Pervasive Function Spaces

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Outline

1. Definitions
2. History
    - Hoffman-Singer
    - Dirichlet algebras
    - Continuous analytic capacity
    - GG-points
    - Harmonic functions and Netuka
3. Cases
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Let X be a compact Hausdorff topological space and $C(X, \mathbb{C})$ (respectively, $C(X, \mathbb{R})$) the Banach algebra of all continuous complex-valued (respectively, real-valued) functions on X endowed with the uniform norm.

A **function space** S on X is a closed subspace of $C(X, \mathbb{C})$.

We denote by $\mathrm{clos}_{C(E,\mathbb{C})}\, S$ the closure in $C(E, \mathbb{C})$ of the function space S, where E is a closed subset of X.

Similarly, we denote by $\mathrm{clos}_{C(E,\mathbb{R})}\, S$ the closure in $C(E, \mathbb{R})$ of the *real subspace* S of $C(X, \mathbb{R})$.

A function space S on X is said to be **complex pervasive** if $\text{clos}_{C(E,\mathbb{C})} S = C(E, \mathbb{C})$ whenever E is a proper non-empty closed subset of X.
Similarly, a real subspace S of $C(X, \mathbb{R})$ is said to be **real pervasive** if $\text{clos}_{C(E,\mathbb{R})} S = C(E, \mathbb{R})$.

Let $U^{\text{open}} \subset \hat{\mathbb{C}}$,
$X = \text{bdy } U$,
$A(U) =$ the algebra of all (restrictions to X of) complex-valued functions continuous on $\hat{\mathbb{C}}$ and analytic on U.
We consider the cases:
(1) $S = A(U)$,
(2) $S = \text{Re } A(U)$.

Example

The disk algebra $A(\mathbb{D})$ is complex pervasive on $\mathbb{S}^1$.

Obviously, if $A(U)$ is complex pervasive on bdy U then Re $A(U)$ is real pervasive on bdy U.
The converse is false:

Example

Let U be the union of two disks having disjoint closures. Then $A(U)$ is **not** complex pervasive on bdy U, but Re $A(U)$ **is** dense in $C(\text{bdy } U, \mathbb{R})$, and hence, *a fortiori*, real pervasive on bdy U.

A uniform algebra A, $A \subset C(X, \mathbb{R})$ is said to be **Dirichlet** on X if Re A is dense in $C(X, \mathbb{R})$
Thus Re $A(U)$ is real pervasive on bdy U if and only if $\mathrm{clos}_{C(E,\mathbb{R})}\, A(U)$ is Dirichlet on E whenever E is a proper closed subset of bdy U.

I first heard of pervasive spaces from
Cerych, at the meeting on Complex Analysis and Applications in Varna in 1981.

Hoffman and Singer

Acta Math 103 (1960):
— introduced the term pervasive
— studied (complex) pervasive uniform algebras, motivated by the analogy with maximal uniform algebras.
— established that $A(U)$ is complex pervasive on bdy U if U is connected and $N \setminus U$ has positive area whenever N is a neighbourhood of a boundary point of U.

Gamelin and Garnett

Journal of Functional Analysis 8 (1971)
— characterized those U for which $A(U)$ is Dirichlet on bdy U.
— It is necessary that each component of U be simply-connected.
— Given that, the condition that $A(U)$ be Dirichlet is rather abstractly characterized by the pointwise bounded density of $A(U)$ in $H^\infty(U)$, and more concretely by a condition involving continuous analytic capacity, α.

Let us say that the point $a \in \mathbb{C}$ is a **GG-point** for U if

$$\liminf_{r \downarrow 0} \frac{\alpha(\mathbb{U}(a,r) \setminus U)}{r} = 0 \ ,$$

where $\mathbb{U}(a,r)$ denote the open disc with center a and radius r.

