$, let $\bar{x}$ denote the coset $x + \bigcap_{n\geq 1}\overline{M^n}$. Then the mapping $\bar{x} \mapsto \sum_{i=0}^{\infty} \pi_i(\bar{x})\, t^i$ is an isomorphism from $A/\bigcap_{n\geq 1}\overline{M^n}$ onto an algebra of formal power series. One can now follow the proof of [15, Theorem 3.1], in order to establish the theorem. □

As a corollary, we have the following result, with $\mathcal{F}$ as a trivial example.

COROLLARY 2.6. *Suppose A has one generator t and that the ideals $\overline{At^n}$ are all distinct. Then $A/\bigcap_{n\geq 1}\overline{At^n}$ is a Fréchet algebra of power series.* $\square$

3 Proof of the Main Theorem.

First, we show that $At^{n+1} \neq At^n$ for each n. Suppose $At^{n+1} = At^n$ for some $n \geq 1$. Then there is $x \in A$ with $t^n(e - xt) = 0$. By the maximality of At, $\hat{t}$ vanishes only at ϕ, whence $(e - xt)\hat{}$ must vanish off $\{\phi\}$. Since $(e - xt)\hat{}(\phi) = 1$, this means that ϕ is isolated, a contradiction. It follows that $At^n = At^{n+1} \oplus \mathbb{C}t^n \supset At^{n+1}$ properly for each $n \in \mathbb{N}$. A simple induction now shows that At^n has co-dimension n in A, and therefore, as the range of the continuous map $x \mapsto xt^n$, is closed. So $\bigcap_{n\geq 1} At^n$ is a closed ideal of A. We have the following conclusions:

(a) $At \cong \varprojlim (At)_m$; $At^n \cong \varprojlim (At^n)_m$; $\bigcap_{n\geq 1} At^n \cong \varprojlim \bigcap_{n\geq 1}(At^n)_m$, by Lemma 2.1.

(b) $A/At^n \cong \varprojlim A_m/(At^n)_m$; $A/\bigcap_{n\geq 1} At^n \cong \varprojlim A_m/\bigcap_{n\geq 1}(At^n)_m$; $At^n/At^{n+1} \cong \varprojlim (At^n)_m/(At^{n+1})_m$, by Lemma 2.2. We have the last Arens-Michael isomorphism as the ideals At^n are all distinct.

(c) By Theorem 2.5, $B = A/\bigcap_{n\geq 1} At^n$ is a Fréchet algebra of power series.

(d) We first recall that each p_m is a norm on A. Then, by Proposition 2.3, $(At)_m$ is a maximal ideal in A_m for sufficiently large m such that $\dim((At^n)_m/(At^{n+1})_m) = 1$ for each n since $\dim(At^n/At^{n+1}) = 1$ for each n. So, by Theorem 2.5, $B_m = A_m/\bigcap_{n\geq 1}(At^n)_m$ is a Banach algebra of power series for sufficiently large m. Hence, by passing to a suitable subsequence of (q_m) defining the same Fréchet topology of B, we conclude, without loss of generality, that each B_m is a Banach algebra of power series. Thus, by [2, p. 144], $B \neq \mathcal{F}$. Hence, by [15, Theorem 3.3], the topology of B is, indeed, defined by a sequence (q_m) of norms. Hence B admits a continuous norm, but not necessarily submultiplicative (A also admits a continuous norm). Not only this, but, by [15, Corollary 4.2], B has a unique Fréchet topology so that each q_m can be taken as a quotient norm induced by the norm p_m.

