Non-Hausdorff Symmetries of C*-Algebras

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Motivation: Model quotient groups

Example (Quantum torus)

Let $\lambda \in \mathbb{T}$. The C*-algebra of the quantum torus A_λ is generated by two unitaries U and V that satisfy the commutation relation $UV = \lambda VU$.
This is the algebra of functions on the quotient group $\mathbb{T}/\lambda^{\mathbb{Z}}$.

Theorem (Piotr Sołtan)

The C-algebra A_λ admits no quantum group structure.*

Question

How can we describe quotient groups such as $\mathbb{T}/\lambda^{\mathbb{Z}}$?
— We use notions from higher category theory:
2-categories, crossed modules

Motivation: Quotient group actions on C*-algebras

Question

How do such quotient groups act on C*-algebras?
— $\mathbb{T}$ acts on A_λ by the gauge action

$$\alpha_z(U) := zU, \qquad \alpha_z(V) := V.$$

Its restriction to $\lambda^{\mathbb{Z}}$ is inner, induced by $u_n := V^{-n}$:

$$V^{-n} x V^n = \alpha_{\lambda^n}(x).$$

Holonomy groupoids of foliations are often non-Hausdorff but locally Hausdorff.
They should therefore be viewed as 2-groupoids.
We currently have no tools to compute their K-theory.
They have too few actions when viewed as groupoids.

Motivation: Weak notions of action

Question

Can we learn something about actions of ordinary groups?
— Yes, we can!
Higher category explains various notions related to group actions on C*-algebras, including

- a notion of weak group action that yields Busby–Smith twisted actions and saturated Fell bundles, and
- a notion of weakly equivariant map that combines equivariant maps and outer equivalence of group actions and contains covariant representations.

From quotient spaces to groupoids

Many interesting moduli spaces, like moduli spaces of Riemann surfaces, are quotient spaces:

- Points are parametrised by a nice space X.
- Reasons for parameters $x, y \in X$ to yield the same point are encoded in a nice space G with two maps $r, s\colon G \rightrightarrows X$.
- Reasons for x, y and y, z to describe the same point yield a reason for x, z to describe the same point. Same for x, y and y, x, and for x, x.
- Thus X and G are the object and arrow spaces of a groupoid.

Example (Quantum torus)

The quotient space $\mathbb{T}/\lambda^{\mathbb{Z}}$ for $\lambda \in \mathbb{T}$ is parametrised by $X = \mathbb{T}$, $G = \mathbb{T} \times \mathbb{Z}$, $s(z, n) := z$, $r(z, n) := \lambda^n \cdot z$.

Quotient groups

Groups describe symmetries of classical spaces.

Question

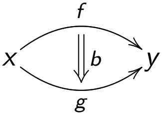
How to describe the symmetries of a quotient space?

- The symmetry group may itself become a quotient.
- It should be the orbit space of a groupoid.

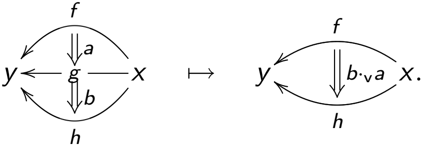
Question

Which extra structure on a groupoid
reflects a group structure on its orbit space?

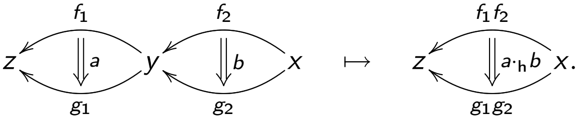
Data of a 2-category

- objects, arrows $x \to y$, and bigons 

- composition of arrows
- vertical composition of bigons



- horizontal composition of bigons



A 2-category of C*-algebras

objects	C^*-algebras
arrows	non-degenerate *-homomorphisms $A \to \mathcal{M}(B)$
bigons	unitary intertwiners: $u\colon f \Rightarrow g$ $u \cdot f(a) = g(a) \cdot u$
arrow composition	clear
vertical composition	multiplication of unitaries
horizontal composition	If $uf_1(b) = g_1(b)u$ and $vf_2(a) = g_2(a)v$, then $wf_1\big(f_2(a)\big) = g_1\big(g_2(a)\big)w$ with $$w := g_1(v)u = uf_1(v).$$

This 2-category encodes the wisdom that
inner automorphisms are almost but not quite trivial.