The Gamelin-Garnett Theorem

Let $U \in \hat{\mathbb{C}}$ be open, and suppose each component of U is simply-connected. Then $A(U)$ is Dirichlet on bdy *U if and only if there are no GG-points for U on* bdy *U.*

Remark

Each GG-point on bdy *U for U is an* **inner** *boundary point of U, i.e. it is not on the boundary of the complement of clos$U$.*

Re $A(U)$ is real pervasive $\Leftrightarrow$
each $A_E = \text{clos}_{C(E,\mathbb{R})} A(U)$ is Dirichlet.
So it is tempting to suppose that the Gamelin-Garnett Theorem settles the matter. This is not so.
However, it is probable that the result of Gamelin and Garnett can be extended to all the so-called T-invariant algebras, with suitable modification, and the algebras A_E are T-invariant, so that one expects that real pervasiveness may be expressed in term of capacities associated to the A_E's. In fact, however, we shall see that a more direct approach may be used, employing the Gamelin-Garnett Theorem as it stands, and yielding a relatively simple and readily checked condition for real pervasiveness.

Netuka

J. Approx. Theory **51** (1987):
— studied the real pervasiveness of spaces of harmonic functions on Euclidean spaces
— showed that if the open set $U \in \mathbb{R}^d$ is bounded and connected, and bdy $U =$ bdy clos U, then the space of functions continuous on clos U and harmonic on U is real pervasive on bdy U.
The present investigation was prompted by the question, whether, when $d = 2$, the space of harmonic functions could be replaced by the space Re $A(U)$ in this result. Realizing that the answer was yes, we proceeded to investigate the necessity of the conditions on U, and eventually were led to a complete characterization of the real pervasiveness of Re $A(U)$ and of the complex pervasiveness of $A(U)$.

Cases:

1. U has **inessential boundary points**, i.e. points that are removable singularities for all elements of $A(U)$ This case reduces rather easily to classical facts.
2. Connected U with **essential** boundary. This is perhaps the most natural situation, and we show that in it $A(U)$ is always complex pervasive on bdy U.
3. General U. We give a complete characterization of complex pervasiveness in topological terms.
4. This is not possible for real pervasiveness. We give a complete characterization involving continuous analytic capacity (deeper).

Given a compact Hausdorff topological space X,

$$C(X,\mathbb{C})^* = \mathrm{M}(X,\mathbb{C}).$$

Similarly $C(X,\mathbb{R})^* = \mathrm{M}(X,\mathbb{R})$.
As remarked by Čeryc, a subspace $S \subset C(X,\mathbb{C})$ is complex pervasive (respectively a subspace $S \subset C(X,\mathbb{R})$ is real pervasive) if and only if each nontrivial measure $\mu \in \mathrm{M}(X,\mathbb{C})$ (respectively $\mathrm{M}(X,\mathbb{R})$) which annihilates S has spt $\mu = X$.

Inessential boundary points

Definition

Let $a \in \text{bdy } U$.
We say that a is an $(A(U)\text{-})$**inessential boundary point** if there exists $r > 0$ such that all functions in $A(U)$ are analytic on $\mathbb{U}(a, r)$.
The $A(U)$-**essential boundary** of U is the set of points in bdy U which are not $A(U)$-inessential boundary points.

Let us define the **regularization** of U to be the set

$$\widetilde{U} = U \ \cup \ \{p \in \text{bdy } U : p \text{ is inessential } \} \ .$$

Proposition

Let $U \subset \hat{\mathbb{C}}$ be open and suppose that the essential boundary of U is nonempty. Let n be the number (possibly infinite) of inessential boundary points of U.

i) *If $n \geq 1$ then $A(U)$ is not complex pervasive on* bdy U.

ii) *If $n > 1$ then* Re $A(U)$ *is not real pervasive on* bdy U.

iii) *If $n = 1$ then* Re $A(U)$ *is real pervasive on* bdy U *if and only if*
a) $A(U)$ is Dirichlet on the essential boundary of U, and
b) the component in $\widetilde{U}$ of the inessential boundary point of U has boundary equal to the essential boundary of U.

The connected, essential case.

Theorem (6)

Let U be a connected open subset of $\hat{\mathbb{C}}$, and let bdy U *be nonempty and essential. Then $A(U)$ is complex pervasive on* bdy U*. A fortiori,* Re $A(U)$ *is real pervasive on* bdy U*.*

Multiple Components: complex pervasiveness.

Theorem (7)

Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on bdy U *if and only if* bdy $U_i =$ bdy U *for each component U_i of U.*

Multiple Components: complex pervasiveness.

Theorem (7)

Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on bdy U *if and only if* bdy $U_i =$ bdy U *for each component U_i of U.*

Remark

The vagaries of plane topology allow up to an infinite number of connected open sets to share a common boundary.