For $x \in A$, let $\bar{x}$ denote the coset $x + \bigcap_{n\geq 1} At^n$ (which is in fact power series $\sum_{i=0}^{\infty} \pi_i(\bar{x})\, t^i$). Then $\bar{t}$ is certainly not a zero divisor. $B\bar{t}$ is the image of At and therefore is a maximal ideal in B. In fact, it is also closed since it cannot be dense as it is contained in $\ker \pi_0$. Since, $\bar{t}$ is not a zero divisor, the mapping $R_{\bar{t}} : \bar{x} \mapsto \bar{x}\,\bar{t}$ is injective. It is, indeed, a homeomorphism, by the open mapping theorem. Now, for each m, the mapping $(R_{\bar{t}})_m : \bar{x} \mapsto$

$\bar{x}\bar{t}$ of $(B,\, q_m)$ into $(B\bar{t},\, q_m)$ is a continuous linear transformation, being a right multiplication operator on the normed algebra $(B,\, q_m)$. Lifting to the completions, for each m, $R_{\bar{t}_m} : \bar{x}_m \mapsto \bar{x}_m \bar{t}_m$ of B_m into $(B\bar{t})_m$ is continuous.

Assuming that there exists $n \in \mathbb{N}$ such that t_m is not a topological divisor of zero in A_m for all $m \geq n$, we have $A_m t_m = (At)_m$ for all $m \geq n$, and so $B_m \bar{t}_m = (B\bar{t})_m$ for all $m \geq n$, hence $R_{\bar{t}_m}$ is injective. As noted in (d) above, $B_m \bar{t}_m$ is a maximal ideal in B_m for sufficiently large m, being the image of $A_m t_m$ a maximal ideal in A_m. Thus, we have, without loss of generality, that each $R_{\bar{t}_m}$ has continuous inverse, by the open mapping theorem. In particular, by the Banach algebra case [13, Theorem 4], we have

$$\| \pi_k^{(1)} \| \leq (2 \| R_{\bar{t}_1}^{-1} \|)^k \quad \text{for } k \in \mathbb{Z}^+.$$

Define $\delta_1 = \| 2R_{\bar{t}_1}^{-1} \|^{-1} \leq \liminf_k \| \pi_k^{(1)} \|^{-1/k}$. Now if Ω_1 is the closed disc centered at zero and radius $\delta_1/2$, then the mapping $\vartheta_1 : \bar{x}_1 \mapsto \sum_{i=0}^{\infty} \pi_i^{(1)}(\bar{x}_1)\, z^i$ of B_1 into $\mathrm{Hol}(\Omega_1)$, a standard disc algebra, is injective, continuous by the closed graph theorem. Since $\mathrm{Hol}(\Omega_1)$ is semisimple, the same holds for B_1. Clearly, the mapping $Q_1 : \bar{x} \mapsto \bar{x}_1$ of B into B_1 is also a continuous, injective homomorphism. This shows that B is a semisimple Fréchet algebra of power series. This proves (i).

For (ii), set $\delta = \delta_1/2$ (note that we can do this as q_1 is a continuous, submultiplicative norm on B), and define functionals $\{\phi_\lambda :\mid \lambda \mid < \delta\}$ on A_1 by $\phi_\lambda : x_1 \mapsto \sum_{i=0}^{\infty} \pi_i(\bar{x}_1)\,\lambda^i$. Then $\Gamma_1 : \lambda \mapsto \phi_\lambda$, mapping $\Delta = \{\lambda : |\lambda| < \delta\}$ into $M(A_1)$, is an analytic disc at ϕ_1 (a point in $M(A_1)$ corresponding to $A_1 t_1$). Then we define functionals $\{\phi_\lambda|A :\mid \lambda \mid < \delta\}$ on A by $\phi_\lambda|A : x \mapsto \sum_{i=0}^{\infty} \pi_i(\bar{x}_1)\,\lambda^i$ (and so $\phi_\lambda|A = \phi_\lambda \circ Q_1$), and then define $\Gamma\,(= Q_1^* \circ \Gamma_1) : \lambda \mapsto \phi_\lambda|A$, mapping $\Delta = \{\lambda : |\lambda| < \delta\}$ into $M(A)$, which is clearly a continuous injection, such that $\Gamma(0) = \phi$ (note that Q_1^* is injective, being the adjoint spectral map (see [8, Lemma 3.2.5], for more details)). Then Γ is an analytic disc at ϕ.