A weak 2-category of C*-algebras

objects	C*-algebras
arrows	Hilbert B-modules with a non-degenerate *-homomorphism $A \to \mathbb{B}(B)$ – correspondences
bigons	unitary intertwiners
composition	balanced tensor product of correspondences
vertical composition	multiplication of unitaries
horizontal composition	balanced tensor product of intertwining unitaries

Warning

The composition of correspondences is only associative up to isomorphism – weak 2-category.

Quotient groups as crossed modules

Definition (Crossed module)

G a group,

H a group,

∂ a group homomorphism $H \to G$,

c an action of G on H by group automorphisms,

such that $\partial(c_g h) = g\partial(h)g^{-1}$ and $c_{\partial(h)}(k) = hkh^{-1}$.

Example (Quotient by a normal subgroup)

Let G be a group, H a normal subgroup.
Let $\partial\colon H \to G$ be the inclusion map, and
let $c\colon G \to \mathrm{Aut}(H)$ be the conjugation action.

Example (Quantum torus)

Let $G = \mathbb{T}$, $H = \mathbb{Z}$, $\partial(n) = \lambda^n$, and $c_z = \mathrm{id}$ for all $z \in \mathbb{T}$.

Some facts about crossed modules

- Crossed modules are equivalent to 2-groups with strictly associative composition of arrows, strict units, and strict inverses.
- $\partial(H)$ is a normal subgroup in G.
- $\ker \partial$ is a central subgroup in H.
- Crossed modules with injective ∂ are equivalent to normal subgroups.
- Crossed modules with surjective ∂ are equivalent to central group extensions.

Crossed modules in topology

J.H.C. Whitehead introduced crossed modules to classify homotopy 2-types, spaces with trivial π_n for $n \neq 1, 2$. Here $\ker \partial = \pi_2$ and $\operatorname{coker} \partial = \pi_1$.

A crossed module of automorphisms

The following crossed module is closely related to the 2-category of C*-algebras that we have already met.

- *G* Aut(A) for a C*-algebra A
- *H* $\mathcal{UM}(A)$, the group of unitary multipliers of A
- ∂ $\mathrm{Ad}\colon \mathcal{UM}(A) \to \mathrm{Aut}(A)$, $\mathrm{Ad}_u(a) := uau^{-1}$
- *c* automorphisms of A act on unitary multipliers

The kernel of Ad is the group of central unitary multipliers.
The cokernel of Ad is the outer automorphism group.
Both may be non-trivial.

Crossed module actions on C*-algebras

Definition

A crossed module (G, H, ∂, c) acts on a C*-algebra A by a homomorphism to the automorphism crossed module of A:

α group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$

u group homomorphism $u\colon H \to \mathcal{UM}(A)$

such that

- $\alpha_{\partial(h)}(a) = u_h a u_h^{-1}$
- $\alpha_g(u_h) = u_{c_g(h)}$

Known special case

If $H \subseteq G$ is a normal subgroup, the above is the same as a twisted action in the sense of P. Green.

An example

Free transitive actions

Any group acts on its underlying space by left translations.
This is, up to isomorphism, the only free transitive actions.
This corresponds to a natural action of a crossed module (G, H, ∂, c) on $C_0(G) \rtimes H$.

Example (A crossed module action on quantum tori)

Let $\lambda \in \mathbb{T}$, let $\mathbb{Z}$ act on $\mathbb{T}$ by rotations by λ^n,
form $A_\lambda = C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$.
Then $\mathbb{T}$ acts on A_λ by the usual gauge action:

$$\alpha_z(U) := z \cdot U, \qquad \alpha_z(V) := V,$$

and $\mathbb{Z}$ acts by $n \mapsto V^{-n}$.

Crossed products

We may generalise definitions of P. Green to define crossed products for actions of crossed modules on C*-algebras.