Multiple Components: real pervasiveness

Theorem (9)

Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U is not connected, and $\mathrm{Re}\, A(U)$ is real pervasive on $\mathrm{bdy}\, U$. Then U has at most one component that is not simply-connected. Furthermore, if U has such a component U_k, then $\mathrm{bdy}\, U_k = \mathrm{bdy}\, U$.

Multiple Components: real pervasiveness

Theorem (9)

Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U is not connected, and $\mathrm{Re}\, A(U)$ is real pervasive on $\mathrm{bdy}\ U$. Then U has at most one component that is not simply-connected. Furthermore, if U has such a component U_k, then $\mathrm{bdy}\ U_k = \mathrm{bdy}\ U$.

Remark

If $U \subset \hat{\mathbb{C}}$ is open, not connected, and several components of U have boundary equal to $\mathrm{bdy}\ U$, then all components of U are simply-connected.

Proof of Theorem

Suppose that Re $A(U)$ is real pervasive and let U_i be a component of U so that bdy $U_i \neq$ bdy U.

Clearly, Re $A(U) \subset$ Re $A(U_i)$. Therefore the restriction of Re $A(U)$ to bdy U_i, Re $A(U)_{|\text{bdy } U_i}$, is dense in $C(\text{bdy } U_i, \mathbb{R})$. Hence $A(U_i)$ is a Dirichlet algebra on bdy U_i, so we can conclude that U_i is simply-connected.

Suppose next that U has at least two different components U_k, U_l that are not simply-connected. Then from the foregoing bdy $U_k =$ bdy $U_l =$ bdy U. Hence U_k and U_l are both components of $\hat{\mathbb{C}} \setminus$ bdy U and by Remark **??**, both are simply-connected, a contradiction.

In the other direction we have

Theorem (11)

Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U has at least one component U_k so that $\text{bdy } U_k = \text{bdy } U$. Then $\text{Re } A(U)$ is real pervasive.

The proof of this theorem involves the theory of T-invariant algebras.

Thus we have completely solved the question of when $A(U)$ is real pervasive except in the case when all components of U are simply-connected, and no component U_i of U has bdy $U_i =$ bdy U. To deal with this, we introduce some terminology.

Definition

We say that a point $p \in$ bdy U **influences** $q \in$ bdy U (with respect to U) if for all $s > 0$ and $r > 0$ there exists U_i, a component of U, such that

$$\mathbb{U}(p,r) \cap U_i \neq \varnothing \quad \text{and} \quad \mathbb{U}(q,r) \cap U_i \neq \varnothing \ .$$

Theorem (13)

Let $U \subset \hat{\mathbb{C}}$ be open and proper, with no inessential boundary points. Suppose all components of U are simply-connected and no component has $\text{bdy } U_i = \text{bdy } U$. Then the following statements are equivalent.

(i) $\text{Re } A(U)$ *is real pervasive on* $\text{bdy } U$.

(ii) *For every* $p \in \text{bdy } U$, *and for every GG-point* q *for* U *on* $\text{bdy } U$, p *influences* q.

Example (14)

Let $a_n \in \mathbb{R}$ and $r_n > 0$ such that the intervals $[a_n - r_n, a_n + r_n]$ are pairwise-disjoint and $\cup_{n=1}^{\infty}[a_n - r_n, a_n + r_n]$ is dense in $\mathbb{R}$. Let

$$\begin{array}{rcl} T & = & \mathbb{R} \cup \bigcup_{n=1}^{\infty} \mathbb{B}(a_n, r_n) \\ U & = & \mathbb{C} \setminus T \,. \end{array}$$

Then U is open and has two components, U_1 and U_2. We can arrange that the r_n are so small that $\mathbb{R}$ has GG-points for U. For instance, 0 will be a GG-point for U if $\sum_{|a_n|<r} r_n < r^2, \forall\ r > 0$. In that case, $A(U)$ is not Dirichlet on bdy U, but Re $A(U)$ is real pervasive on bdy U, by Theorem 13, since all GG-points lie on $\mathbb{R}$ and are influenced by each boundary point of U. Theorem 7 tells us that $A(U)$ is not complex pervasive on bdy U.