To prove (iii), define linear operators on A by $T_\lambda : x \mapsto (t - \lambda\, e)\, x$. Since At is a principal closed maximal ideal of A, clearly $T_0 = L_t : x \mapsto t\,x$ is a semi-Fredholm operator on A. Since T_0 has deficiency 1, there is $\eta > 0$ such that $T_\lambda = T_0 - \lambda\, e$ has deficiency ≤ 1 for $\mid \lambda \mid < \eta$. Let $\epsilon = \min(\delta, \eta)$. Then if $\mid \lambda \mid < \epsilon$, $T_\lambda(A) \subset \ker \phi_\lambda$ and $\operatorname{codim} T_\lambda(A) \geq \operatorname{codim} \ker \phi_\lambda$. So $\ker \phi_\lambda = T_\lambda(A)$.

Let $\Delta_1 := \{z \in \mathbb{C} :\mid z \mid < \epsilon\}$, $U := \{\psi \in M(A) :\mid \psi(t) \mid < \epsilon\}$. Then from what we have just shown, $\Gamma : \Delta_1 \to U$ is a continuous injection. So if $x \in \bigcap_{n \geq 1} At^n$, then $\hat{x}\mid_U = 0$. Conversely, suppose $\hat{x}\mid_V = 0$ for some neighbourhood V of ϕ. Since ϕ is not isolated point, $U \cap V$ contains infinitely many points. But $\hat{x}\mid_{U\cap V} = 0$, so that $\phi_\lambda(x) = 0$ for infinitely many points in Δ. Since $\phi_\lambda(x) = \hat{x} \circ \Gamma(\lambda)$ and $\hat{x} \circ \Gamma \in \mathrm{Hol}(\Delta)$, $\phi_\lambda(x) \equiv 0$ on Δ. Thus $x \in \bigcap_{n\geq 1} At^n$. □

4 Some remarks on the Main Theorem and open questions.

4.1. If ϕ were isolated, then (ii) is clearly impossible, except in a trivial sense, and in the semisimple case, by Shilov's idempotent theorem for Fréchet algebras, there exists $f \in A$ such that $\hat{f}(\phi) = 0$; $\hat{f}(\psi) = 1$, $\psi \in M(A) \setminus \{\phi\}$. But then $At = Af$ and so $At^n = At$ for all $n \geq 1$. Thus $\bigcap_{n\geq 1} At^n = At$ so that (i) is impossible and (iii) is trivially true. In fact, $A/At \cong \mathbb{C}$ via Gel'fand-Mazur theorem for Fréchet algebras.

4.2. We now show that the assumption that there exists $n \in \mathbb{N}$ such that t_m is not a topological divisor of 0 in A_m for all $m \geq n$, is essential for the existence of analytic disc at ϕ. We give two examples here - in the semisimple case and in the local case.

(a) Take $A = F(W)$ as in §4 of [3]. $A^\infty(\Gamma)$, Γ the unit circle, is a particular example of $F(W)$ (see [2, Example 1.5]). Then A is a Fréchet algebra of power series having a power series generator X ($e^{i\theta}$ in the case of $A^\infty(\Gamma)$) in the sense of [2], with $M(A) = D$, the closed unit disc. By [3, Proposition 4.5], every closed maximal ideal is principal. Let $z \in \Gamma$. It is clear that R_{X-ze} has a continuous inverse and so $X - ze$ is not a strong topological divisor of 0 in A; but, by [14, Proposition 11.8], $X - ze$ is a topological divisor of 0 in A. In fact, since Γ is the boundary of the maximal ideal space of A_m for each m, $X - ze$ (as an element of A_m) is a topological divisor of 0 in A_m, so that R_{X-ze} does not admit a continuous inverse on A_m. Hence no analytic discs can exist at the points of Γ, except in a trivial sense.