Theorem

Let (G, H, ∂, c) be a crossed module and let B be a C-algebra with an action β of G. Then $B \rtimes_{\beta\circ\partial} H$ carries an induced action of (G, H, ∂, c), and the resulting crossed product is*

$$(B \rtimes_{\beta\circ\partial} H) \rtimes (G, H) \cong B \rtimes_\beta G.$$

Example (Quantum tori once again)

The crossed module $(\mathbb{T}, \mathbb{Z}, \lambda^?)$ acts on A_λ in this way. The crossed product is

$$A_\lambda \rtimes (\mathbb{T}, \mathbb{Z}, \lambda^?) \cong \mathrm{C}(\mathbb{T}) \rtimes \mathbb{T} \cong \mathbb{K}(L^2\mathbb{T}).$$

Weakening categorical notions

The general ideal

In 2-category theory, do not ask for arrows to be equal, ask that they be equivalent, that is, related by a bigon.

Definition

An arrow f in a 2-category is an equivalence if there is an arrow g in the opposite direction and invertible bigons that relate fg and gf to identity maps.

Example

Homotopy equivalences between topological spaces are equivalences in the homotopy 2-category of topological spaces.

Example

Equivalences of categories are equivalences in the 2-category of categories, which has functors as arrows and natural transformations as bigons.

Weak actions

Definition

A strict group action of a group G on a C*-algebra A is a map $\alpha_g \colon G \to \mathrm{Aut}(A)$ such that $\alpha_{g_1 g_2} = \alpha_{g_1}\alpha_{g_2}$ for all $g_1, g_2 \in G$.

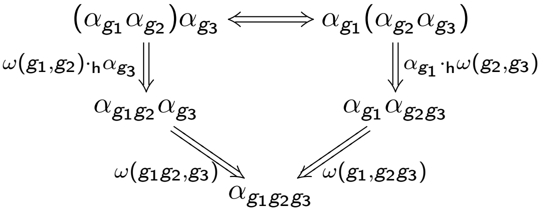
When we weaken this, we add bigons as additional data for all basic equations and coherence laws as additional constraints whenever there are two ways of deducing an equation from the basic equations.

A weak group action of G on A consists of:

- a map $\alpha_g \colon G \to \mathrm{Aut}(A)$
- a map $\omega \colon G \times G \to \mathcal{UM}(A)$ such that $\omega_{g_1,g_2} \colon \alpha_{g_1}\alpha_{g_2} \Rightarrow \alpha_{g_1 g_2}$ is a unitary intertwiner:

$$\omega_{g_1,g_2}\alpha_{g_1}\big(\alpha_{g_2}(a)\big) = \alpha_{g_1 g_2}(a)\omega_{g_1,g_2}$$

satisfying a coherence law that corresponds to the two ways of idenifying $\alpha_{g_1}\alpha_{g_2}\alpha_{g_3} = \alpha_{g_1 g_2}\alpha_{g_3} = \alpha_{g_1 g_2 g_3}$ and $\alpha_{g_1}\alpha_{g_2}\alpha_{g_3} = \alpha_{g_1}\alpha_{g_2 g_3} = \alpha_{g_1 g_2 g_3}$.



When these conditions are unravelled, we get the notion of a Busby–Smith twisted action.
When we use similar recipes in the weak 2-category of C^*-algebras with correspondences as arrows,
we get saturated Fell bundles.

Weakly equivariant maps

Definition

Let G be a group and let (A, α) and (B, β) be group actions. An equivariant map between them is a map $f\colon A \to B$ with $\alpha_g f = f \beta_g$.

Definition

Let (A, α, ω^A) and (B, β, ω^B) be weak actions of a group G. A weakly equivariant map between them is a

- *-homomorphism $f\colon A \to B$ with
- unitary intertwiners $V_g\colon \alpha_g f \Rightarrow f \beta_g$ for all $g \in G$

that satisfy a coherence law for the two ways of deducing
$f \alpha_{g_1} \alpha_{g_2} = f \alpha_{g_1 g_2} = \beta_{g_1 g_2} f$ and
$f \alpha_{g_1} \alpha_{g_2} = \beta_{g_1} f \beta_{g_2} = \beta_{g_1} \beta_{g_2} f = \beta_{g_1 g_2} f$.

The coherence law amounts to the usual cocycle conditions for outer equivalence of actions if $f = \mathrm{id}$.

Summary

- Quotient groups may be described as crossed modules or as 2-categories.
- Roughly speaking, their actions on C^*-algebras are actions by *-homomorphisms that are only well-behaved up to specified unitary intertwiners, subject to certain cocycle conditions.
- Standard notions from higher category theory give new interpretations for known concepts about group actions on C^*-algebras.