Example (15)

If we modify Example 14 so that $\cup_{n=1}^{\infty}[a_n - r_n, a_n + r_n]$ has for its closure $[-2,-1]\cup[1,2]$ and take

$$\begin{aligned} T &= [-2,-1]\cup[1,2]\cup\bigcup_{n=1}^{\infty}\mathbb{B}(a_n,r_n) \\ U &= \mathbb{C}\setminus T\ . \end{aligned}$$

Then U is connected, so $A(U)$ is complex pervasive on bdy U by Theorem 6, but U is not simply-connected, so $A(U)$ is not Dirichlet on bdy U.

Example (16)

If, instead, we take T as in Example 15 and let

$$U' = (\mathbb{C} \setminus T) \cup \bigcup_{n=1}^{\infty} \mathbb{U}(a_n, r_n) \ ,$$

we obtain U' with components $\mathbb{C} \setminus T$, $\mathbb{U}(a_n, r_n)$ (n=1, 2, 3,. . .). This U' has some inessential boundary (some circle arcs). Applying a suitable homeomorphism of the plane, we can convert these circle arcs to nowhere-differentiable arcs, and thus convert U' to a set U with essential boundary. In that case $A(U)$ is not Dirichlet on bdy U, Re $A(U)$ is real pervasive by Theorem 11, but $A(U)$ is not complex pervasive on bdy U by Theorem 7.

Example (17)

Let T be as in Example 14 and let

$$S = T \cup \{iz : z \in T\} .$$

Then $U = \mathbb{C} \setminus S$ has four components. We can arrange that there are GG-points for U on the positive and negative real and imaginary axes. In that case, for each point $p \in \text{bdy } U$ there exists a GG-point q not influenced by p. Thus $A(U)$ is not real pervasive on bdy U.

Proof of Theorem 6:

Theorem (6)

Let U be a connected open subset of $\hat{\mathbb{C}}$, and let $\operatorname{bdy} U$ be nonempty and essential. Then $A(U)$ is complex pervasive on $\operatorname{bdy} U$. A fortiori, $\operatorname{Re} A(U)$ is real pervasive on $\operatorname{bdy} U$.

Tools:

Let $m =$ Lebesgue measure on $\mathbb{C}$.

μ be a complex measure with compact support.

The **Cauchy transform** of μ:

$$\hat{\mu}(\xi) = \frac{1}{\pi} \int \frac{d\mu(z)}{\xi - z} .$$

$R(K) =$ the uniform closure on $\hat{\mathbb{C}}$ of the algebra of all continuous functions on $\hat{\mathbb{C}}$ that are analytic near K.

Theorem (19)

$\hat{\mu}$ is defined m-a.e.
$\hat{\mu}$ is holo on $\mathbb{C} \setminus \mathrm{spt}\ \mu$.
If $\hat{\mu} = 0$ m-a.e., then $\mu = 0$.
Let $K^{compact} \subset \hat{\mathbb{C}}$. Then $\hat{\mu} = 0$ off K if and only if $\mu \perp R(K)$.
If $K^{compact} \subset \mathbb{C}$ and $\mu = m_{|K}$, then $\hat{\mu}$ is continuous.

Proof of Theorem 6. Let $\mu \in \mathrm{M}(\text{bdy } U, \mathbb{C})$, $\mu \perp A(U)$ and suppose that spt $\mu \neq$ bdy U. We shall prove that $\mu = 0$.
As $\mu \perp A(U)$, it follows that $\mu \perp R(\text{clos } U)$ so by (iv) of Theorem 19, $\hat{\mu} = 0$ in $\hat{\mathbb{C}} \setminus \text{clos } U$.
Suppose now that $a \in \text{bdy } U \setminus \text{spt } \mu$, $a \neq \infty$. Choose $r > 0$ sufficiently small so that $\mathbb{B}(a, r) \cap \text{spt } \mu = \varnothing$. By hypothesis, given a compact set $K \subset \text{bdy } U \cap \mathbb{B}(a, r)$, the continuous analytic capacity $\alpha(K)$ of K is positive, so there exists $f_n \in A(\hat{\mathbb{C}} \setminus K)$, f_n nonconstant and $\|f_n\| = 1$. After a rotation of the Riemann sphere, $f_n(p_n) = 1$ for some $p_n \in K$ and $\|f_n\| < 1$ off $\mathbb{B}(a, r)$ by the maximum modulus principle. Note that $\hat{\mu}$ is analytic near $\mathbb{B}(a, r)$.