More interestingly, the example $A^\infty(\Gamma)$ exhibits a significant difference between the Banach case and the Fréchet case. This is in view of the fact that Crownover proved for the sup norm algebra case that At being a maximal ideal (corresponding to ϕ) in the Shilov boundary implies that ϕ is isolated in the Gel'fand topology ([5, Theorem 1]); he posed an open question ([5, p. 304]) for an arbitrary commutative semisimple Banach algebra with identity, with a remark that a negative answer would allow for power series representations on the Shilov boundary, and solved the question in the affirmative ([6, Lemma 2.1]). We remark from the example $A^\infty(\Gamma)$ that Theorem 1 of

[5] does not hold true for an arbitrary commutative semisimple non-Banach Fréchet algebra, and allows for power series representations on the (Shilov) boundary Γ. Finally, this example shows that the assumption on t is sufficient (but possibly not necessary) condition for B being semisimple, and is also sufficient for the existence of analytic disc at ϕ.

(b) Take $A = \ell^1(\mathbb{Z}^+, W)$, $W = (\omega_n)$, an increasing sequence of local weights on $\mathbb{Z}^+$, as in Example 1.2 of [2]. Then A is a local Fréchet algebra of power series having a power series generator X in the sense of [2], with its unique closed maximal ideal $M = \ker \pi_0$. For each m, A_m is also a local Banach algebra of power series having a power series generator X, with its unique closed maximal ideal $M_m = \ker \pi_0^m$, and X (as an element of A_m) is clearly a topological divisor of 0 in A_m, being quasinilpotent element. Hence X is a topological divisor of 0 in A. But, since $AX = M$, by [17, Lemma 4] and the fact that a closed ideal of a power series generated Fréchet algebra is also power series generated, X is not a strong topological divisor of 0 in A. Thus it is of interest to construct a counterexample of A whose quotient B is a local Fréchet algebra of power series in the absence of the condition on t. We expect that such example should exist, but have been unable to construct it.

4.3. We show that the assumption that (p_m) is a sequence of norms, is essential. For example, take $A = C^\infty(\mathbb{R})$ with the usual topology defined by the sequence (p_m) of proper seminorms (note that by a *proper* seminorm we mean a seminorm that is not a norm), where $p_m(f) = \sum_{n=0}^{m} \frac{1}{n!}\max\{| f^{(n)}(x) | : x \in [-m, m]\}$. Then A is a semisimple Fréchet algebra, singly generated by the identity function $f(x) = x$ $(x \in \mathbb{R})$, and every closed maximal ideal is principal [4]. The closed maximal ideal at 0 is Af, and $\bigcap_{n\geq 1} Af^n = \{g \in A : g^{(n)}(0) = 0 \,\text{for all}\, n \in \mathbb{Z}^+\}$, which is a closed, prime ideal of infinite codimension in A. $A/\bigcap_{n\geq 1} Af^n$ is the algebra of Taylor series about 0 of functions in A in which the series can have arbitrarily small discs of convergence. In fact, following the argument given in the proof of Theorem 4.4.9 (ii) of [7], we can show that $A/\bigcap_{n\geq 1} Af^n$ is topologically isomorphic to $\mathcal{F}$, so that both (i) and (ii) are clearly impossible. Since $M(A) = \mathbb{R}$, it contains no analytic discs.

4.4. We remark that this result extends Theorem 2.6 of [4] to nonuniform Fréchet algebras, characterizing local Riemann algebras in the following corollary. Carpenter considered only commutative, unital uniform Fréchet algebras with locally compact Gel'fand space; there is a positive real δ, where $\delta = \min\{| f(t) | : t \in \partial B_m \;, m \,\text{a fixed positive integer}\}$, under the given hypothesis, by [4, Lemmas 2.2 and 2.3]. Note that ∂B_m is the Shilov boundary of B_m the completion of the algebra $A|X_m$ with respect to the supremum norm on X_m. In our case, $\delta_1 :=\| 2R_{\bar{t}_1}^{-1} \|^{-1}$. In fact, by [5, Remark, p. 302], both are same. We note that if A is a uniform Fréchet algebra with locally compact spectrum, then t is not a topological divisor of 0 (w.l.o.g.), since it is not a strong topological divisor of 0, by [4, Lemmas 2.2-2.4], hence we can drop the hypothesis on t in this case.