Next, $\hat{\mu}(p_n) = 0$ because otherwise

$$\nu = \frac{1}{\hat{\mu}(p_n)} \frac{\mu}{z - p_n}$$

is a complex representing measure for p_n on $A(U)$ and

$$1 = f_n^k(p_n) = \int f_n^k \, d\nu \longrightarrow 0 \quad \text{as } k \uparrow +\infty$$

which is a contradiction.
Consequently, a is an accumulation point of zeros of $\hat{\mu}$. By (ii) of Theorem 19 we can conclude that $\hat{\mu} = 0$ on $\mathbb{B}(a, r)$, and therefore, since U is connected, $\hat{\mu} = 0$ on U. Hence $\hat{\mu} = 0$ on $\hat{\mathbb{C}} \setminus \text{spt } \mu$.

Finally, let $E \subset \text{bdy } U$ be compact. Let $\lambda = m_{|E}$. By (v) of Theorem 19, $\hat{\lambda}$ is continuous and therefore $\hat{\lambda} \in A(U)$, so by Fubini's Theorem

$$0 = \int \hat{\lambda}\, d\mu = -\int \hat{\mu}\, d\lambda = \int_E \hat{\mu}\, dm\ ,$$

so $\hat{\mu} = 0$ m-almost everywhere on bdy U.
As spt $\mu \subset$ bdy U it follow then that $\hat{\mu} = 0$ m-almost everywhere on $\hat{\mathbb{C}}$, so by (iii) of Theorem 19, $\mu = 0$.

Proof of Theorem 7

Theorem (7)

Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on $\mathrm{bdy}\ U$ *if and only if* $\mathrm{bdy}\ U_i = \mathrm{bdy}\ U$ *for each component U_i of U.*

Proof of Theorem 7

Theorem (7)

Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on bdy U *if and only if* bdy $U_i =$ bdy U *for each component U_i of U.*

The “if” direction is proved by essentially the same argument as that for Theorem 6.

Proof of Theorem 7

Theorem (7)

Suppose U is a (possibly disconnected) proper open subset of $\hat{\mathbb{C}}$ without inessential boundary points. Then $A(U)$ is complex pervasive on bdy U *if and only if* bdy $U_i =$ bdy U *for each component U_i of U.*

The "if" direction is proved by essentially the same argument as that for Theorem 6.

To see the "only if" direction, suppose U has a component U_i with bdy $U_i \neq$ bdy U. We may choose a nonzero annihilating measure μ for $A(U_i)$ supported on bdy U_i, which is a proper subset of bdy U. Then μ annihilates $A(U)$, and this shows that $A(U)$ is not complex pervasive on bdy U.

Proof of Theorem 11

Theorem (11)

Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U has at least one component U_k so that $\text{bdy } U_k = \text{bdy } U$. Then $\text{Re } A(U)$ is real pervasive.

Proof of Theorem 11

Theorem (11)

Suppose $U \subset \hat{\mathbb{C}}$ is open and proper, with no inessential boundary points. Suppose U has at least one component U_k so that $\text{bdy } U_k = \text{bdy } U$. Then $\text{Re } A(U)$ is real pervasive.

Steps:

1. We review the basic notation and ideas about T-invariance.
2. Lemma about $A(U) + R(K)$.
3. Davie's Theorem.
4. Technical approximation lemmas.

For a continuous function $f \in C(\mathbb{C}, \mathbb{C})$, having compact support, we define the Cauchy transform

$$\mathcal{C}f = \widehat{f\ m},$$

where m, as before, denotes the Lebesgue measure on $\mathbb{C}$. We have

$$\frac{\partial}{\partial \overline{z}}(\mathcal{C}f) = f$$

in the sense of distributions, so that (by Weyl's Lemma), $\mathcal{C}f$ is holomorphic off spt f.

For a bump function φ and $f \in C(\mathbb{C}, \mathbb{C})$, we define

$$T_\varphi f = \varphi\ f - \mathcal{C}\left(f\ \frac{\partial \varphi}{\partial \overline{z}}\right)\ .$$

The linear operator T_φ (the **Vitushkin localization operator**) is continuous from $C(\mathbb{C}, \mathbb{C})$ into itself.

A subalgebra $A \subset C(\mathbb{C}, \mathbb{C})$ is said to be T-**invariant** if

$$T_\varphi f \in A\ , \quad \forall\ f \in A\ , \quad \forall\ \varphi.$$

We note that

$$\frac{\partial}{\partial \bar{z}} \; T_\varphi f = \varphi \; \frac{\partial f}{\partial \bar{z}} \tag{1}$$

in the sense of distributions, so that $T_\varphi f$ is holomorphic whenever f is holomorphic and off spt φ. This is the basis for the utility of T_φ in localizing singularites of analytic functions. It is obvious from this observation that $A(U)$ is a T-invariant algebra, whenever $U \subset \hat{\mathbb{C}}$ is open. So also is $\mathcal{O}(K)$, the algebra of all functions continuous on $\hat{\mathbb{C}}$ and holomorphic near K, whenever $K \subset \hat{\mathbb{C}}$. Since T_φ is continuous on $C(\mathbb{C}, \mathbb{C})$ it follows that $\mathrm{clos}_{C(\hat{\mathbb{C}},\mathbb{C})}\, \mathcal{O}(K)$ is also T-invariant. But this closure is, by Runge's Theorem, equal to $R(K)$, whenever K is compact.

Lemma

Let $U \in \hat{\mathbb{C}}$ be open and $K \in \mathbb{C}$ compact. Then

$$B = \mathrm{clos}_{C(\hat{\mathbb{C}},\mathbb{C})} \left(A(U) + R(K)\right)$$

is a T-invariant algebra.

(It is obvious that $A(U) + R(K)$ is T-invariant, and hence so is B. It is less obvious, but true, that its closure is an algebra.)

It is possible to associate a capacity γ_A to each T-invariant algebra A (Davie), generalizing the association of $E \longmapsto \alpha(E \setminus U)$ to $A(U)$. Davie showed that closed T-invariant algebras are uniquely determined by their corresponding capacities.

Davie's Theorem

(TAMS 171 (1971)) Let A_0 denote the algebra of all bounded borel functions on $\mathbb{C}$ which are analytic outside some compact set, and let φ be a test function. Let A_1 and A_2 be T_φ-invariant subalgebras of A_0, and suppose all functions in A_1 are continuous on $\mathbb{C}$. Suppose also that for all $z \in \mathbb{C}$ we can find $m,\ r,\ \delta_0 > 0$ with $\gamma_{A_1}(\mathbb{U}(z,\delta)) \le m\ \gamma_{A_2}(\mathbb{U}(z,r\delta))$ and $0 < \delta < \delta_0$. Let $f \in A_1$. Then f is in the uniform closure of A_2.

Using Davie's result we establish an approximation lemma. This is the key ingredient in the proof of the theorem.

Lemma (23)

Let $U \subset \hat{\mathbb{C}}$ be open and proper, and suppose bdy U *is essential. Let $\{U_i : \ i \in I\}$ be the set of connected components of U. Let $a \in$* bdy U *and $r > 0$. Let*

$$\begin{aligned} V &= \mathbb{U}(a,r) \cup \bigcup_{i\in I} \{U_i : \ U_i \cap \mathbb{U}(a,r) \neq \varnothing\} \\ K &= \hat{\mathbb{C}} \setminus V \\ W &= \bigcup_{i\in I} \{U_i : \ U_i \cap \mathbb{U}(a,r) = \varnothing\}, \\ \text{and } B &= A(U) + R(K) \ . \end{aligned}$$

Then B is dense in $A(W)$.

The proof of Theorem 13 also involves capacity estimates, and comes down to the following lemma:

Lemma

Suppose U, a, r, V, K and W are as in Lemma 23. Suppose that all components of W are simply-connected. Then there are no GG-points for U on $\mathrm{bdy}\ W \setminus \mathrm{bdy}\ V$ if and only if $A(W)$ is Dirichlet on $\mathrm{bdy}\ W$.

Remarks

- Netuka's theorem also holds if the assumption that U be the interior of the closure of U is relaxed to the assumption that the boundary of U is essential for continuous functions harmonic on U.
- The theorem on complex pervasiveness extends to open sets U with compact closure on an arbitrary open Riemann surfaces.