We also remark that, in order to obtain a stronger form of the main theorem, one can replace the hypothesis on t by the same (but weaker) hypothesis on $\bar{t}$: $\bar{t}$ has the property that there exists $n \in \mathbb{N}$ such that $\bar{t}_m$ is not a topological divisor of zero in B_m for all $m \geq n$, since one actually uses this fact (together with Proposition 2.3) in the proof. But we prefer the hypotheses in terms of the objects A, ϕ, p_m, t rather than in terms of the objects B, q_m, $\bar{t}$ occurring in the conclusion. Further, in the Banach algebra case, we do not require the hypothesis on $\bar{t}$, since it is automatic in that case (see [13, Theorem 4]). As a special case of the main theorem, we have the following result, whose proof we omit. (Similar results appeared in [4], [6]; proved for commutative semisimple Banach algebras, and in [9].)

COROLLARY 4.1. *Let A be a Fréchet algebra, with its topology defined by a sequence (p_m) of norms. Suppose that $M(A)$ contains a subspace Y*

such that: (i) Y has no isolated points; (ii) every closed maximal ideal (corresponding to a point in Y) is principal, such that the generator t has the property that t_m is not a topological divisor of zero in A_m for all sufficiently large m. Then Y can be given the structure of a Riemann surface in such a way that, for each $x \in A$, the restriction of $\hat{x}$ to Y is analytic. In particular, if Y is locally compact and connected, such that the conditions (i) and (ii) hold, then A is a local Riemann algebra (that is, the completion of $\hat{A}|Y$ with respect to the compact open topology is topologically and algebraically isomorphic to Hol(Y)). Conversely, if A is a local Riemann algebra, then every closed maximal ideal (corresponding to a point in Y) is principal. □

Clearly, if $A\,(\neq \mathbb{C})$ is a local Riemann algebra, then Y above is locally compact and connected with no isolated points, but the condition (ii) above need not hold as is seen in the case of $A^{\infty}(\Gamma)$. We note that $F(W)$, in particular, the nuclear Fréchet algebras $F(\widetilde{W})$ (see [3, §6]), Banach algebras satisfying Theorem 4 of [13], Riemann algebras, and the algebra $\mathcal{A}_2$ of all holomorphic functions on the analytic set (Neil's parabola) $\{(x, y) \in \mathbb{C}^2 : x^3 - y^2 = 0\}$ (see [9, §3]) are local Riemann algebras whereas $C^{\infty}(\mathbb{R})$, $\ell^1(\mathbb{Z}^+, W)$ (see 4.2 (b) above) are not local Riemann algebras. Also, it is easy to see that $A^{\infty}(\Gamma)$ is a nuclear, planar local Riemann algebra, with the open unit disc as its Riemann surface, and which allows for power series representations on the whole Gel'fand space D (which is compact). Thus $A^\infty(\Gamma)$ exhibits a significant difference between Riemann algebras and local Riemann algebras, since if R is a compact Riemann surface, then one obtains the trivial Riemann algebra $\mathbb{C}$ [9]. Note that the disc algebra $A(D)$ is a non-nuclear, planar local Riemann algebra, with the open unit disc as its Riemann surface, and which does not allow for power series representations on its boundary Γ. In the literature, there are other examples of complex function algebras with no analytic structure in their spectra, but in the case $n = 2$ (see [8, Remark, p. 235] for more references).

4.5. It is clear from the proofs that the results obtained in the main theorem and Corollary 4.1 are independent of the Arens-Michael representation which is chosen, in the sense that if (p''_m) is any other sequence of norms defining the Fréchet topology of A, then the proofs are valid with that sequence. Of course, the B''_m, obtained using the sequence (p''_m), may be different Banach algebras of power series in an Arens-Michael representation of B. For example, two different Arens-Michael representations of $\mathrm{Hol}(U)$, U the open unit disc, $\mathrm{Hol}(\mathbb{C})$ and $A^\infty(\Gamma)$, are discussed in [2, Examples 1.4 and 1.5], containing different Banach algebras of power series. The other obvious question is: does such result remain valid for finitely generated ideals in a commutative unital Fréchet algebra A? The affirmative answer would give us a characterization of local Stein algebras.

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